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ANALYSIS

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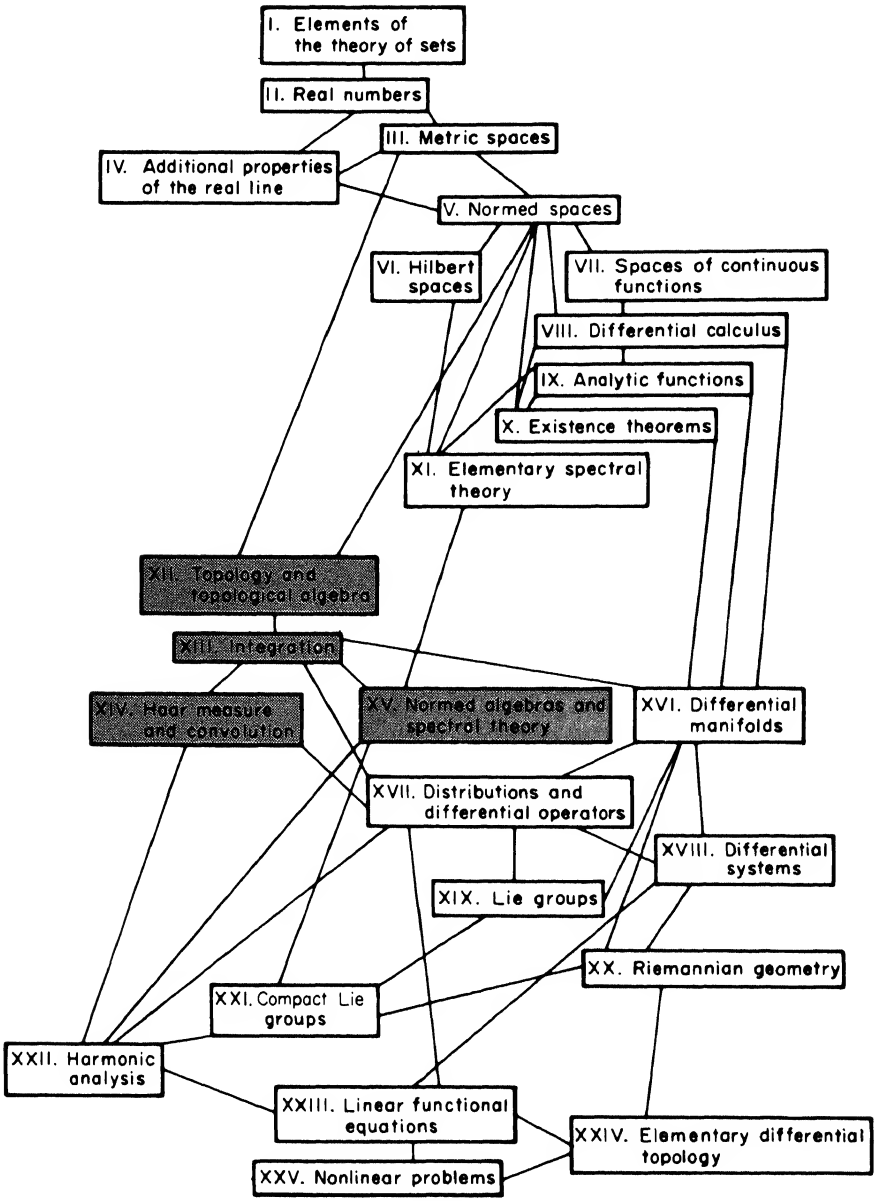

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NOTATION

In the following definitions, the first number is the number of the chapter and the second number is the number of the section within that chapter.

$\mathrm{Supp}(f)$	support of a function: 12.6
φ_A	characteristic function of a subset A of a set: 12.7
$\sup_\alpha f_\alpha$, $\inf_\alpha f_\alpha$	upper and lower envelopes of a family of real-valued functions: 12.7
$\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$	upper and lower limit of a sequence of real-valued functions: 12.7
G^0	opposite of a group G: 12.8
$\mathbf{GL}(E)$	group of automorphisms of a topological vector space E: 12.8
A^{-1}	set of elements x^{-1} where $x \in A$ (where A is a subset of a group in which the group law is written multiplicatively): 12.8
AB	set of elements xy with $x \in A$ and $y \in B$, where A and B are subsets of a group in which the group law is written multiplicatively: 12.8
$\mathscr{N}(H)$, $\mathscr{Z}(H)$	normalizer and centralizer of a subset H of a group: 12.8
V^∞	set of finite products $x_1 x_2 \dots x_n$ (with n arbitrary) of elements of a subset V of a group: 12.8

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V^∞	set of finite products $x_1 x_2 \dots x_n$ (with n arbitrary) of elements of a subset V of a group: 12.8

$\mathbf{Z}_p$	set of p-adic integers: 12.9, Problem 4
$\mathbf{T}_p$	p-adic solenoid: 12.9, Problem 4
G/H, H\G	homogeneous space of left (right) cosets of a subgroup H in a group G: 12.10 and 12.11
$G \cdot A$	union of the orbits of the points of a subset A of E, with respect to an action of G on E: 12.10
E/G	space of orbits with respect to G: 12.10
$\mathscr{C}_{\mathbf{C}}(X)$	space of continuous complex-valued functions on X: 12.14
E/F	quotient normed space of a normed space E by a vector subspace F: 12.14
$\langle x, x' \rangle$, $\langle x', x \rangle$	$x'(x)$ for a vector $x \in E$ and a continuous linear form $x' \in E'$ (the dual of E): 12.15
${}^t u$	transpose of a continuous linear mapping: 12.15
$\mathbf{U}(E)$	unitary group of a Hilbert space E: 12.15, Problem 8
$\mathscr{K}(X; K)$, $\mathscr{K}_{\mathbf{C}}(X; K)$	space of continuous complex-valued functions with support contained in the compact subset K of X: 13.1
$\mathscr{K}(X)$, $\mathscr{K}_{\mathbf{C}}(X)$	space of continuous complex-valued functions on X with compact support: 13.1
ε_x	Dirac measure at the point x: 13.1
$g \cdot \mu$	measure with density g with respect to μ: 13.1 and 13.13
$\pi(\mu)$	image of a measure μ under a proper continuous mapping π: 13.1 and 13.1, Problem 8
μ_U, $\mu \mid U$	measure induced on an open set U by a measure μ (or restriction of μ to U): 13.1
$\mathrm{M}_{\mathbf{C}}(X)$, $\mathrm{M}(X)$	space of complex measures on X: 13.1
$\langle f, \mu \rangle$, $\mu(f)$, $\int f\, d\mu$, $\int f(x)\, d\mu(x)$	integral of $f \in \mathscr{K}(X)$ with respect to μ: 13.1
$\mathscr{K}_{\mathbf{R}}(X; K)$	space of continuous real-valued functions with support contained in the compact subset K of X: 13.2

$\mathscr{K}_{\mathbf{R}}(X)$	space of continuous real-valued functions on X with compact support: 13.2
$\bar{\mu}$	conjugate of the measure μ: 13.2
$M_{\mathbf{R}}(X)$	space of real measures on X: 13.2
$\mathscr{R}\mu, \mathscr{I}\mu$	real and imaginary parts of a complex measure: 13.2
$M_+(X)$	set of positive measures on X: 13.3
f^+, f^-	positive and negative parts of a real-valued function f: 13.3
$\mu \leqq \nu$	order relation between real measures: 13.3
$\lvert\mu\rvert$	absolute value of a complex measure: 13.3
$\mathscr{I}, \mathscr{I}(X)$	set of lower semicontinuous functions on X which are bounded below by a function belonging to $\mathscr{K}_{\mathbf{R}}(X)$: 13.5
$\mu^*(f), \int^* f\,d\mu, \int^* f(x)\,d\mu(x)$	upper integral of f: 13.5
$\sum_n t_n$	sum of a sequence (t_n) of elements $\geqq 0$ of $\overline{\mathbf{R}}$: 13.5
$\mathscr{S}, \mathscr{S}(X)$	set of upper semicontinuous functions on X which are bounded above by a function belonging to $\mathscr{K}_{\mathbf{R}}(X)$: 13.5
$\mu_*(f), \int_* f\,d\mu, \int_* f(x)\,d\mu(x)$	lower integral of f: 13.5
$\mu^*(A), \mu_*(A)$	outer and inner measures of $A \subset X$: 13.5
$\tilde{f}$	equivalence class of f (with respect to μ): 13.6
$\tilde{f} \leqq \tilde{g}$	order relation between equivalence classes: 13.6
$\tilde{f} + \tilde{g}, \tilde{f}\tilde{g}$	sum and product of equivalence classes: 13.6
$\int f\,d\mu, \int f(x)\,d\mu(x),$ $\mu(f), \langle f, \mu\rangle$	integral of a μ-integrable function: 13.7
$\mathscr{L}^1_{\mathbf{R}}(X, \mu), \mathscr{L}^1_{\mathbf{R}}(\mu), \mathscr{L}^1_{\mathbf{R}}$	space of (finite) real-valued μ-integrable functions: 13.7
$\mu(A)$	measure of a μ-integrable set: 13.7
$\mu(\tilde{f})$	integral of an equivalence class: 13.7
$\int_A f\,d\mu, \int_A f(x)\,d\mu(x)$	integral of f over A: 13.9
$\int_A^* f\,d\mu$	upper integral of f over A: 13.9
μ_Y	measure induced by μ on a closed subspace Y: 13.9

$H(\alpha)$	entropy of a finite partition α: 13.9, Problem 27
$H(\alpha/\beta)$	entropy of a finite partition α relative to a finite partition β: 13.9, Problem 27
$\bigvee_{j=1}^{n} \alpha_j$, $\alpha_1 \vee \ldots \vee \alpha_n$	least upper bound of a finite sequence of finite partitions: 13.9, Problem 27
$h(u, \alpha)$	entropy of a mapping u relative to a finite partition α: 13.9, Problem 28
$h(u)$	entropy of a mapping u: 13.9, Problem 28
$\int \mathbf{f}\, d\mu$, $\int \mathbf{f}(x)\, d\mu(x)$	integral of a vector-valued function: 13.10
$\mathscr{L}^1_{\mathbf{C}}(X, \mu)$, $\mathscr{L}^1_{\mathbf{C}}(\mu)$, $\mathscr{L}^1_{\mathbf{C}}$	space of complex-valued μ-integrable functions: 13.10
$N_1(f)$	$\int^* \lvert f\rvert\, d\mu$: 13.11
$N_2(f)$	$\left(\int^* \lvert f\rvert^2\, d\mu\right)^{1/2}$: 13.11
$\mathscr{L}^2_{\mathbf{R}}(X, \mu)$, $\mathscr{L}^2_{\mathbf{R}}(\mu)$, $\mathscr{L}^2_{\mathbf{R}}$	space of (finite) real-valued square-integrable functions: 13.11
$\mathscr{N}$	space of μ-negligible functions: 13.11
$L^p_{\mathbf{R}}(X, \mu)$, $L^p_{\mathbf{R}}(\mu)$, $L^p_{\mathbf{R}}$	space of classes of pth power integrable functions: 13.11 and 13.11, Problem 12
$N_p(\tilde{f})$	$N_p(f)$ for any f in the class $\tilde{f}$: 13.11
$\mathscr{L}^2_{\mathbf{C}}(X, \mu)$, $\mathscr{L}^2_{\mathbf{C}}(\mu)$, $\mathscr{L}^2_{\mathbf{C}}$	space of complex-valued square-integrable functions: 13.11
$L^2_{\mathbf{C}}(X, \mu)$, $L^2_{\mathbf{C}}(\mu)$, $L^2_{\mathbf{C}}$	space of classes of complex-valued square-integrable functions: 13.11
$M_\infty(f)$, $m_\infty(f)$, $\operatorname{ess\,sup}_{x \in X} f(x)$, $\operatorname{ess\,inf}_{x \in X} f(x)$	maximum and minimum in measure of f: 13.12
$\mathscr{L}^\infty_{\mathbf{R}}(X, \mu)$, $\mathscr{L}^\infty_{\mathbf{R}}(\mu)$, $\mathscr{L}^\infty_{\mathbf{R}}$	space of real-valued μ-measurable functions bounded in measure: 13.12
$\mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$, $\mathscr{L}^\infty_{\mathbf{C}}(\mu)$, $\mathscr{L}^\infty_{\mathbf{C}}$	space of complex-valued μ-measurable functions bounded in measure: 13.12
$L^\infty_{\mathbf{R}}(X, \mu)$, $L^\infty_{\mathbf{R}}(\mu)$, $L^\infty_{\mathbf{R}}$	space of classes of real-valued μ-measurable functions bounded in measure: 13.12
$L^\infty_{\mathbf{C}}(X, \mu)$, $L^\infty_{\mathbf{C}}(\mu)$, $L^\infty_{\mathbf{C}}$	space of classes of complex-valued μ-measurable functions bounded in measure: 13.12
$\mathscr{S}(X, \mu)$	space of (finite) real-valued μ-measurable functions: 13.12, Problem 2
$S(X, \mu)$	space of classes of (finite) real-valued μ-measurable functions: 13.12, Problem 2

$\mathscr{L}^1_{\mathrm{loc},\mathbf{R}}(X,\mu)$, $\mathscr{L}_{\mathrm{loc}}(X,\mu)$, $\mathscr{L}_{\mathrm{loc}}(X)$, $\mathscr{L}_{\mathrm{loc}}$	space of real-valued locally μ-integrable functions: 13.13
$\mathscr{L}^1_{\mathrm{loc},\mathbf{C}}(X,\mu)$	space of complex-valued locally μ-integrable functions: 13.13
$\mathrm{L}^1_{\mathrm{loc},\mathbf{R}}(X,\mu)$, $\mathrm{L}^1_{\mathrm{loc},\mathbf{C}}(X,\mu)$	space of classes of real (complex) valued locally μ-integrable functions: 13.13
$\int f\,d\lambda$, $\int f(x)\,d\lambda(x)$	integral with respect to a complex measure: 13.16
$\mathrm{Supp}(\mu)$	support of a measure: 13.19
$\|\mu\|$	norm of a measure: 13.20
$\mathrm{M}^1_{\mathbf{R}}(X)$, $\mathrm{M}^1_{\mathbf{C}}(X)$	space of bounded real (complex) measures: 13.20
$\mathscr{C}^0_{\mathbf{R}}(X)$, $\mathscr{C}^0_{\mathbf{C}}(X)$	space of real (complex) valued continuous functions which tend to 0 at infinity: 13.20
$\int d\mu(y)\int h(x,y)\,d\lambda(x)$, $\iint h\,d\lambda\,d\mu$, $\iint h(x,y)\,d\lambda(x)\,d\mu(y)$	integral with respect to the product measure $\lambda\otimes\mu$: 13.21
$\lambda\otimes\mu$	product measure: 13.21
$\iint^* h\,d\lambda\,d\mu$, $\iint^* h(x,y)\,d\lambda(x)\,d\mu(y)$	upper integral with respect to a product measure: 13.21
$\iint_* h\,d\lambda\,d\mu$, $\iint_* h(x,y)\,d\lambda(x)\,d\mu(y)$	lower integral with respect to a product measure: 13.21
$f\otimes g$	the function $(x,y)\mapsto f(x)g(y)$: 13.21
$\mu_1\otimes\mu_2\otimes\cdots\otimes\mu_n$, $\bigotimes_{i=1}^{n}\mu_i$	product of n measures: 13.21
$\iint\cdots\int f\,d\mu_1\cdots d\mu_n$, $\iint\cdots\int f(x_1,\ldots,x_n)\,d\mu_1(x_1)\cdots\cdots d\mu_n(x_n)$	integral with respect to the product of the n measures: 13.21
$\bigotimes_{n=1}^{\infty}\mu_n$	product of an infinite sequence of measures: 13.21, Problem 9
$\gamma(s)f$, $\delta(s)f$	translates of a function: 14.1
$\gamma(s)\mu$, $\delta(s)\mu$	translates of a measure: 14.1
$\check{f}$, $\check{\mu}$	transforms of a function or measure under $s\mapsto s^{-1}$: 14.1

$\mathbf{T}$	one-dimensional torus: 14.2
$\Delta_G(s)$, $\Delta(s)$	modulus function on a group: 14.3
$\mathrm{mod}_G(u)$, $\mathrm{mod}(u)$	modulus of an automorphism of G: 14.3
$\mathbf{Q}_p$	field of p-adic numbers: 14.3, Problem 6
$f^\flat$	average of f over a fibre: 14.4, Problem 2
$\mu_1 * \mu_2 * \cdots * \mu_n$	convolution of n measures: 14.5
$\mathrm{M}_{\mathbf{C}}^c(\mathrm{X})$	space of measures with compact support: 14.7
$\mu^\natural$	centralization of a measure: 14.7, Problem 6
$\mu * f$, $\mu *^\beta f$, $f * \mu$, $f *^\beta \mu$	convolution of a measure and a function: 14.8
$f * g$, $f *^\beta g$	convolution of two functions: 14.10
$\mathrm{Sp}_A(x)$, $\mathrm{Sp}(x)$	spectrum of an element x of an algebra A: 15.2
$\rho(x)$	spectral radius of x: 15.2
$\mathbf{X}(\mathrm{A})$	spectrum of an algebra A: 15.3
$\mathscr{G}$	Gelfand transformation: 15.3
$\mathscr{H}^p(\mu)$, $\mathscr{H}^\infty(\mu)$, $\mathrm{H}^p(\mu)$, $\mathrm{H}^\infty(\mu)$	Hardy spaces: 15.3, Problem 15
x^*	adjoint (in an algebra with involution): 15.4
$\|u\|_2$	Hilbert-Schmidt norm: 15.4
$\check{\mu}$	involution in $\mathrm{M}_{\mathbf{C}}^1(\mathrm{G})$: 15.4
$\mathscr{L}_2(\mathrm{E})$	algebra of Hilbert-Schmidt operators: 15.4
$p(x)$, $p_A(x)$	$(\rho(x^*x))^{1/2}$: 15.4, Problem 18
$U(x)$, $U_A(x)$	regular representation of a complete Hilbert algebra: 15.8
$\mathbf{H}(\mathrm{A})$	space of hermitian characters: 15.9
$M_\mu(u)$	multiplication by the class of u: 15.10
$\mathscr{U}_{\mathbf{C}}(\mathrm{K})$	space of bounded universally measurable complex-valued functions on K: 15.10
$H \geqq 0$	$(H \cdot x \mid x) \geqq 0$ for all x: 15.11
$f(N)$	function of a normal operator: 15.11
$H^{1/2}$	square root of a positive hermitian operator: 15.11
$\mathrm{abs}(T)$	absolute value of an operator: 15.11, Problem 6
$\mathrm{Tr}(T)$	trace of a nuclear operator: 15.11, Problem 7

$\|T\|_1$	nuclear norm: 15.11, Problem 7
$\mathscr{L}_1(E)$	space of nuclear operators: 15.11, Problem 7
$\mathrm{Spa}(T)$	approximative point-spectrum of an operator: 15.11, Problem 9
$\mathrm{dom}(T)$	domain of an unbounded operator: 15.12
T^*	adjoint of an unbounded operator with dense domain: 15.12
$\mathrm{Sp}(T)$	spectrum of an unbounded operator: 15.12
$f(N)$	function of an unbounded normal operator: 15.12

CHAPTER XII

TOPOLOGY AND TOPOLOGICAL ALGEBRA

As we said in the Introduction, we have sought to limit the material in this chapter to the minimum that the reader will require. We shall not stay to explore the refinements of general topology (filters, uniformities, separation axioms); instead, we shall pass as soon as possible to the category of uniformizable spaces, which are the only ones we shall meet in later chapters. Usually these spaces occur merely as "ambient spaces" in which it is convenient to operate, and we shall be mainly concerned—in conformity with the spirit of this book—with their separable metrizable subspaces. For this reason we have included many criteria of metrizability and separability (12.3.6, 12.4.6, 12.4.7, 12.5.8, 12.9.1, 12.10.10, 12.11.3, 12.14.6.2, 12.15.7, 12.15.9, 12.15.10). These, together with Baire's theorem and its consequences, are the only results in the chapter whose proofs are not straightforward. We have also included an account of various purely topological techniques which were not needed in the first volume (partitions of unity (12.6), semi-continuous functions (12.7)), and more than half the chapter is devoted to elementary concepts of topological algebra (topological groups, spaces with operators, topological vector spaces). All of this will be used constantly in the succeeding chapters.

1. TOPOLOGICAL SPACES

A *topology* on a set E is a set $\mathfrak{O}$ of subsets of E (in other words, a subset of $\mathfrak{P}(E)$) satisfying the following two conditions:

(O_I) *The union of any family* $(A_\lambda)_{\lambda \in L}$ *of sets belonging to* $\mathfrak{O}$ *belongs to* $\mathfrak{O}$;

(O_{II}) *The set* E *belongs to* $\mathfrak{O}$, *and the intersection of any two sets belonging to* $\mathfrak{O}$ *belongs to* $\mathfrak{O}$.

A *topological space* is a set E endowed with a topology $\mathfrak{O}$. The subsets of E which belong to $\mathfrak{O}$ are called the *open sets* of the topological space E. Taking $L = \varnothing$ in (O_I), it follows that the empty subset $\varnothing$ of E is always an open set, and from (O_{II}) we see that E itself is an open set.

Examples

(12.1.1) The set $\mathfrak{O}$ of open sets in a metric space E satisfies (O_I) and (O_{II}) (3.5.2 and 3.5.3). This topology $\mathfrak{O}$ is called the topology of the metric space E (or is said to be *defined by the distance* given on E). Two *topologically equivalent* distances (3.12) define the *same* topology. A topological space is said to be *metrizable* if its topology can be defined by a distance (and then this topology is also said to be *metrizable*).

On any set E, the set $\mathfrak{O} = \{\varnothing, E\}$ is a topology, called the *chaotic* topology. It is not metrizable if E has at least two elements, because otherwise there would exist an open ball containing one of these elements and not the other, and this is impossible because E is the only nonempty open set.

On a set $E = \{a, b\}$ consisting of two elements, the set $\mathfrak{O} = \{\varnothing, \{a\}, E\}$ is a nonmetrizable topology.

If $\mathfrak{O}_1, \mathfrak{O}_2$ are two topologies on the same set E, we say that $\mathfrak{O}_2$ is *finer* than $\mathfrak{O}_1$ (or that $\mathfrak{O}_1$ is *coarser* than $\mathfrak{O}_2$) if $\mathfrak{O}_1 \subset \mathfrak{O}_2$. Two topologies on E are said to be *comparable* if one is finer than the other. The chaotic topology is coarser than all others. The *discrete* topology (i.e., the topology defined by the metric (3.2.5), for which $\mathfrak{O} = \mathfrak{P}(E)$) is finer than all others. Two topologies on E are not necessarily comparable; for example, let $E = \{a, b\}$ be a set with two elements, and consider the two topologies $\mathfrak{O}_1 = \{\varnothing, \{a\}, E\}$ and $\mathfrak{O}_2 = \{\varnothing, \{b\}, E\}$.

2. TOPOLOGICAL CONCEPTS

We have already remarked (3.12), in the context of metric spaces, that the notions of *closed set*, *cluster point* of a set, *interior point* of a set, *frontier point* of a set, *neighborhood of a point* (or of a set), *dense set*, *continuous function* and *homeomorphism* are defined entirely in terms of the notion of an *open set* in a metric space. Their definitions can therefore be transferred, without any change, to the context of arbitrary topological spaces.

Moreover, all the properties involving these notions which were proved for metric spaces, and whose statements do not involve the distance (cf. (3.5) to (3.12)) remain valid for arbitrary topological spaces (*and we shall therefore make use of them in the general case*) with the following exceptions:

(1) The properties (3.8.11) and (3.8.12) are not true for arbitrary topological spaces, as is shown by the example of the space $E = \{a, b\}$ with the topology $\{\varnothing, \{a\}, E\}$. The existence of a distance is essential to the proof of these two propositions, although it does not feature in the enunciation.

(2) We may define as in (3.9) the notion of a *basis* of open sets (or of the topology) of a topological space E. The criterion (3.9.3) is valid in general. We can also prove as in (3.9.4) that if there exists a *denumerable* basis for the topology of a topological space E, then there exists an at most denumerable set which is dense in E; but the converse is not valid for arbitrary topological spaces (Section 12.4, Problem 6).

By definition, a basis $\mathfrak{B}$ of the topology of a topological space E has the following property:

The intersection of any two sets of $\mathfrak{B}$ *is a union of sets of* $\mathfrak{B}$.

Conversely, let $\mathfrak{B}$ be a set of subsets of a set E. If $\mathfrak{B}$ has the above property and if $E \in \mathfrak{B}$, then the set $\mathfrak{O}$ of (arbitrary) *unions* of sets of $\mathfrak{B}$ is a topology for which $\mathfrak{B}$ is a basis. For it is immediately seen that $\mathfrak{O}$ satisfies (O_I) and (O_{II}).

If E is a topological space and if F is any subset of E, then the set of intersections $U \cap F$, where U runs through the set of open subsets of E, satisfies the axioms (O_I) and (O_{II}) and is therefore a *topology* on F. This topology on F is said to be *induced* by the topology on E. The set F, endowed with this topology, is called a *subspace* of E. With these definitions, all the propositions of (3.10) are valid for arbitrary topological spaces, with the exception of (3.10.9), which has to be stated in the following form: if the topology of E has a *denumerable basis*, then so does the topology induced on any subset of E.

We have already remarked that the properties of *compactness* and *local compactness* for a metric space depend only on the topology, and not on the distance which defines the topology. We may therefore speak unambiguously of *compact* and *locally compact metrizable spaces.*

We observe also that all the definitions and results of (3.19) relating to *connectedness* are valid for arbitrary topological spaces.

(12.2.1) *Let* $\mathfrak{O}_1$, $\mathfrak{O}_2$ *be two topologies on a set* E. *Then the following properties are equivalent*:

(a) $\mathfrak{O}_2$ *is finer than* $\mathfrak{O}_1$;

(b) *if* E_i *denotes the topological space obtained by giving* E *the topology* $\mathfrak{O}_i$ $(i = 1, 2)$, *then the identity mapping of* E_2 *onto* E_1 *is continuous*;

(c) *for all* $x \in E$, *every neighborhood of* x *in the topology* $\mathfrak{O}_1$ *is a neighborhood of* x *in* $\mathfrak{O}_2$.

The equivalence of (a) and (b) follows from the criterion of continuity (3.11.4(b)), and the equivalence of (b) and (c) from the definition of a continuous function.

Remark

(12.2.2) Let $(U_\alpha)_{\alpha \in I}$ be an *open* covering (3.16) of a topological space E. For a set $G \subset E$ to be *open* in E it is necessary and sufficient that each of the sets $G \cap U_\alpha$ should be *open in the subspace* U_α. This follows immediately from the axioms (O_I) and (O_{II}) and the relation $G = \bigcup_{\alpha \in I} (G \cap U_\alpha)$. Taking complements, it follows that a set $F \subset E$ is *closed* in E if and only if each of the sets $F \cap U_\alpha$ is *closed in the subspace* U_α.

(12.2.3) *Let* L *be a subset of a topological space* E. *Then the following properties are equivalent*:

(a) *For each* $x \in L$ *there exists a neighborhood* V *of* x *in* E *such that* $L \cap V$ *is closed in* V;

(b) L *is an open subset of the subspace* $\bar{L}$ (*the closure of* L *in* E);

(c) L *is the intersection of an open subset and a closed subset of* E.

It is clear that (b) implies (c), L being the intersection of $\bar{L}$ with an open subset of E. Equally clearly, (c) implies (a). Let us show that (a) implies (b). For each $x \in L$, we have $V \cap L = V \cap \bar{L}$, because $V \cap L$ is closed in V; this shows that in the subspace $\bar{L}$ the point x is an interior point of L, and therefore L is open in $\bar{L}$.

When L satisfies the equivalent conditions of (12.2.3), it is said to be a *locally closed* subset of E.

(12.2.4) A procedure for constructing topological spaces which is frequently used is that of "*patching together*" a family of topological spaces $(E_\lambda)_{\lambda \in L}$ in such a way that, in the topological space E so obtained, the E_λ are identified with *open sets* of E. Since pairs of these sets may very well intersect, this identification requires that, for each pair of indices (λ, μ), we are given a *homeomorphism* of an open subset of E_λ onto an open subset of E_μ.

To be precise, suppose that for each pair $(\lambda, \mu) \in L \times L$ we are given:

(1) an open subset $A_{\lambda\mu}$ of E_λ;

(2) a homeomorphism $h_{\mu\lambda}: A_{\lambda\mu} \to A_{\mu\lambda}$,

satisfying the following conditions:

(I) $A_{\lambda\lambda} = E_\lambda$, and $h_{\lambda\lambda}$ is the identity mapping 1_{E_λ};

(II) for each triple of indices (λ, μ, ν) and each $x \in A_{\lambda\mu} \cap A_{\lambda\nu}$, we have $h_{\mu\lambda}(x) \in A_{\mu\nu}$ and

$$h_{\nu\lambda}(x) = h_{\nu\mu}(h_{\mu\lambda}(x)) \tag{12.2.4.1}$$

(*patching condition*).

Let F be the *sum* of the E_λ, which are therefore pairwise disjoint subsets of F (1.8). In F, consider the relation

$$R(x, y) : \text{“there exist } \lambda, \mu \text{ such that } x \in A_{\lambda\mu}, y \in A_{\mu\lambda}, \text{ and } y = h_{\mu\lambda}(x).\text{”}$$

This is an *equivalence relation*. It is reflexive by virtue of condition (I); it is symmetric because $h_{\mu\lambda}$ and $h_{\lambda\mu}$ are inverses of each other (by applying (II) with $\nu = \lambda$, and then (I)); finally, it is transitive, because if

$$x \in A_{\lambda\mu}, \qquad y = h_{\mu\lambda}(x) \in A_{\mu\lambda} \cap A_{\mu\nu}, \qquad z = h_{\nu\mu}(y),$$

then we have $x = h_{\lambda\mu}(y)$ and therefore $x \in A_{\lambda\mu} \cap A_{\lambda\nu}$ by (II), and it follows now from (12.2.4.1) that $z = h_{\nu\lambda}(x)$. We remark also that, by condition (I), the intersection of an E_λ and an equivalence class of R consists of *at most one point*. If $E = F/R$ is the set of equivalence classes of R and if

$$\pi : F \to F/R = E$$

is the canonical mapping, then each of the restrictions $\pi_\lambda = \pi \mid E_\lambda : E_\lambda \to E$ is *injective*. Moreover, the sets $\pi_\lambda(E_\lambda)$ form a *covering* of E.

Now consider the set $\mathfrak{O}$ of subsets X of E with the following property: for each $\lambda \in L$, the set $\pi_\lambda^{-1}(X \cap \pi_\lambda(E_\lambda))$ is *open* in E_λ. It is clear that $\mathfrak{O}$ satisfies axioms (O_I) and (O_{II}), and is therefore a *topology* on E. We shall show that in this topology the sets $\pi_\lambda(E_\lambda)$ are *open* in E and that π_λ is a *homeomorphism* of E_λ onto the subspace $\pi_\lambda(E_\lambda)$ of E, for each $\lambda \in L$. In view of the definition of $\mathfrak{O}$ and of the fact that π_λ is a bijection of E_λ onto $\pi_\lambda(E_\lambda)$, it is enough to establish the equivalence of the following two properties of a subset X_λ of E_λ:

(a) X_λ is an open set in the topological space E_λ;

(b) for each $\mu \in L$, the set $\pi_\mu^{-1}(\pi_\lambda(X_\lambda) \cap \pi_\mu(E_\mu))$ is open in E_μ.

Taking $\mu = \lambda$, we see that (b) implies (a). Conversely, if (a) is satisfied, then since we have $\pi_\mu^{-1}(\pi_\lambda(E_\lambda) \cap \pi_\mu(E_\mu)) = A_{\mu\lambda}$ by definition, the condition (b) signifies that, for each $\mu \in L$, the set $h_{\mu\lambda}(X_\lambda \cap A_{\lambda\mu})$ is open in E_μ. Now $X_\lambda \cap A_{\lambda\mu}$ is open in $A_{\lambda\mu}$, and $h_{\mu\lambda}$ is a homeomorphism of $A_{\lambda\mu}$ onto an open subset of E_μ. Hence (a) implies (b).

The topological space E so defined is said to be obtained by *patching together the* E_λ *along the* $A_{\lambda\mu}$ *by means of the* $h_{\lambda\mu}$.

3. HAUSDORFF SPACES

The notion of a *limit* is defined as in (3.13) for arbitrary topological spaces, and the criterion (3.13.1) is merely another formulation of the definition. However, the conclusion of (3.13.3) is no longer *necessarily true*: for example, if E is a set endowed with the chaotic topology (12.1.1), *every* sequence of points of E has *every* point of E as a limit.

A topological space E is said to be *Hausdorff*, and its topology is said to be a *Hausdorff topology*, if it satisfies the following "Hausdorff axiom":

Given any two distinct *points a, b in* E, *there exists a neighborhood* U *of a and a neighborhood* V *of b which do not intersect.*

Every metrizable space is Hausdorff.

(12.3.1) *Let* E *be a topological space,* A *a subset of* E, *and let a be a cluster point of* A. *Then a mapping f of* A *into a* Hausdorff *space* E′ *has at most one limit at the point a with respect to* A.

For if a', b' were two distinct limits of f at the point a, there would exist neighborhoods U′, V′ of a', b', respectively, having no point in common. But by hypothesis there would be a neighborhood W of a in E such that $f(W \cap A) \subset U'$ and $f(W \cap A) \subset V'$, and this is absurd because $W \cap A \neq \varnothing$.

Hence we may continue to use the notation $\lim_{x \in A,\, x \to a} f(x)$ to denote the unique limit of f at the point a. The propositions of (3.13) which do not refer to distances will remain valid for mappings (in particular for sequences) from an *arbitrary* topological space to a *Hausdorff* space, with the exception of (3.13.13) and (3.13.14).

(12.3.2) *Every subspace of a Hausdorff space is Hausdorff.*

(12.3.3) *Every topology which is finer than a Hausdorff topology is Hausdorff.*

These are immediate consequences of the definitions.

(12.3.4) *In a Hausdorff space* E, *every finite subset is closed.*

For if $(a_i)_{1 \leqq i \leqq n}$ is a finite sequence of points in E, a point b distinct from all the a_i cannot be a cluster point of the set of the a_i, because for each i there

exists a neighborhood V_i of b which does not contain a_i, and $V = \bigcap_{i=1}^{n} V_i$ is then a neighborhood of b which does not contain any of the a_i.

(12.3.5) *Let f, g be two continuous mappings of a topological space* E *into a Hausdorff space* E'. *Then the set* A *of points $x \in E$ such that $f(x) = g(x)$ is closed in* E.

The proof is similar to that of (3.15.1) (which is a special case of (12.3.5)). If $a \notin A$, then $f(a) \neq g(a)$, and so there are disjoint neighborhoods U' of $f(a)$ and V' of $g(a)$. Since $f^{-1}(U')$ and $f^{-1}(V')$ are neighborhoods of a in E, the same is true of their intersection W, and it is clear that $f(x) \neq g(x)$ for all $x \in W$. Hence $E - A$ is open.

The principle of extension of identities (3.15.2) therefore remains true for continuous mappings of any topological space into a *Hausdorff* space. Proposition (3.15.3) and its corollary (3.15.4) (the "principle of extension of inequalities") are also valid—the proofs are the same—for arbitrary topological spaces.

(12.3.6) *Let* E *be a compact metrizable space,* F *a Hausdorff space, and f a continuous mapping of* E *into* F. *Then f(E) is closed in* F. *If f is injective, it is a homeomorphism of* E *onto the subspace f(E) of* F.

Let y be a point of the complement of $f(E)$. For each $z \in f(E)$, there exist disjoint open neighborhoods $V(z)$ of z and $W(z)$ of y. The inverse images $f^{-1}(V(z))$ form an open covering of E as z runs through $f(E)$ (3.11.4). Hence by compactness there exists a finite number of points $z_i \in f(E)$ such that the $V(z_i)$ form an open covering of $f(E)$ in F. But then $U = \bigcap_i W(z_i)$ is an open neighborhood of y which does not intersect $f(E)$. This shows that $f(E)$ is closed in F. It follows that if A is any closed subset of E, then $f(A)$ is closed in F (and therefore also in $f(E)$), because A is compact (3.17.3). If f is injective, this establishes the continuity of the inverse mapping $f(E) \to E$ (3.11.4).

(12.3.7) *If a Hausdorff topology is coarser than the topology of a compact metrizable space, then the two topologies coincide.*

(12.3.8) *Let* E *be a compact metrizable space,* F *a Hausdorff space, f a continuous mapping of* E *into* F. *If b is any point of f(E) and* U *is any open set in* E *containing $f^{-1}(b)$, then there exists a neighborhood* V *of b in* F *such that $f^{-1}(V) \subset U$.*

For $E - U$ is closed in E, hence $f(E - U)$ is closed in F by (12.3.6). We have $b \notin f(E - U)$ by definition, hence the complement V of $f(E - U)$ in F is an open neighborhood of b, and clearly $U \supset f^{-1}(V)$.

PROBLEMS

1. Find all possible topologies on a set of 2 or 3 elements.

2. Let $\mathfrak{F}$ be a set of subsets of a set E. Show that $\mathfrak{F}$ is the set of closed sets for a topology on E if and only if $\mathfrak{F}$ satisfies the following two conditions: (1) every intersection of a family of sets belonging to $\mathfrak{F}$ belongs to $\mathfrak{F}$; (2) the empty set belongs to $\mathfrak{F}$, and the union of two sets belonging to $\mathfrak{F}$ belongs to $\mathfrak{F}$.

3. Let E be a set and for each $x \in E$ let $\mathfrak{B}(x)$ be a set of subsets of E. Then there exists a topology $\mathscr{T}$ on E such that, for all $x \in E$, $\mathfrak{B}(x)$ is the set of neighborhoods of x with respect to $\mathscr{T}$, if and only if the sets $\mathfrak{B}(x)$ satisfy the following conditions:

   (V_I) Every subset of E which contains a set belonging to $\mathfrak{B}(x)$ belongs to $\mathfrak{B}(x)$.
   (V_{II}) The intersection of two sets belonging to $\mathfrak{B}(x)$ belongs to $\mathfrak{B}(x)$.
   (V_{III}) For all $x \in E$ and all $V \in \mathfrak{B}(x)$, we have $x \in V$.
   (V_{IV}) For all $x \in E$ and all $V \in \mathfrak{B}(x)$, there exists a set $W \in \mathfrak{B}(x)$ such that $V \in \mathfrak{B}(y)$ for all $y \in W$.

   The topology $\mathscr{T}$ is then unique.

4. Let A be a commutative ring with an identity element $1 \neq 0$. An ideal $\mathfrak{p}$ of A is *prime* if $A/\mathfrak{p}$ is an integral domain (and therefore $\neq 0$). The set of prime ideals of A is called the *spectrum* of A and is denoted by Spec(A) (it can be shown to be nonempty). If $\mathfrak{a}$ is an ideal in A, the set of prime ideals $\mathfrak{p} \supset \mathfrak{a}$ is denoted by $V(\mathfrak{a})$. Show that $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any two ideals $\mathfrak{a}, \mathfrak{b}$ in A. Deduce that the subsets $V(\mathfrak{a})$ of Spec(A) are the closed sets in a topology, called the *spectral topology*, on Spec(A). If x, y are two distinct points of Spec(A), show that either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x. Under what conditions is a set $\{x\}$ consisting of a single point closed in Spec(A)? Consider the case where $A = \mathbf{Z}$, and the case where A is a discrete valuation ring.

   Let A′ be another ring and let $h: A \to A'$ be a ring homomorphism such that $h(1) = 1$. Show that the mapping $\mathfrak{p}' \mapsto h^{-1}(\mathfrak{p}')$ of Spec(A′) into Spec(A) is continuous with respect to the spectral topologies.

5. (a) The following conditions on a nonempty topological space E are equivalent: (1) the intersection of any two nonempty open sets in E is nonempty; (2) every nonempty open set is dense in E; (3) every open set in E is connected. The space E is then said to be *irreducible*. A nonempty subset F of a topological space E is said to be an *irreducible set* if the subspace F is irreducible.
   (b) Show that in a Hausdorff space every irreducible set consists of a single point.

(c) In a topological space E, a subset F is irreducible if and only if its closure $\overline{F}$ is irreducible. In particular, for each $x \in E$, the set $\overline{\{x\}}$ is irreducible. If an irreducible set F in E is of the form $\overline{\{x\}}$, then x is said to be a *generic point* of F.

(d) Let A be an integral domain. Show that the topological space Spec(A) (Problem 4) is irreducible and that {0} is its unique generic point.

In Spec(**Z**), show that the complement of the generic point is an irreducible set which has no generic point.

(e) If E is an irreducible space, every nonempty open set in E is irreducible.

(f) Let $(U_\alpha)_{\alpha \in A}$ be an open covering of a topological space E, such that $U_\alpha \cap U_\beta \neq \varnothing$ for all pairs of indices (α, β). Show that if the sets U_α are irreducible, then E is irreducible.

(g) Let E, F be two topological spaces and let f be a continuous mapping of E into F. If A is an irreducible subset of E, show that $f(A)$ is an irreducible subset of F.

(h) Let E, F be two irreducible spaces, each of which has at least one generic point. Suppose also that F has a unique generic point b. Let f be a continuous mapping of E into F. Show that $f(E)$ is dense in F if and only if $f(x) = b$ for each generic point x of E.

6. A topological space E is said to be *quasi-compact* if it satisfies the Borel–Lebesgue axiom (3.16) and *compact* if it is quasi-compact and Hausdorff. A subset F of a topological space E is said to be a *quasi-compact set* (resp. a *compact set*) if the subspace F of E is quasi-compact (resp. compact). Every finite set is quasi-compact.

(a) In a quasi-compact space, every closed set is quasi-compact.

(b) Let E be a Hausdorff space and let A, B be two disjoint compact sets in E. Show that there exist two disjoint open sets U, V, such that $A \subset U$ and $B \subset V$. (Consider first the case where A consists of a single point.) Deduce that a compact set in a Hausdorff space is closed.

(c) In a topological space E, any finite union of quasi-compact sets is quasi-compact.

(d) If E is a quasi-compact space and $f: E \to F$ is a continuous mapping, then the set $f(E)$ is quasi-compact.

(e) A *filter base* on a set E is a set $\mathfrak{B} \subset \mathfrak{P}(E)$ of nonempty subsets of E such that, whenever X and Y belong to $\mathfrak{B}$, there exists $Z \in \mathfrak{B}$ such that $Z \subset X \cap Y$. Show that if E is a quasi-compact space and all the sets of $\mathfrak{B}$ are *closed*, then the intersection of the sets of $\mathfrak{B}$ is nonempty (consider the complements of the sets of $\mathfrak{B}$, and argue by contradiction).

(f) If A is a commutative ring with an identity element $1 \neq 0$, show that the topological space Spec(A) (Problem 4) is quasi-compact.

(g) Let E be the union of the interval $]0, 1]$ and two elements α, β. Let $\mathfrak{B}$ be the set of finite intersections of sets of the forms $]a, 1], \{\alpha\} \cup]0, a[, \{\beta\} \cup]0, a[$, where $0 < a < 1$. Show that $\mathfrak{B}$ is a basis of a non-Hausdorff topology on E, for which E is quasi-compact and every set consisting of a single point is closed in E. Every point of E has a compact metrizable neighborhood, but there are such neighborhoods of α (or β) which are not closed in E. Show that the intersection of a compact metrizable neighborhood of α and a compact metrizable neighborhood of β is not quasi-compact. The topology of E has a denumerable basis consisting of separable metrizable subspaces.

(h) Let E be the sum (1.8) of N and an infinite set A. For each $x \in N$, write $\mathfrak{S}(x) = \{x\}$. For each $x \in A$, let $S_{x,n}$ denote the union of $\{x\}$ and the set of integers $\geqq n$, and let $\mathfrak{S}(x)$ denote the set of sets $S_{x,n}$ as n runs through N. Show that there exists a non-Hausdorff topology on E such that $\mathfrak{S}(x)$ is a fundamental system of neighborhoods of x, for each $x \in E$. In this topology, every set consisting of a single point is closed, and E has a dense compact subspace, although E itself is not quasi-compact.

4. UNIFORMIZABLE SPACES

A pseudo-distance on a set E is a mapping d of $E \times E$ into $\mathbf{R}$ which is such that $d(x, x) = 0$ for all $x \in E$, and satisfies axioms (I), (III), and (IV) of (3.1), but *not necessarily axiom* (II). It is clear that a pseudo-distance satisfies the inequalities

$$d(x_1, x_n) \leqq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n),$$
$$|d(x, z) - d(y, z)| \leqq d(x, y)$$

for all x_i and x, y, z in E.

Examples

(12.4.1) If f is *any* mapping of a set E into $\mathbf{R}$, the mapping

$$(x, y) \mapsto |f(x) - f(y)|$$

is a pseudo-distance.

Let φ be a mapping of the interval $[0, +\infty[$ into itself which satisfies the following three conditions:

(1) $\varphi(0) = 0$; (2) φ is increasing; (3) $\varphi(u + v) \leqq \varphi(u) + \varphi(v)$ for all $u \geqq 0$ and $v \geqq 0$.

Then if d is a pseudo-distance on a set E, the same is true of the composite mapping $\varphi \circ d : (x, y) \mapsto \varphi(d(x, y))$. We have only to check the triangle inequality, and by conditions (2) and (3) we have

$$\varphi(d(x, z)) \leqq \varphi(d(x, y) + d(y, z)) \leqq \varphi(d(x, y)) + \varphi(d(x, z)).$$

If (d_n) is a sequence of pseudo-distances on a set E, such that the series $\sum_n d_n(x, y)$ converges for all $(x, y) \in E \times E$, then its sum $d(x, y)$ is a pseudo-distance on E (3.15.4).

Now consider a family $(d_\alpha)_{\alpha \in I}$ of pseudo-distances on a set E. For each $a \in E$, each *finite* family $(\alpha_j)_{1 \leqq j \leqq m}$ of elements of I and each finite family $(r_j)_{1 \leqq j \leqq m}$ of real numbers >0, put

$$B(a; (\alpha_j), (r_j)) = \{x \in E \mid d_{\alpha_j}(a, x) < r_j \quad \text{for} \quad 1 \leqq j \leqq m\},$$
$$B'(a; (\alpha_j), (r_j)) = \{x \in E \mid d_{\alpha_j}(a, x) \leqq r_j \quad \text{for} \quad 1 \leqq j \leqq m\}.$$

Let $\mathfrak{O}$ denote the set of subsets U of E such that, for each $x \in U$, there exists a finite family $(\alpha_j)_{1 \leqq j \leqq m}$ of elements of I and a finite family $(r_j)_{1 \leqq j \leqq m}$ of strictly positive real numbers such that $B(x; (\alpha_j), (r_j)) \subset U$. It is immediately verified that $\mathfrak{O}$ is a *topology* on E. This topology is said to be *defined by the*

family of pseudo-distances $(d_\alpha)_{\alpha \in I}$. A topology which can be defined by a family of pseudo-distances is said to be *uniformizable*, and a space equipped with such a topology is said to be *uniformizable*. It is obvious that a metrizable space is uniformizable, but the converse is false (for example, the family consisting only of the pseudo-distance $d = 0$ defines the chaotic topology). There exist Hausdorff topologies which are not uniformizable (Problems 3 and 4); but throughout this book (and in the majority of situations in analysis) uniformizable spaces are the only ones which we shall have to consider.

(12.4.2) *Let* E *be a uniformizable space whose topology is defined by a family of pseudo-distances* $(d_\alpha)_{\alpha \in I}$. *Then the sets*

$$B(a; (\alpha_j), (r_j)) \qquad (\text{resp. } B'(a; (\alpha_j), (r_j)))$$

are open (resp. *closed*) *in* E.

Let $x \in B(a; (\alpha_j), (r_j))$, and put $s_j = d_{\alpha_j}(a, x)$. Then $s_j < r_j$ for $1 \leqq j \leqq m$, hence $B(a; (\alpha_j), (r_j))$ contains the set $B(x; (\alpha_j), (r_j - s_j))$ by virtue of the triangle inequality for each of the d_{α_j}. This shows that $B(a; (\alpha_j), (r_j))$ is an open set. If $x \notin B'(a; (\alpha_j), (r_j))$, then there exists an index k such that $1 \leqq k \leqq m$ and $d_{\alpha_k}(a, x) = s_k > r_k$, and hence the triangle inequality for d_{α_k} shows that the set $B(x; \alpha_k, s_k - r_k)$ does not meet $B'(a; (\alpha_j), (r_j))$. This shows that $B'(\alpha;(\alpha_j), (r_j))$ is closed.

(12.4.3) *In a uniformizable space, the closed neighborhoods of a point form a fundamental system of neighborhoods of the point.*

This follows from (12.4.2) and the fact that

$$B'(a; (\alpha_j), (r_j/2)) \subset B(a; (\alpha_j), (r_j)).$$

(12.4.4) *Let* E *be a uniformizable space and let* $(d_\alpha)_{\alpha \in I}$ *be a family of pseudo-distances defining the topology of* E. *Then* E *is Hausdorff if and only if, for each pair of distinct points* x, y *in* E, *there exists* $\beta \in I$ *such that* $d_\beta(x, y) \neq 0$.

If E is Hausdorff, then by definition there exists a set $B(x; (\alpha_j), (r_j))$ not containing y, and therefore for at least one index j we have $d_{\alpha_j}(x, y) \geqq r_j > 0$. Conversely, if $d_\beta(x, y) = t > 0$, then the open neighborhoods $B(x; \beta, t/2)$ and $B(y; \beta, t/2)$ of x and y, respectively, are disjoint, by virtue of the triangle inequality for d_β.

If (d_α) is a family of pseudo-distances defining a Hausdorff topology on a set E, then to say that a sequence (x_n) of points of E has a point a as *limit* with respect to this topology means that $\lim_{n \to \infty} d_\alpha(a, x_n) = 0$ *for all* α.

Two families of pseudo-distances $(d_\alpha)_{\alpha \in I}$, $(d'_\lambda)_{\lambda \in L}$ on the same set E are said to be *topologically equivalent* if they define the same topology on E.

(12.4.5) *If $(d_\alpha)_{\alpha \in I}$ is any family of pseudo-distances on a set* E, *there exists a topologically equivalent family $(d'_\alpha)_{\alpha \in I}$ such that $0 \leqq d'_\alpha \leqq 1$ for all $\alpha \in I$.*

Let $\varphi(u) = \inf(u, 1)$ for $0 \leqq u < +\infty$. Then $\varphi(0) = 0$, and φ is an increasing function on $[0, +\infty[$ and satisfies the inequality

$$\varphi(u + v) \leqq \varphi(u) + \varphi(v).$$

(This last assertion is clear if $u > 1$ or $v > 1$ or if $u + v \leqq 1$; and if $u \leqq 1$ and $v \leqq 1$ and $u + v > 1$ we have $\varphi(u + v) = 1 < \varphi(u) + \varphi(v)$.) It follows now from (12.4.1) that $d'_\alpha = \varphi \circ d_\alpha$ is a pseudo-distance on E, for each $\alpha \in I$. That the topologies defined by the families (d_α) and (d'_α) are the same follows from (12.2.1) and the fact that, when $0 < r < 1$, the relations $d_\alpha(x, y) < r$ and $d'_\alpha(x, y) < r$ are equivalent.

(12.4.6) *A Hausdorff uniformizable space* E, *whose topology can be defined by a sequence (d_n) of pseudo-distances, is metrizable.*

Without loss of generality we may assume that the sequence (d_n) is infinite, and by (12.4.5) that $0 \leqq d_n \leqq 1$ for all n. Then the series

$$(12.4.6.1) \qquad d(x, y) = \frac{1}{2} d_1(x, y) + \frac{1}{2^2} d_2(x, y) + \cdots + \frac{1}{2^n} d_n(x, y) + \cdots$$

converges for all pairs (x, y) of elements of E, and d is a pseudo-distance on E (12.4.1). Moreover, d is a *distance*, because $d(x, y) = 0$ implies that $d_n(x, y) = 0$ for all n, and therefore that $x = y$ because E is Hausdorff (12.4.4). If $B_0(x; r)$ is the open ball with center x and radius r with respect to the distance d, then it follows immediately from (12.4.6.1) that $B_0(x; r) \subset B(x; n, 2^n r)$ for all n. Conversely, if n is so large that $2^{n-1} \geqq 1/r$, then we have

$$\sum_{k=1}^{\infty} 2^{-n-k} d_{n+k}(x, y) \leqq 2^{-n} \leqq \tfrac{1}{2} r$$

for all x, y in E, and therefore

$$B(x; (1, 2, \ldots, n), (\tfrac{1}{2}r, \ldots, \tfrac{1}{2}r)) \subset B_0(x, r).$$

By (12.2.1), the proof is complete.

(12.4.7) *Let* E *be a topological space and* (U_n) *a finite or denumerable open covering of* E *such that the subspaces* $\bar{U}_n$ *are separable and metrizable. Then* E *is separable and metrizable.*

We begin by showing that the topology of E is *Hausdorff*. Let x, y be two distinct points of E. If there exists an index n such that $x \in U_n$ and $y \notin \bar{U}_n$, then $V = U_n$ and $W = E - \bar{U}_n$ are open neighborhoods of x and y respectively which do not intersect. If on the other hand $x \in U_n$ and $y \in \bar{U}_n$ for some n, then in the metrizable subspace $\bar{U}_n$ there exists an open neighborhood V_0 of x and an open neighborhood W_0 of y which do not intersect. The set $V = V_0 \cap U_n$ is an open neighborhood of x in E, and there exists an open set W in E such that $W \cap \bar{U}_n = W_0$. Hence W is an open neighborhood of y in E, and $V \cap W = \varnothing$.

For each n, let d_n be a distance defining the topology of $\bar{U}_n$, and let $(V_{mn})_{m \geq 1}$ be a *basis* for the topology of U_n, where V_{mn} is an open ball with center a_{mn} and radius r_{mn} (3.9.4). Let f_{mn} be the function which is equal to $r_{mn} - d_n(a_{mn}, x)$ in V_{mn} and is 0 in the complement of V_{mn} in E. Then f_{mn} is *continuous on* E, since it vanishes at the frontier points of V_{mn}. Now let

$$d_{mn}(x, y) = |f_{mn}(x) - f_{mn}(y)|$$

for all x, y in E. By virtue of (12.4.6), the proof will be complete if we show that the pseudo-distances d_{mn} define the topology of E.

For each $x_0 \in E$, every set defined by an inequality of the form $d_{mn}(x_0, x) < \alpha$ is open in E (3.11.4). Conversely, there exists an integer n such that $x_0 \in U_n$, and for each neighborhood W of x_0 in E there exists an integer m such that V_{mn} is an open neighborhood of x_0 contained in W (3.9.3). Hence $f_{mn}(x_0) = \beta > 0$, and the set of all $x \in E$ such that $d_{mn}(x_0, x) < \frac{1}{2}\beta$ is therefore contained in W. By virtue of (12.2.1), this completes the proof.

We remark that the conclusion of (12.4.7) is not valid without the hypothesis of denumerability on the open covering (U_λ) (Section 12.16, Problem 22); again, the hypotheses cannot be weakened by assuming merely that the *open sets* U_n are separable and metrizable (section 12.3, Problem 6(f)).

PROBLEMS

1. Let E be a topological space such that, for each $x \in E$, the neighborhoods of x which are both open and closed in E form a fundamental system of neighborhoods of x. Show that E is uniformizable. (Observe that the characteristic function of a set which is both open and closed in E is continuous on E.)

2. Let E be a *denumerable* Hausdorff uniformizable space. Show that, for each $x \in E$, the open-and-closed neighborhoods of x form a fundamental system of neighborhoods of x.

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3. Let θ be a positive irrational number. For each point $(x, y) \in \mathbf{Q} \times \mathbf{Q}_+$ (where $\mathbf{Q}_+ = \mathbf{Q} \cap \mathbf{R}_+$) and each integer $n > 0$, let $B_n(x, y)$ denote the set consisting of (x, y) and the points $(z, 0) \in \mathbf{Q} \times \mathbf{Q}_+$ such that $|z - (x + \theta y)| < 1/n$ or $|z - (x - \theta y)| < 1/n$. As (x, y) runs through $\mathbf{Q} \times \mathbf{Q}_+$ and n runs through the set of strictly positive integers, show that the sets $B_n(x, y)$ form a (denumerable) basis of a topology $\mathscr{T}$ on $\mathbf{Q} \times \mathbf{Q}_+$. Show that $\mathscr{T}$ is Hausdorff but that, if a and b are any two points of $\mathbf{Q} \times \mathbf{Q}_+$, every *closed* neighborhood of a meets every *closed* neighborhood of b. Deduce that $\mathbf{Q} \times \mathbf{Q}_+$, endowed with the topology $\mathscr{T}$, is *connected* and that every continuous mapping of $\mathbf{Q} \times \mathbf{Q}_+$ into $\mathbf{R}$ is *constant*. This shows that the topology $\mathscr{T}$ is not uniformizable.

4. Let $\mathbf{Q}$ be the set of rational numbers endowed with the topology $\mathscr{T}_0$ induced by that of $\mathbf{R}$. Let $\mathfrak{M}$ be the set of subsets A of $\mathbf{Q}$ such that the closure $\bar{A}$ of A (in the topology $\mathscr{T}_0$) has only finitely many nonisolated points (3.10.10). Let $\mathfrak{B}$ be the set consisting of all open intervals in $\mathbf{Q}$, all complements of sets belonging to $\mathfrak{M}$ and all intersections of these complements with open intervals in $\mathbf{Q}$. Show that $\mathfrak{B}$ is a basis for a topology $\mathscr{T}$ on $\mathbf{Q}$, and that $\mathscr{T}$ is finer than $\mathscr{T}_0$ and therefore Hausdorff. Show that, with respect to this topology, a convergent sequence in $\mathbf{Q}$ has only finitely many distinct terms, although the topology $\mathscr{T}$ is not discrete. In the topology $\mathscr{T}$, no point of $\mathbf{Q}$ has a denumerable fundamental system of neighborhoods, although $\mathbf{Q}$ is denumerable and every point of $\mathbf{Q}$ is the intersection of a denumerable family of neighborhoods of the point. By using (12.4.3), show that $\mathscr{T}$ is not uniformizable, although it is finer than a metrizable topology.

5. Let E be a topological space satisfying the following condition: for each $x \in E$ and each neighborhood V of x in E, there exists a continuous mapping $f: E \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in E - V$. Show that E is uniformizable.

   Deduce that if E is a topological space such that every point of E has a *closed* neighborhood in E which is a uniformizable subspace, then E is uniformizable. Can the word "closed" be omitted from this proposition? (Cf. Section 12.3, Problem 6(f).)

6. For each $x \in \mathbf{R}$ and each integer $n > 0$, let $U_n(x)$ denote the union of the intervals $[x, x + 1/n[$ and $]-x - 1/n, -x[$. Show that there exists a topology $\mathscr{T}$ on $\mathbf{R}$ such that the $U_n(x)$ form a fundamental system of neighborhoods of x, and that $\mathscr{T}$ is not comparable with the usual topology of $\mathbf{R}$. Show that $\mathscr{T}$ is uniformizable (use Problem 5). In this topology, every point has a denumerable fundamental system of neighborhoods, and there exists a denumerable dense subset. But there exists no denumerable basis of open sets for $\mathscr{T}$, and consequently $\mathscr{T}$ is not metrizable. Show also that the topology induced by $\mathscr{T}$ on the interval $E = [-1, 1]$ of $\mathbf{R}$ is not metrizable, although E is quasi-compact and Hausdorff in this topology.

5. PRODUCTS OF UNIFORMIZABLE SPACES

Let E_1, E_2 be two topological spaces. In the product set $E = E_1 \times E_2$ let $\mathfrak{O}$ be the set of (arbitrary) unions of sets of the form $A_1 \times A_2$, where A_1 is open in E_1 and A_2 is open in E_2. The set $\mathfrak{O}$ is a *topology* on E, because it clearly satisfies axiom (O_I), and it satisfies (O_{II}) by reason of the relation

$$(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \cap B_2),$$

where A_i and B_i are open sets in E_i $(i = 1, 2)$. This topology $\mathfrak{T}$ is called the *product* of the topologies on E_1 and E_2, and the set E endowed with this topology is called the *product* of the topological spaces E_1, E_2. If $x = (x_1, x_2)$ is any point of E, the sets $V_1 \times V_2$ (where V_i runs through a fundamental system of neighborhoods of x_i, for $i = 1, 2$) form a fundamental system of neighborhoods of x in E (3.9.3). From this it follows that the continuity criterion (3.20.4) is valid for the product of two arbitrary topological spaces.

The relation $\overline{A_1 \times A_2} = \bar{A}_1 \times \bar{A}_2$ (where $A_1 \subset E_1$ and $A_2 \subset E_2$) also remains true; for if $(a_1, a_2) \in E$ and if V_i is a neighborhood of a_i in E_i $(i = 1, 2)$, the set $(V_1 \times V_2) \cap (A_1 \times A_2) = (V_1 \cap A_1) \times (V_2 \cap A_2)$ is empty if and only if one of the sets $V_i \cap A_i$ is empty.

For each $a_1 \in E_1$, the set $(\{a_1\} \times E_2) \cap (A_1 \times A_2)$ is either empty or equal to $\{a_1\} \times A_2$, and therefore the mapping $x_2 \mapsto (a_1, x_2)$ is a *homeomorphism* of E_2 onto the subspace $\{a_1\} \times E_2$ of E. Likewise, if $a_2 \in E_2$, the mapping $x_1 \mapsto (x_1, a_2)$ is a homeomorphism of E_1 onto the subspace $E_1 \times \{a_2\}$ of E. From this it follows immediately that the propositions (3.20.12) and (3.20.13), and the continuity criteria (3.20.14) and (3.20.15), are valid in general.

If E_1 and E_2 are *Hausdorff* then so is $E_1 \times E_2$. For if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are distinct, then either $x_1 \neq y_1$ or $x_2 \neq y_2$. If the latter, there is a neighborhood U of x_2 and a neighborhood V of y_2 in E_2 which do not intersect, and then $E_1 \times U$ and $E_1 \times V$ are disjoint neighborhoods of x and y, respectively, in E.

If $E_2 = E_1$, the *canonical symmetry* $(x_1, x_2) \mapsto (x_2, x_1)$ is a homeomorphism of $E_1 \times E_1$ onto itself, and is equal to its inverse.

The product of any finite number of topological spaces is defined in the same way. It follows immediately from the definition that the canonical "associativity" mappings such as $(E_1 \times E_2) \times E_3 \to E_1 \times (E_2 \times E_3)$ are homeomorphisms.

We shall be concerned especially with the case in which E_1 and E_2 are *uniformizable*. In this case the product space E is also uniformizable. To be precise, let $(d_\lambda^{(1)})_{\lambda \in L}$, $(d_\mu^{(2)})_{\mu \in M}$ be families of pseudo-distances which define the topologies of E_1, E_2 respectively, and let

$$e_\lambda^{(1)}(x, y) = d_\lambda^{(1)}(\mathrm{pr}_1 x, \mathrm{pr}_1 y), \qquad e_\mu^{(2)}(x, y) = d_\mu^{(2)}(\mathrm{pr}_2 x, \mathrm{pr}_2 y).$$

Then the $e_\lambda^{(1)}$ and $e_\mu^{(2)}$ are pseudo-distances on E, and the definition of neighborhoods given above shows that the family of pseudo-distances $e_\lambda^{(1)}$ and $e_\mu^{(2)}$, where $\lambda \in L$ and $\mu \in M$, defines the product topology on E.

This leads to a generalization of the notion of a product to arbitrary (not necessarily finite) families of topological spaces. We shall restrict ourselves to uniformizable spaces.

Let $(E_\alpha)_{\alpha \in I}$ be any family of uniformizable spaces, and let $E = \prod_{\alpha \in I} E_\alpha$. For each $\alpha \in I$, let $(d_{\alpha,\lambda})_{\lambda \in L_\alpha}$ be a family of pseudo-distances defining the topology of E_α. If we put

$$e_{\alpha,\lambda}(x, y) = d_{\alpha,\lambda}(\mathrm{pr}_\alpha x, \mathrm{pr}_\alpha y) \tag{12.5.1}$$

for each pair (x, y) of elements of $E = \prod_{\alpha \in I} E_\alpha$, then it is immediately checked that the $e_{\alpha,\lambda}$ are pseudo-distances on E. The *product* of the topologies of the E_α is then the topology on E defined by the $e_{\alpha,\lambda}$ as α runs through I and, for each $\alpha \in I$, λ runs through L_α. The set E endowed with this topology is called the *product* of the spaces E_α. When $I = \{1, 2\}$ this definition agrees with that given above for two spaces E_1, E_2.

For every finite family $(\alpha_i, \lambda_i)_{1 \leqq i \leqq n}$ and every finite family $(r_i)_{1 \leqq i \leqq n}$ of strictly positive real numbers, the set $B(x; ((\alpha_i, \lambda_i)), (r_i))$ can be written in the form $\prod_{\alpha \in I} B_\alpha$ where, if $\alpha = \alpha_i$, we have $B_\alpha = B(\mathrm{pr}_{\alpha_i} x; (\lambda_j), (r_j))$, where j runs through the indices for which $\alpha_j = \alpha_i$, and $B_\alpha = E_\alpha$ if α is not equal to any α_i. From this and (3.6.4) we deduce immediately:

(12.5.2) *For each* $\alpha \in I$, *the projection* pr_α *is a continuous mapping of* E *onto* E_α, *and the image under* pr_α *of any open set in* E *is an open set in* E_α.

(12.5.3) *For each* finite *subset* H *of* I, *and each family* $(U_\alpha)_{\alpha \in H}$, *where* U_α *is open in* E_α *for each* $\alpha \in H$, *the set* $\left(\prod_{\alpha \in H} U_\alpha\right) \times \left(\prod_{\alpha \in I - H} E_\alpha\right)$ *is open in* E.

For it is the intersection of the sets $\mathrm{pr}_\alpha^{-1}(U_\alpha)$, where $\alpha \in H$, and these sets are open (12.5.2) and finite in number.

The sets described in (12.5.3) are called *elementary sets* in E. The description given above of the neighborhoods shows that (having regard to (3.9.3)) these sets form a *basis* for the topology of E. Incidentally, this shows also that the topology of E depends only on the topologies of the E_α and not on the families of pseudo-distances $(d_{\alpha,\lambda})$. Moreover, if $\mathfrak{B}_\alpha$ is a *basis* for the topology of E_α, the elementary sets such that $U_\alpha \in \mathfrak{B}_\alpha$ for all α form a *basis* for the topology of E. It should be noted that, if I is *infinite*, a product $\prod_{\alpha \in I} U_\alpha$ with U_α open in E_α, nonempty and $\neq E_\alpha$ for *all* $\alpha \in I$, *is not open* in E.

(12.5.4) *Let* A_α *be a subset of* E_α, *for each* α, *and let* $A = \prod_{\alpha \in I} A_\alpha$. *Then* $\bar{A} = \prod_{\alpha \in I} \bar{A}_\alpha$. *In particular, if each* A_α *is closed in* E_α, *then* A *is closed in* E.

We have $\mathrm{pr}_\alpha(\bar{A}) \subset \overline{\mathrm{pr}_\alpha(A)} = \bar{A}_\alpha$ for all $\alpha \in I$ ((12.5.2) and (3.11.4)), hence $\bar{A} \subset \prod_{\alpha \in I} \bar{A}_\alpha$. Conversely, if $a = (a_\alpha) \in \prod_{\alpha \in I} \bar{A}_\alpha$, and if we consider an elementary set $U = \prod_{\alpha \in H} U_\alpha \times \prod_{\alpha \in I - H} E_\alpha$ containing a, its intersection with A is

$$\prod_{\alpha \in H} (A_\alpha \cap U_\alpha) \times \prod_{\alpha \in I - H} A_\alpha .$$

Since none of the $\bar{A}_\alpha$ is empty, no A_α is empty and therefore $U \cap A \neq \varnothing$, which shows that $a \in \bar{A}$.

(12.5.5) *Let $z \mapsto f(z) = (f_\alpha(z))_{\alpha \in I}$ be a mapping of a topological space* F *into* E. *For f to be continuous at a point $z_0 \in F$, it is necessary and sufficient that each f_α should be continuous at z_0.*

For any elementary set $U = \prod_{\alpha \in H} U_\alpha \times \prod_{\alpha \in I - H} E_\alpha$, we have $f^{-1}(U) = \bigcap_{\alpha \in H} f_\alpha^{-1}(U_\alpha)$.

(12.5.6) *Let $(x^{(n)})$ be a sequence of points of* E. *For $a = (a_\alpha)$ to be a limit of the sequence $(x^{(n)})$, it is necessary and sufficient that a_α should be a limit of the sequence $(\mathrm{pr}_\alpha x^{(n)})_{n \geqq 1}$, for each $\alpha \in I$.*

This is an immediate consequence of (12.5.5) and the definition of a limit.

In particular, if all the E_α are equal to the *same space* F, then to say that a mapping $u \in F^I$ of I into F is a limit of a sequence of mappings $(u_n)_{n \geq 1}$ of I into F, with respect to the product topology, means that for each $\alpha \in I$ the sequence $(u_n(\alpha))$ has $u(\alpha)$ as a limit in F. In this situation we say that the mapping u is a *simple limit* of the sequence (u_n), or that the latter *converges simply* to u.

(12.5.7) *If each space E_α is Hausdorff, then so is the product space* E.

For if $x \neq y$, there exists an index $\alpha \in I$ such that $\mathrm{pr}_\alpha x \neq \mathrm{pr}_\alpha y$; hence, using (12.4.4), there exists $\lambda \in L_\alpha$ such that $d_{\alpha,\lambda}(\mathrm{pr}_\alpha x, \mathrm{pr}_\alpha y) \neq 0$, and therefore $e_{\alpha,\lambda}(x, y) \neq 0$. The result now follows from (12.4.4).

(12.5.8) *The product* E *of a* denumerable *family of metrizable* (resp. *separable metrizable*) *spaces is metrizable* (resp. *separable and metrizable*).

First of all, E is Hausdorff by (12.5.7). To show that E is metrizable, apply (12.4.6) and the definition of pseudo-distances on a product. Now suppose that $E = \prod_{n=1}^{\infty} E_n$, where the E_n are separable and metrizable. Then each

E_n has a denumerable basis $(U_{mn})_{m \geqq 0}$ of open sets. For each $n \geqq 1$, let $\mathfrak{B}_n$ be the set of elementary sets in E of the form $\prod_{j=1}^{n} U_{f(j),\,j} \times \prod_{k=n+1}^{\infty} E_k$, where f is any mapping of $\{1, 2, \ldots, n\}$ into $\mathbf{N}$. Since $\mathbf{N}^n$ is denumerable (1.9.3), the set $\mathfrak{B}_n$ is at most denumerable (1.9.2). We have seen that the union $\mathfrak{B}$ of the sets $\mathfrak{B}_n$ is a basis for the topology of E (12.5.3); since $\mathfrak{B}$ is denumerable (1.9.4), it follows that E is separable (3.9.4).

(12.5.9) *The product of a denumerable family of compact metrizable spaces is compact and metrizable.*

Let (E_n) be a sequence of compact metric spaces. Since we know already (12.5.8) that $E = \prod_{n=1}^{\infty} E_n$ is metrizable, it is enough to show that every sequence (x_n) of points of E has a cluster value (3.16.1). We define by induction on $m \geqq 0$ a family of sequences $(x_n^{(m)})_{n \geqq 1}$ of points of E, as follows: $x_n^{(0)} = x_n$; for $m \geqq 1$, the sequence $(x_n^{(m)})_{n \geqq 1}$ is a *subsequence* of the sequence $(x_n^{(m-1)})_{n \geqq 1}$ (in other words, there exists a strictly increasing mapping φ_m: $\mathbf{N} \to \mathbf{N}$ such that $x_n^{(m)} = x_{\varphi_m(n)}^{(m-1)}$) such that the sequence $(\mathrm{pr}_m x_n^{(m)})_{n \geqq 1}$ *converges* in E_m to a point a_m. This construction is possible because E_m is compact. Now consider the sequence (y_n) in E, where $y_n = x_n^{(n)}$ (Cantor's "diagonal trick"). If we put $\psi_n = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1$, we have $y_n = x_{\psi_n(n)}$; since $\psi_n(n) > \psi_{n-1}(n-1)$ by definition, it follows that (y_n) is a subsequence of (x_n). Furthermore, for each m, the sequence $(y_n)_{n \geqq m}$ is a subsequence of the sequence $(x_n^{(m)})_{n \geqq 1}$, and therefore the sequence $(\mathrm{pr}_m y_n)_{n \geqq m}$ converges to a_m. Hence so does the sequence $(\mathrm{pr}_m y_n)_{n \geqq 1}$, since it differs from the former by only a finite number of terms. Hence the sequence (y_n) converges to the point $a = (a_m)$ (12.5.6).

PROBLEMS

1. Show that the product of two quasi-compact topological spaces (Section 12.3, Problem 6) is quasi-compact.

2. Show that a topological space E is Hausdorff if and only if the diagonal (1.4.2) of $E \times E$ is closed in $E \times E$.

3. Let $(E_\alpha)_{\alpha \in I}$ be a family of arbitrary (not necessarily uniformizable) topological spaces, and let $E = \prod_{\alpha \in I} E_\alpha$. Show that the elementary sets, defined as in (12.5.3), form a basis of a topology on E. This topology is called the *product* of the topologies of the E_α. Generalize (12.5.2) to (12.5.7) to this situation.

4. Let $E = \prod_{\alpha \in I} E_\alpha$ be a product of Hausdorff topological spaces, such that each E_α contains at least two distinct points a_α, b_α. For each $\alpha \in I$, let c_α be the point of E such that $\mathrm{pr}_\alpha c_\alpha = b_\alpha$ and $\mathrm{pr}_\beta c_\alpha = a_\beta$ for all $\beta \neq \alpha$. Show that every point of the set $\{c_\alpha\}_{\alpha \in I}$ is an isolated point.

Deduce that the topology of E has a denumerable basis if and only I is denumerable and each of the E_α has a denumerable basis.

Show that if I is not denumerable, the point $a = (a_\alpha)$ has no denumerable fundamental system of neighborhoods. If $E_\alpha = \{a_\alpha, b_\alpha\}$ for each α, and if $F \subset E$ is the set of all $x \in E$ such that $\mathrm{pr}_\alpha x = b_\alpha$ for all but denumerably many indices α, show that F is dense in E and that $a = (a_\alpha)$ is not a limit of any sequence of elements of F.*

5. Let K be the discrete space $\{0, 1\}$, let A be an infinite set, and let E be the product space K^A. Let V be a nonempty elementary set in E. If h is the (finite) number of indices α such that $\mathrm{pr}_\alpha V \neq K$, let $\mu(V) = 2^{-h}$ (cf. (13.21)).

(a) Show that, if $U_1, \ldots, U_n$ are nonempty mutually disjoint elementary sets, then $\sum_{k=1}^{n} \mu(U_k) \leqq 1$. (Write U_k in the form $W_k \times K^B$, where B is the same for all k and is the complement of a finite subset of A.)

(b) Deduce from (a) that if $(U_\lambda)_{\lambda \in L}$ is a family of nonempty pairwise disjoint open sets in E, then L is *at most denumerable*, although there exist sets C of *arbitrary* cardinal in E (for a suitable choice of A) all of whose points are isolated. Show also that if the cardinal of A is strictly greater than that of $\mathfrak{P}(N)$, then E contains no denumerable dense subset.

6. With the notation of Problem 5, let $\mathfrak{B}$ be the set of all subsets of K^A of the form $\prod_{\alpha \in A} M_\alpha$, where $M_\alpha = K$ except for *at most denumerably many* indices α. Show that $\mathfrak{B}$ is a basis for a Hausdorff topology on K^A which is finer than the product topology, and which is not discrete provided that A is infinite and *non-denumerable*. In this topology, every *denumerable* intersection of open sets is open; no point has a denumerable fundamental system of neighborhoods; and every quasi-compact subset is *finite*. The projections pr_α are continuous with respect to this topology. Deduce that K^A is uniformizable with respect to this topology (Section 12.4, Problem 5).

7. Let I be the interval $[0, 1]$ in $\mathbf{R}$, with the induced topology. Show that every separable metrizable space E is homeomorphic to a subspace of the product I^N. (Reduce to the case where the distance d on E is $\leqq 1$, and consider a sequence (a_n) which is dense in E, and the functions $x \mapsto d(a_n, x)$.)

8. With the notation of Problem 7, show that in the uniformizable product space I^I, the subspace of continuous mappings of I into I is dense. Deduce that I^I has a denumerable dense subset, although there is no denumerable basis of open sets (Problem 4).

9. Show that if I is a nonempty open interval in $\mathbf{R}$, there exists no nonconstant mapping f of I into the product space N^N with the following property: for each $x \in I$ and each

* This example shows that in general topological spaces (and even in Hausdorff uniformizable spaces) the convergent sequences do not determine the topology, as they do in metrizable spaces (cf. (3.13.3) and (3.13.4)). To get corresponding results in general, it is necessary to replace the notion of sequence by the more general notion of a *filter* (cf. [5]).

integer $n > 0$, there exists a neighborhood V of x in I such that, for all $y \in V$, the first n terms of the sequence $f(y)$ are the same as the first n terms of the sequence $f(x)$. (Show that this condition implies that f is continuous, and use (3.19.7).)

10. (a) Let E be a topological space and A a nonempty closed subset of E. Let E′ be the sum of $E - A$ and a set $\{\omega\}$ consisting of a single element, and let $\mathfrak{O}'$ be the set of subsets of E′ which are either of the form U where U is an open set in $E - A$, or of the form $(V - A) \cup \{\omega\}$, where V is an open set in E which contains A. Show that $\mathfrak{O}'$ is a topology on E′, and denote by E/A the set E′ endowed with this topology. Let $\varphi(x) = x$ if $x \in E - A$, and $\varphi(x) = \omega$ if $x \in A$. Show that φ is a continuous mapping of E onto E/A. Every continuous mapping f of E into a topological space F, which is constant on A, can be written uniquely in the form $f = g \circ \varphi$, where g: E/A → F is continuous.

(b) Suppose that E is metrizable and compact; let d be a distance defining the topology of E, and let $(W_n)_{n \geq 1}$ be a basis for the topology of $E - A$. Let $f_0(x) = d(x, A)$, $f_n(x) = d(x, E - W_n)$ for $n \geq 1$, and $f(x) = (f_n(x))_{n \geq 0} \in \mathbf{R}^{\mathbf{N}}$. Show that $f(E)$ is a compact subspace of $\mathbf{R}^{\mathbf{N}}$, homeomorphic to E/A. This shows that E/A is compact and metrizable.

(c) Show that if we take $E = \mathbf{R}$ and $A = \mathbf{Z}$, then the space E/A defined in (a) is not metrizable (cf. Section 3.6, Problem).

6. LOCALLY FINITE COVERINGS AND PARTITIONS OF UNITY

In a topological space E, a family $(A_\alpha)_{\alpha \in I}$ of subsets of E is said to be *locally finite* if for each point $x \in E$ there exists a neighborhood U of x in E such that $U \cap A_\alpha = \varnothing$ for all but a *finite* number of indices $\alpha \in I$. If E is metrizable and K is a *compact* subset of E, it follows that there is a covering of K by a *finite* number of neighborhoods of points of K in E, each of which meets only a *finite* number of the sets A_α. In particular, K meets only a *finite* number of the A_α.

If $(A_\lambda)_{\lambda \in L}$, $(B_\mu)_{\mu \in M}$ are two coverings of a topological space E, the covering (B_μ) is said to be *finer* than (A_λ) if, for each $\mu \in M$, there exists $\lambda \in L$ such that $B_\mu \subset A_\lambda$.

(12.6.1) *Let* E *be a separable, locally compact, metrizable space and let* $\mathfrak{B}$ *be a basis of open sets in* E. *If* $(A_\lambda)_{\lambda \in L}$ *is any open covering of* E, *there exists a denumerable locally finite open covering* (B_n) *of* E *which is finer than* (A_λ), *and such that the sets* B_n *are relatively compact and belong to* $\mathfrak{B}$. *Consequently each* B_n *meets only finitely many of the sets* B_m.

From (3.18.3) we know that there exists an increasing sequence $(U_n)_{n \geq 0}$ of relatively compact open sets such that $\bar{U}_n \subset U_{n+1}$ for all n, and $E = \bigcup_n U_n$.

Put $K_n = \bar{U}_n - U_{n-1}$, which is a compact set for all $n \geqq 0$ (we put $U_n = \varnothing$ for all $n < 0$). For each $n \geqq 0$, the open set $U_{n+1} - \bar{U}_{n-2}$ is a neighborhood of K_n. Hence, for each $x \in K_n$, there exists an open neighborhood $W_x^{(n)} \in \mathfrak{B}$ of x, contained in $U_{n+1} - \bar{U}_{n-2}$ and contained in one of the sets A_λ. There exists a finite number of points $x_i \in K_n$ ($1 \leqq i \leqq p_n$) such that the sets $W_{x_i}^{(n)}$ cover K_n. Arrange the sets $W_{x_i}^{(m)}$ ($m \geqq 0$, $1 \leqq i \leqq p_m$ for each m) in a sequence in any way, and let (B_n) denote this sequence. It is clear that (B_n) is an open covering of E which is finer than $(A_\lambda)_{\lambda \in L}$, and that the sets B_n are relatively compact. Thus all that has to be checked is that the covering (B_n) is *locally finite*. Let z be any point of E, and let n be the least integer such that $z \in U_n$. Since $z \notin U_{n-1}$, there exists a neighborhood T of z contained in U_n and not meeting $\bar{U}_{n-2}$. Consequently T can intersect only the sets $W_{x_i}^{(m)}$ for which $n - 2 \leqq m \leqq n + 1$, and the number of these sets is finite.

(12.6.2) *Let (A_n) be a denumerable locally finite open covering of a metrizable space* E. *Then there exists an open covering (B_n) of* E *such that $\bar{B}_n \subset A_n$ for all n.*

We shall define the family (B_n) by induction on n, in such a way that $\bar{B}_n \subset A_n$ for each n, and such that for each n the family consisting of the B_k with $k \leqq n$ and the A_j with $j > n$ is an open covering of E. Suppose that the sets B_n have been defined for $n < m$. Then the B_n with $n < m$ and the A_j with $j \geqq m$ cover E. Let C be the open set which is the union of the B_n with $n < m$ and the A_j with $j > m + 1$, so that we have $E - A_m \subset C$. We shall show that there exists an open set V such that $E - A_m \subset V \subset \bar{V} \subset C$. If $E = A_m$, we may take $V = \varnothing$. If $C = E$, we may take $V = E$. If neither $E - A_m$ nor $E - C$ is empty, there exists a continuous mapping f of E into the interval $[0, 1]$ of $\mathbf{R}$, which is equal to 0 on $E - A_m$ and to 1 on $E - C$ (4.5.2). We then take V to be the open set of points y such that $f(y) < \frac{1}{2}$, and then $\bar{V}$ is contained in the closed set of points y such that $f(y) \leqq \frac{1}{2}$, and hence $\bar{V} \subset C$. Put $B_m = E - \bar{V}$. Then we have $\bar{B}_m \subset E - V \subset A_m$ and $B_m \cup C = E$, so that the sets B_n with $n \leqq m$ satisfy the required conditions, and the induction can proceed. For each $x \in E$, there exists an integer n such that $x \notin A_m$ for all $m > n$, and therefore $x \in B_k$ for some $k \leqq n$. Hence the sets B_n cover E.

It is clear that the same argument will apply if the covering (A_n) is *finite*.

If E is a topological space and f is a mapping of E into a real vector space F or into $\overline{\mathbf{R}}$, the *support* of f (denoted by $\mathrm{Supp}(f)$) is defined to be the smallest *closed* set S in E such that f vanishes on $E - S$. In other words, $\mathrm{Supp}(f)$ is the *closure* in E of the set of points $x \in E$ such that $f(x) \neq 0$; or again it is the set of points $x \in E$ with the property that every neighborhood of x in E contains a point y such that $f(y) \neq 0$.

Let $(f_\alpha)_{\alpha \in I}$ be a family of mappings of E into F (resp. $\overline{\mathbf{R}}$) whose supports form a *locally finite* family. Then the sum $\sum_{\alpha \in I} f_\alpha(x)$ is defined for all $x \in E$, because it contains at most finitely many non-zero terms. We denote by $\sum_{\alpha \in I} f_\alpha$ the function $x \mapsto \sum_{\alpha \in I} f_\alpha(x)$. If F is $\overline{\mathbf{R}}$ or a normed space (or more generally a topological vector space (12.13)), and if each f_α is *continuous* on E, then so is $f = \sum_{\alpha \in I} f_\alpha$: for given $x \in E$ there exists a neighborhood V of x which meets only finitely many of the $\mathrm{Supp}(f_\alpha)$, and hence there is a finite subset H of I such that $f(y) = \sum_{\alpha \in H} f_\alpha(y)$ for all $y \in V$.

A *continuous partition of unity* on E is by definition a family $(f_\alpha)_{\alpha \in I}$ of continuous mappings of E into $[0, 1]$, such that the supports of the f_α form a locally finite family, and such that $\sum_{\alpha \in I} f_\alpha(x) = 1$ for all $x \in E$. If $(A_\alpha)_{\alpha \in I}$ is an open covering of E indexed by the same set I, then the partition of unity $(f_\alpha)_{\alpha \in I}$ is said to be *subordinate* to the covering $(A_\alpha)_{\alpha \in I}$ if we have $\mathrm{Supp}(f_\alpha) \subset A_\alpha$ for all $\alpha \in I$.

(12.6.3) *Let* (A_n) *be an at most denumerable locally finite open covering of a metrizable space* E. *Then there exists a continuous partition of unity* (f_n) *on* E *subordinate to* (A_n).

Let (B_n) be an open covering of E, such that $\bar{B}_n \subset A_n$ for all n (12.6.2). It is clear that the covering (B_n) is locally finite. By (4.5.2), for each n there exists a continuous mapping h_n: $E \to [0, 1]$ such that h_n is equal to 1 on B_n and equal to 0 on $E - A_n$. If we put $g_n = (h_n - \frac{1}{2})^+$, then $\mathrm{Supp}(g_n)$ is contained in the set of points x such that $h_n(x) \geqq \frac{1}{4}$, and hence is contained in A_n. Let $g = \sum_n g_n$. Since the sets B_n cover E, we have $g(x) > 0$ for all $x \in E$, and therefore the functions $f_n = g_n/g$ are defined and continuous on E, and form a partition of unity with the required properties.

(12.6.4) *Let* E *be a metrizable space,* K *a compact subset of* E *and* $(A_k)_{1 \leqq k \leqq m}$ *a finite covering of* K *by open sets of* E. *Then there exist m continuous mappings* f_k: $E \to [0, 1]$ *such that* $\mathrm{Supp}(f_k) \subset A_k$ *for* $1 \leqq k \leqq m$, *and such that* $\sum_{k=1}^{m} f_k(x) \leqq 1$ *for all* $x \in E$ *and* $\sum_{k=1}^{m} f_k(x) = 1$ *for all* $x \in K$.

Take a continuous partition of unity $(f_k)_{0 \leqq k \leqq m}$ subordinate to the open covering of E consisting of $A_0 = E - K$ and the A_k $(1 \leqq k \leqq m)$.

Remark

(12.6.5) Let $\mathscr{F}$ be a subset of the set of continuous mappings of E into $\mathbf{R}$, with the following three properties:

(1) for each pair of disjoint nonempty subsets M, N of E, with M compact and N closed, there exists a function $f \geqq 0$ in $\mathscr{F}$ which is equal to 0 on N and $\geqq 1$ on M;

(2) if $(f_\alpha)_{\alpha \in I}$ is a family of functions in $\mathscr{F}$ whose supports form a locally finite family, then $\sum_{\alpha \in I} f_\alpha$ belongs to $\mathscr{F}$;

(3) if $f \in \mathscr{F}$ is such that $f(x) > 0$ for all $x \in E$, then $g/f \in \mathscr{F}$ for all $g \in \mathscr{F}$.

Then the conclusions of (12.6.3) and (12.6.4) are still valid, when the A_n are relatively compact, if we impose the extra condition that the functions f_n should *belong to* $\mathscr{F}$. The proofs are unaltered.

7. SEMICONTINUOUS FUNCTIONS

In this section, and above all in Chapter XIII, we shall need to consider mappings of a set A into the *extended real line* $\overline{\mathbf{R}}$. Such mappings we shall call (by abuse of language) *real-valued functions* on A. Such a function f is said to be *finite* if its value at each point $a \in A$ is finite, i.e., if $f(A) \subset \mathbf{R}$. The function f is said to be *majorized* or *bounded above* (resp. *minorized* or *bounded below*) if there exists a *finite* majorant (resp. a finite minorant) of $f(A)$. If f is both majorized and minorized, it is said to be *bounded* (which clearly implies that f is finite).

We recall ((4.1.8), (4.1.9), and (3.15.5)) that in $\overline{\mathbf{R}} \times \overline{\mathbf{R}}$ the function $(x, y) \mapsto x + y$ (resp. $(x, y) \mapsto xy$) is *defined and continuous except at the points* $(+\infty, -\infty)$ *and* $(-\infty, +\infty)$ (resp. $(+\infty, 0)$, $(-\infty, 0)$, $(0, +\infty)$, $(0, -\infty)$). The function $x \mapsto 1/x$ on $\overline{\mathbf{R}}$ is defined and continuous *except* at the point $x = 0$. In the interval $[0, +\infty]$ of $\overline{\mathbf{R}}$, the function $x \mapsto 1/x$ (defined on $]0, +\infty]$) can be extended by continuity by giving it the value $+\infty$ at the point 0.

Let E be a topological space and let f be a mapping of E into the extended real line $\overline{\mathbf{R}}$. The function f is said to be *lower* (resp. *upper*) *semicontinuous* at a point $x_0 \in E$ if, for each $\alpha \in \overline{\mathbf{R}}$ such that $\alpha < f(x_0)$ (resp. $\alpha > f(x_0)$), there exists a neighborhood V of x_0 in E such that, for all $x \in V$, we have $\alpha < f(x)$ (resp. $\alpha > f(x)$). The function f is said to be *lower* (resp. *upper*) *semicontinuous* on E if it is so at every point $x_0 \in E$.

Clearly, if f is lower semicontinuous at x_0, then $-f$ is upper semicontinuous at this point. Hence we need consider only lower semicontinuous functions.

If φ is a mapping of a topological space F into E which is *continuous* at a point $y_0 \in F$, and if f is lower semicontinuous at the point $x_0 = \varphi(y_0)$, then $f \circ \varphi$ is lower semicontinuous at y_0. In particular, if F is a *subspace* of E, and if f is lower semicontinuous at a point $y_0 \in F$, then so is the function $f|F$.

Examples

(12.7.1) A mapping $f: E \to \overline{\mathbf{R}}$ is said to have a *relative minimum* at a point $x_0 \in E$ if there exists a neighborhood V of x_0 such that $f(x) \geqq f(x_0)$ for all $x \in V$. If so, then f is lower semicontinuous at the point x_0. This is clearly the case whenever $f(x_0) = -\infty$.

For each $x \in \mathbf{R}$, put $f(x) = 0$ if x is irrational, and $f(x) = 1/q$ if x is rational and equal to p/q (where p, q are coprime integers and $q > 0$). For each integer $n \geqq 0$, the subspace of rational numbers p/q with $q \leqq n$ is closed in $\mathbf{R}$ and discrete. Hence for each irrational x there exists a neighborhood V of x such that $f(y) \leqq 1/n$ for all $y \in V$, and therefore f is continuous at the point x. On the other hand, f has a relative maximum at each rational point, and is therefore *upper semicontinuous* on $\mathbf{R}$.

(12.7.2) *A mapping* $f: E \to \overline{\mathbf{R}}$ *is lower semicontinuous on* E *if and only if, for each* $\alpha \in \mathbf{R}$, *the set* $f^{-1}(]\alpha, +\infty])$ *of points* x *at which* $f(x) > \alpha$ *is open in* E (or, equivalently, the set $f^{-1}([-\infty, \alpha])$ of points x such that $f(x) \leqq \alpha$ is *closed* in E).

For to say that f is lower semicontinuous on E signifies that, for each $\alpha \in \mathbf{R}$, the set $f^{-1}(]\alpha, +\infty])$ is a neighborhood of each of its points (3.6.4).

If A is any subset of a set E, the *characteristic function* of A (usually denoted by φ_A) is the mapping of E into $\mathbf{R}$ such that $\varphi_A(x) = 1$ for all $x \in A$ and $\varphi_A(x) = 0$ for all $x \in E - A$. So we have $\varphi_E = 1$, $\varphi_\varnothing = 0$, and the formulas

$$\varphi_{E-A} = 1 - \varphi_A, \qquad \varphi_A \varphi_B = \varphi_{A \cap B},$$

$$\varphi_A + \varphi_B = \varphi_{A \cup B} + \varphi_{A \cap B}, \qquad \varphi_{A \cap \complement B} = \varphi_A - \varphi_A \varphi_B, \tag{12.7.3}$$

$$|\varphi_A - \varphi_B| = \varphi_{A \cap \complement B} + \varphi_{B \cap \complement A},$$

$$\inf_\lambda (\varphi_{A_\lambda}) = \varphi_{\bigcap_\lambda A_\lambda}, \qquad \sup_\lambda (\varphi_{A_\lambda}) = \varphi_{\bigcup_\lambda A_\lambda},$$

where A, B are any two subsets of E, and (A_λ) is any family of subsets of E.

(12.7.4) *A subset* A *of a topological space* E *is open* (resp. *closed*) *in* E *if and only if* φ_A *is lower* (resp. *upper*) *semicontinuous on* E.

This follows immediately from (12.7.2).

(12.7.5) *Let* f, g *be two mappings of a topological space* E *into* $\mathbf{R}$, *each of which is lower semicontinuous at a point* $x_0 \in E$. *Then the functions* $\sup(f, g)$ *and* $\inf(f, g)$ *are lower semicontinuous at* x_0. *The same is true for* $f+g$ *if the sum* $f(x)+g(x)$ *is defined for all* $x \in E$ (4.1.8), *and for* fg *if* f *and* g *are both* $\geqq 0$ *and the product* $f(x)g(x)$ *is defined for all* $x \in E$ (4.1.9).

We shall give the proof for $f+g$; the other cases are analogous. The result is obvious if $f(x_0)$ or $g(x_0)$ is equal to $-\infty$. If not, then we have $f(x_0)+g(x_0) > -\infty$. Every number $\alpha \in \mathbf{R}$ such that $\alpha < f(x_0)+g(x_0)$ can be written in the form $\alpha = \beta + \gamma$, with $\beta < f(x_0)$ and $\gamma < g(x_0)$ (it is enough to choose γ such that $\alpha - f(x_0) < \gamma < g(x_0)$). By hypothesis, there exists a neighborhood V of x_0 such that, for all $x \in V$, we have $\beta < f(x)$ and $\gamma < g(x)$. It follows that $\alpha = \beta + \gamma < f(x) + g(x)$ for all $x \in V$. Hence the result.

If f is *lower* semicontinuous at a point x_0, and if $f \geqq 0$ on E, then the function $1/f$ (where $1/f(x)$ is taken to be $+\infty$ if $f(x) = 0$) is *upper* semicontinuous at x_0. The proof is analogous.

Given a set E and any family $(f_\alpha)_{\alpha \in I}$ of mappings of E into $\overline{\mathbf{R}}$, the *upper* (resp. *lower*) *envelope* of the family is defined to be the mapping $x \mapsto \sup_{\alpha \in I} f_\alpha(x)$ (resp. $x \mapsto \inf_{\alpha \in I} f_\alpha(x)$) of E into $\overline{\mathbf{R}}$. It is denoted by $\sup_{\alpha \in I} f_\alpha$ (resp. $\inf_{\alpha \in I} f_\alpha$). We have $\sup_{\alpha \in I}(-f_\alpha) = -\inf_{\alpha \in I} f_\alpha$.

(12.7.6) *Let* E *be a topological space and let* $(f_\alpha)_{\alpha \in I}$ *be a family of mappings of* E *into* $\overline{\mathbf{R}}$. *If each* f_α *is lower semicontinuous at a point* $x_0 \in E$, *then the upper envelope* $f = \sup_{\alpha \in I} f_\alpha$ *is lower semicontinuous at* x_0.

Given any $\lambda < f(x_0) = \sup_{\alpha \in I} f_\alpha(x_0)$, there exists by hypothesis an index $\beta \in I$ such that $\lambda < f_\beta(x_0)$. Since f_β is lower semicontinuous at the point x_0, there exists a neighborhood V of x_0 such that $\lambda < f_\beta(x)$ for all $x \in V$, and therefore $\lambda < f(x)$ for all $x \in V$.

In particular:

(12.7.7) *The upper* (resp. *lower*) *envelope of a family of continuous mappings of* E *into* $\overline{\mathbf{R}}$ *is lower* (resp. *upper*) *semicontinuous.*

The *lower* envelope of an *infinite* family of *lower* semicontinuous functions is not necessarily lower semicontinuous. As an example, for each rational number r let f_r denote the characteristic function of $\mathbf{R} - \{r\}$, which is lower semicontinuous (12.7.4). The lower envelope f of the denumerable family $(f_r)_{r \in \mathbf{Q}}$ takes the value 0 at each rational number and the value 1 at each irrational number (in other words, it is the "Dirichlet function" $1 - \varphi_{\mathbf{Q}}$, cf. (3.11.6)), and is not lower semicontinuous at irrational points.

The following proposition is a partial converse of (12.7.7):

(12.7.8) *Let* E *be a separable, metrizable, locally compact space, and let* f *be a lower semicontinuous mapping of* E *into* $\overline{\mathbf{R}}$. *Suppose that there exists a finite real valued function* g, *continuous on* E *and with compact support, such that* $f \geqq g$ (which implies that $f(x) \geqq 0$ in the complement of some compact set). *Then there exists an increasing sequence* (f_n) *of finite real valued functions, continuous on* E *and with compact supports, such that* $f = \sup_n f_n$.

Replacing f if necessary by $f - g$ (which is everywhere defined), we may assume that $f \geqq 0$. It is then enough to show that there exists a *denumerable* family of finite real-valued continuous functions with compact supports, with f as upper envelope. For it we arrange this family in a sequence (g_n) and take $f_n = \sup(g_1, \ldots, g_n)$, it is clear that the f_n satisfy the required conditions.

Let (U_n) be an increasing sequence of relatively compact open sets in E, such that $\mathrm{E} = \bigcup_n \mathrm{U}_n$ (3.18.3). If g_n is the function which agrees with f on U_n and is zero on $\mathrm{E} - \mathrm{U}_n$, then clearly $f = \sup_n g_n$, and g_n is lower semicontinuous. Hence if g_n is the upper envelope of a denumerable set D_n of finite real-valued continuous functions with compact supports, then f will be the upper envelope of $\bigcup_n \mathrm{D}_n$ (2.3.6). Hence we may restrict ourselves to the case where f has *compact support.*

Suppose first of all that $f = \varphi_{\mathrm{A}}$, where A is a relatively compact *open* set. Put $f_n(x) = n \cdot \inf(d(x, \mathrm{E} - \mathrm{A}), 1/n)$. This is a finite, continuous, positive-valued function on E, with compact support (3.11.8). Moreover, we have $f_n(x) = f(x)$ whenever $x \in \mathrm{E} - \mathrm{A}$ or $d(x, \mathrm{E} - \mathrm{A}) \geqq 1/n$. It follows immediately that $f = \sup_n f_n$ (3.8.9).

In the case where $f \geqq 0$ and has compact support, we can restrict ourselves to the case where $0 \leqq f \leqq 1$, provided that the functions f_n are chosen so that $0 \leqq f_n(x) < 1$ for all $x \in \mathrm{E}$. For in the general case the function $h = f/(1 + f)$ (taken to have the value 1 at points where $f(x) = +\infty$) is lower semicontinuous (cf. (3.3.2)). If we have $h = \sup_n h_n$, with h_n continuous and $h_n(x) < 1$ for

all $x \in E$, then $f = \sup_n f_n$ with $f_n = h_n/(1 - h_n)$, and f_n is finite and continuous on E.

So we are now in the situation where $0 \leqq f \leqq 1$ and f has compact support. For each integer $n \geqq 1$, consider the decreasing finite sequence of open sets $A_{kn} = f^{-1}(]k/n, +\infty[)$ (12.7.2), where $0 \leqq k \leqq n-1$. These sets A_{kn} are relatively compact. The function $g_n = (1/n)\sum_{k=1}^{n-1} \varphi_{A_{kn}}$ is lower semicontinuous (12.7.5), and we have $0 \leqq f(x) - g_n(x) \leqq 1/n$ for all $x \in E$, hence f is the upper envelope of the sequence (g_n). On the other hand, since each $\varphi_{A_{kn}}$ is the upper envelope of a sequence $(h_{kmn})_{m \geqq 1}$ of continuous functions, it follows that g_n is the upper envelope of the sequence $(h_{mn})_{m \geqq 1}$, where $h_{mn} = (1/n)\sum_{k=1}^{n-1} h_{kmn}$ (2.3.11), and the h_{mn} are continuous. Since $g_n \leqq 1 - (1/n)$, the functions h_{mn} do not take the value 1, and it is clear that f is the upper envelope of the "double" family $(h_{mn})_{m \geqq 1, n \geqq 1}$ (2.3.6).

(12.7.9) *Let* E *be a nonempty metrizable compact space and let f be a lower semicontinuous mapping of* E *into* $\overline{\mathbf{R}}$. *Then there exists at least one point $a \in E$ such that $f(a) = \inf_{x \in E} f(x)$* (in other words, *$f$ attains its lower bound on* E).

Let $\mu = \inf_{x \in E} f(x)$. Then there exists a decreasing sequence (λ_n) of elements of $\overline{\mathbf{R}}$, belonging to $f(E)$ and such that $\inf_n \lambda_n = \mu$. The set $f^{-1}(\mu)$ is therefore the intersection of the nonempty closed sets $F_n = f^{-1}([-\infty, \lambda_n])$ (12.7.2), which form a decreasing sequence. If their intersection were empty, the open sets $U_n = E - F_n$ would cover E. But E is compact and the sequence (U_n) is increasing; hence we should have one of the U_n equal to E, which is absurd.

In particular, if f does not take the value $-\infty$ on E, then f is *minorized* on E.

Let (x_n) be any sequence of elements of $\overline{\mathbf{R}}$, and put

$$y_n = \inf_{p \geqq 0} x_{n+p}, \qquad z_n = \sup_{p \geqq 0} x_{n+p}.$$

The sequence (y_n) (resp. (z_n)) is increasing (resp. decreasing), hence has a limit (4.2.1). We write

$$\text{(12.7.10)} \quad \liminf_{n \to \infty} x_n = \lim_{n \to \infty} \left(\inf_{p \geqq 0} x_{n+p} \right), \qquad \limsup_{n \to \infty} x_n = \lim_{n \to \infty} \left(\sup_{p \geqq 0} x_{n+p} \right)$$

Clearly we have $\limsup_{n \to \infty} x_n = -\liminf_{n \to \infty} (-x_n)$, so that we need only consider the limit inferior.

(12.7.11) *For every sequence* (x_n) *in* $\overline{\mathbf{R}}$, $\liminf_{n\to\infty} x_n$ (resp. $\limsup_{n\to\infty} x_n$) *is the least* (resp. *the greatest*) *of the cluster values of the sequence* (x_n).

Let $a = \liminf_{n\to\infty} x_n$, and suppose first that a is finite. If $c < a < c'$ and if m is any integer, there exists $n > m$ such that $c < \inf_{p\geqq 0} x_{n+p} < c'$, hence there exists $p \geqq 0$ such that $c < x_{n+p} < c'$. This shows that a is a cluster value of the sequence (x_n) (3.13.11). If $a = +\infty$ (resp. $a = -\infty$), the argument is the same if we replace c' (resp. c) by a. Conversely, if b is a cluster value of (x_n), then it is also a cluster value of every sequence $(x_{n+p})_{p\geqq 0}$, and therefore $b \geqq \inf_{p\geqq 0} x_{n+p}$ (3.13.7). Hence $b \geqq a$.

This result shows that the existence of a *limit* of the sequence (x_n) is equivalent to the relation $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, and that the common value of the two sides is then the limit of (x_n) (3.16.4).

It follows from the definition that if (x_{n_k}) is any subsequence of a sequence (x_n) in $\overline{\mathbf{R}}$, then we have

(12.7.12) $$\liminf_{k\to\infty} x_{n_k} \geqq \liminf_{n\to\infty} x_n .$$

(12.7.13) *Let* E *be a Hausdorff topological space and let* f *be a mapping of* E *into* $\overline{\mathbf{R}}$ *which is lower semicontinuous at a point* $a \in$ E. *Then for every sequence* (x_n) *of points of* E *such that* $\lim_{n\to\infty} x_n = a$, *we have*

(12.7.13.1) $$\liminf_{n\to\infty} f(x_n) \geqq f(a).$$

For every $\alpha < f(a)$ there exists a neighborhood V of a in E such that $f(y) \geqq \alpha$ for all $y \in$ V; but there exists n_0 such that $x_n \in$ V for all $n \geqq n_0$. Hence $f(x_n) \geqq \alpha$ for all $n \geqq n_0$, which proves the result.

PROBLEMS

1. Let E be a topological space. For each closed subset A of the product space $\mathrm{E} \times \overline{\mathbf{R}}$, show that the mapping $x \mapsto \inf(\mathrm{A}(x))$ of $\mathrm{pr}_1\mathrm{A}$ into $\overline{\mathbf{R}}$ is lower semicontinuous. Conversely, if f: $\mathrm{E} \to \overline{\mathbf{R}}$ is lower semicontinuous, then the subset B of $\mathrm{E} \times \overline{\mathbf{R}}$ consisting of all pairs (x, y) such that $f(x) \leqq y$ is closed in $\mathrm{E} \times \overline{\mathbf{R}}$.

2. (a) Let E, F be two metrizable locally compact spaces and let $\pi: E \to F$ be a continuous map. The mapping π is said to be *proper* if, for each compact subset K of F, the inverse image $\pi^{-1}(K)$ is compact. Show that, if π is proper and $y \in \pi(E)$, every neighborhood in E of the set $\pi^{-1}(y)$ contains a neighborhood of the form $\pi^{-1}(U)$, where U is a neighborhood of y in F.

(b) Let g be a lower semicontinuous real-valued function on E. For each $y \in F$, let $f(y)$ be the greatest lower bound of g on the set $\pi^{-1}(y)$ (so that $f(y) = +\infty$ if $\pi^{-1}(y) = \varnothing$). Show that f is lower semicontinuous on F.

(c) Let g be the continuous function $(x_1, x_2) \mapsto |x_1 x_2 - 1|$ on $\mathbf{R} \times]0, +\infty[$. For each $x \in \mathbf{R}$, let $f(x_1) = \inf_{x_2 > 0} g(x_1, x_2)$. Show that f is not lower semicontinuous on $\mathbf{R}$.

3. For each point $t = (t_1, \ldots, t_n) \in \mathbf{C}^n$, put $P_t(X) = X^n + t_1 X^{n-1} + \cdots + t_n$. Show that there exists a continuous mapping $t \mapsto x(t)$ of $\mathbf{C}^n$ into $\mathbf{R}$ and a lower semicontinuous mapping $t \mapsto y(t)$ of $\mathbf{C}^n$ into $\mathbf{R}$, such that $P_t(x(t) + iy(t)) = 0$ for all $t \in \mathbf{C}^n$. (Take $x(t)$ to be the largest of the real parts of the roots of P_t, and use (9.17.4)).

4. Let f be any mapping of a metric space E into a metric space F. For each $x \in E$, let $\Omega(x)$ be the oscillation of f at x with respect to E (3.14), so that $\Omega(x)$ is a positive real number or $+\infty$. Show that the mapping $x \mapsto \Omega(x)$ of E into $\bar{\mathbf{R}}$ is upper semicontinuous.

5. Let E be a metric space and let $f: E \to \mathbf{R}$ be lower semicontinuous. Let $a \in E$ be a point at which the oscillation $\Omega(a)$ of f is finite. Show that for each $\varepsilon > 0$, there exists a neighborhood V of a in E such that $\inf_{x \in V} \Omega(x) \leqq \varepsilon$. (Show that otherwise there would exist points x arbitrarily close to a at which $f(x)$ took arbitrarily large values.)

6. Let $(a_n)_{n \geqslant 1}$ be an infinite sequence of distinct points in the interval $[0, 1[$. For each integer $N > 1$, let $b_1, \ldots, b_N$ denote the sequence obtained by arranging the set $\{a_1, \ldots, a_N\}$ in *increasing* order. The intervals $[0, b_1[$, $[b_1, b_2[, \ldots, [b_{N-1}, b_N[, [b_N, 1[$ are said to form the N*th subdivision* of $[0, 1[$ corresponding to the sequence (a_n). Let u_N (resp. v_N) denote the minimum (resp. maximum) of the lengths of the intervals of the Nth subdivision. Let $\lambda = \liminf_{N \to \infty} N u_N$, $\mu = \limsup_{N \to \infty} N v_N$.

(a) For each $\varepsilon > 0$, let n_0 be such that $u_N \geqq (\lambda - \varepsilon)/N$ and $v_N \leqq (\mu + \varepsilon)/N$ for all $N \geqq n_0$. Show that, for each $N \geqq n_0$ and each integer r such that $0 \leqq r \leqq N$, there exist $2r$ intervals of the $(N + r)$th subdivision whose lengths are greater than or equal to the $2r$ numbers

$$\frac{\lambda - \varepsilon}{N+1}, \frac{\lambda - \varepsilon}{N+1}, \frac{\lambda - \varepsilon}{N+2}, \frac{\lambda - \varepsilon}{N+2}, \cdots, \frac{\lambda - \varepsilon}{N+r}, \frac{\lambda - \varepsilon}{N+r}$$

and such that the other intervals of the $(N + r)$th subdivision are intervals of the Nth subdivision (use induction on r).

(b) Show that, if $N \geqq n_0$, the intervals of the Nth subdivision can be arranged in order in such a way that their lengths are less than or equal to the $N + 1$ numbers

$$\frac{\mu + \varepsilon}{N}, \frac{\mu + \varepsilon}{N+1}, \cdots, \frac{\mu + \varepsilon}{2N}.$$

(For each interval I of the Nth subdivision, let $s(I)$ be the largest integer $\leqq 2N$ such that I is an interval of the $s(I)$th subdivision, and arrange the intervals I in order of increasing $s(I)$.)

(c) Deduce from (a) and (b) that

$$\lambda \leqq 1/\log 4, \qquad \mu \geqq 1/\log 2$$

(use the fact that $\sum_{r=1}^{N} 1/(N+r)$ tends to $\log 2$ as $N \to +\infty$).

(d) Take $a_n = \log_2(2n-1) - [\log_2(2n-1)]$. Show that we have $\lambda = 1/\log 4$ and $\mu = 1/\log 2$ for this sequence. (Observe that, if $N = 2^p$, the set $\{a_1, \ldots, a_N\}$ is the same as the set $\{c_N, \ldots, c_{2N-1}\}$ where $c_n = \log_2 n - [\log_2 n]$.) (Cf. Section 13.21, Problem 15.)

7. Let E be a metrizable space. Show that a nonempty closed subset S of E is the support of a lower semicontinuous (resp. continuous) real-valued function $\geqq 0$ on E if and only if S is the closure of its interior.

8. Let E be a metrizable space and f a mapping of E into $\bar{\mathbf{R}}$. The *lower semicontinuous regularization* of f is defined to be the upper envelope of the continuous functions g on E such that $g \leqq f$.

(a) For each $x \in E$ we denote by $\liminf_{y \to x} f(y)$ the greatest lower bound of the numbers $\liminf_{n \to \infty} f(y_n)$, for all sequences (y_n) tending to x. Show that the function $x \mapsto \liminf_{y \to x} f(y)$ is the lower semicontinuous regularization of f.

(b) Suppose that E is an open set in $\mathbf{R}$. For each $x \in E$, let $\liminf_{y \to x,\ y \geqq x} f(y)$ denote the greatest upper bound of the numbers $\liminf_{n \to \infty} f(y_n)$ for all sequences (y_n) such that $y_n \to x$ and $y_n \geqq x$ for all n. Show that the set of all $x \in E$ such that

$$\liminf_{y \to x,\ y \geqq x} f(y) > \liminf_{y \to x} f(y)$$

is at most denumerable. (For each pair (p, q) of rational numbers with $p > q$, show that the set of points $x \in E$ such that

$$\liminf_{y \to x,\ y \geqq x} f(y) > p > q > \liminf_{y \to x} f(y)$$

is at most denumerable, by following the method of Section 3.9, Problem 3.)

9. A *Dirichlet series* is a series whose general term is of the form $a_n e^{-\lambda_n s}$, where (λ_n) is an increasing sequence of real numbers, tending to $+\infty$, and (a_n) is any sequence of complex numbers, and s is complex.

(a) Show that if the series is convergent for $s = s_0$, then it is uniformly convergent in the angular sector consisting of the points $s = s_0 + \rho e^{i\theta}$ with $\rho \geqq 0$ and $-\alpha \leqq \theta \leqq \alpha$, where α is such that $0 < \alpha < \pi/2$. (Reduce to the case $s_0 = 0$; show first that if $\mathscr{R}s = \sigma$, then

$$|e^{-as} - e^{-bs}| \leqq |s| \sigma^{-1}(e^{-a\sigma} - e^{-b\sigma})$$

whenever a and b are real and $a < b$, by considering the integral $\int_a^b e^{-xs}\,dx$. For each integer m and each $n \geqq m$, put $S_{m,n} = a_m + a_{m+1} + \cdots + a_n$ and use the identity (*Abel's partial summation formula*)

$$\sum_{k=m}^{n} a_k e^{-\lambda_k s} = \sum_{k=m}^{n-1} S_{m,k}(e^{-\lambda_k s} - e^{-\lambda_{k+1} s}) + S_{m,n} e^{-\lambda_n s}).$$

(b) Deduce from (a) that there are the following three alternatives: either the series $(a_n e^{-\lambda_n s})$ does not converge for any value of s, or it converges for all values of s, or else there exists a real number σ_0 such that the series converges for $\mathscr{R}s > \sigma_0$ and does not converge when $\mathscr{R}s < \sigma_0$. In the first (resp. second) case we put $\sigma_0 = +\infty$ (resp. $-\infty$). In all cases, the (extended real) number σ_0 is called the *abscissa of convergence* of the series. The sum of the series is an analytic function of s in the region $\mathscr{R}s > \sigma_0$.
(c) Show that if $\sigma_0 \geqq 0$, then $\sigma_0 = \limsup_{n\to\infty} (\log |S_{0,n}|)/\lambda_n$. (Show first that if the series with general term $a_n e^{-\lambda_n s} = b_n$ is convergent (with s real and >0), then we have $|S_{0,n}| \leqq K e^{\lambda_n s}$ where K is a constant, by writing $a_n = b_n e^{\lambda_n s}$. Then show that if $\gamma = \limsup_{n\to\infty} (\log |S_{0,n}|)/\lambda_n$, the series $(a_n e^{-\lambda_n s})$ converges when $s = \gamma + \delta$, where δ is real and >0, by arguing as in (a).)
(d) Let σ_1 be the abscissa of convergence of the Dirichlet series with general term $|a_n| e^{-\lambda_n s}$. Then $\sigma_1 \geqq \sigma_0$. If $\sigma_0 < +\infty$, show that

$$\sigma_1 - \sigma_0 \leqq \limsup_{n\to\infty} \frac{\log n}{\lambda_n}.$$

(Remark that, given $\varepsilon > 0$, we have $|a_n| \leqq e^{(\sigma_0+\varepsilon)\lambda_n}$ for all sufficiently large n.) Consider the case where $\lambda_n = n$ (cf. Section 9.1, Problem 1). Show that, for the series with general term

$$\frac{(-1)^n}{\sqrt{n}} e^{-s \log\log n},$$

we have $\sigma_0 = -\infty$ and $\sigma_1 = +\infty$.

10. Let f be a real-valued function defined on the interval $[0, +\infty[$, satisfying the following conditions:
(1) $f(s+t) \geqq f(s) + f(t)$;
(2) there exists a number $M > 0$ such that $|f(t)| \leqq Mt$ for all t.
Show that the limits

$$\alpha = \lim_{t\to 0} \frac{f(t)}{t}, \qquad \beta = \lim_{t\to+\infty} \frac{f(t)}{t}$$

exist and are finite, and that $\alpha t \leqq f(t) \leqq \beta t$ for all $t \geqq 0$. (To establish the existence of α it is enough to show that, if we take $\alpha = \limsup_{t\to 0} f(t)/t$ (Problem 8), then $\alpha \leqq f(t)/t$ for all $t > 0$. Take a decreasing sequence (t_n) tending to 0, such that $\lim_{n\to\infty} f(t_n)/t_n = \alpha$, and let k_n be the integral part of t/t_n. Show that

$$f(t_n) \leqq \frac{f(t)}{k_n} + M\left(\frac{t}{k_n} - t_n\right).$$

The proof of the existence of $\lim_{t\to+\infty} f(t)/t$ is similar; put $\beta = \liminf_{t\to+\infty} f(t)/t$.)

11. Let f be a continuous mapping of an open set U in a Banach space E into a Banach space F. For each $x \in U$, let

$$D^+f(x) = \limsup_{y\to x,\ y\neq x} \frac{\|f(y) - f(x)\|}{\|y - x\|},$$

$$D^-f(x) = \liminf_{y\to x,\ y\neq x} \frac{\|f(y) - f(x)\|}{\|y - x\|}.$$

We have $0 \leqq D^-f(x) \leqq D^+f(x) \leqq +\infty$.

(a) If the function f is differentiable at a point $x \in U$, then $D^+f(x) = \|f'(x)\|$. If $f'(x)$ is not a linear homeomorphism of E onto a subspace of F, we have $D^-f(x) = 0$. In the contrary case, we have $D^-f(x) = \|(f'(x))^{-1}\|^{-1}$, where $(f'(x))^{-1}$ denotes the inverse homeomorphism.

(b) Suppose that the segment with endpoints a, b is contained in U, and that $D^+f(x) \leqq M$ for all points x in this segment. Prove that $\|f(b) - f(a)\| \leqq M\|b - a\|$ (see the proof of 8.5.1).)

(c) Take $U = E = \mathbf{R}^2$ and $F = \mathbf{R}^3$, and let f be defined as follows:

$$f(\xi_1, \xi_2) = \left(\frac{\xi_1(\xi_1^2 - \xi_2^2)}{\xi_1^2 + \xi_2^2}, \frac{\xi_2(\xi_1^2 - \xi_2^2)}{\xi_1^2 + \xi_2^2}, \xi_2 \right)$$

if $(\xi_1, \xi_2) \neq (0, 0)$; and $f(0, 0) = 0$. Show that the greatest lower bound of D^-f on U is strictly positive, but that there is no neighborhood of 0 in U on which f is injective.

(d) For the remainder of this problem, suppose that there exist two numbers m, M with $0 < m < M < +\infty$, such that $m \leqq D^-f(x) \leqq M$ for all $x \in U$. Suppose also that for each $x \in U$ there exists an open neighborhood V of x in U such that $f|V$ is a *homeomorphism* of V onto an open set in F; then $f(U)$ is open in F. Let a be a point of U, and for each line $D \subset F$ containing $f(a)$ let I_D be the connected component of the point $f(a)$ in the open set $D \cap f(U)$ in D. The union S_a of the sets I_D is the largest star-shaped open set with respect to $f(a)$ contained in $f(U)$. For each line D in F passing through $f(a)$, there exists a unique continuous mapping $g_D : I_D \to U$ such that $g_D(f(a)) = a$ and $f(g_D(y)) = y$ for all $y \in I_D$ (the proof is the same as in Section 10.2, Problem 6(c).) If y, y' are points of I_D, we have $\|g_D(y') - g_D(y)\| \leqq m^{-1}\|y' - y\|$. Deduce that as y tends to an endpoint of I_D (when there is one), $g_D(y)$ has a limit belonging to Fr(U).

(e) Let $\gamma : J \to U$ be a path in U with origin a and extremity b. Show that, if $f(\gamma(J)) \subset S_a$, then we have $b = g_D(f(b))$, where D is the line through $f(a)$ and $f(b)$, and $\|f(b) - f(a)\| \geqq m\|b - a\|$.

(f) Deduce from (d) and (e) that if $U = E$ then $S_a = F$, and f is a homeomorphism of E onto F.

(g) Let $k = M/m$. Let a, b be two points of U such that the set $E_{k,a,b}$ of points $z \in E$ such that $\|z - a\| + \|z - b\| \leqq k\|b - a\|$ is contained in U. If L is the closed segment with endpoints a, b, show that $f(L) \subset S_a$, and hence that

$$\|f(b) - f(a)\| \geqq m\|b - a\|.$$

(Proof by contradiction: consider the least $t \in [0, 1]$ such that $y = f(a + t(b - a)) \notin S_a$. If D is the line through $f(a)$ and y, there is an endpoint u of I_D belonging to the open segment with endpoints $f(a)$ and y. When $t' < t$ tends to t, there is a point u' of the open segment with endpoints $f(a)$ and $y' = f(a + t'(b - a))$ which tends to u. Let D' be the line through $f(a)$ and y', and let $z' = g_{D'}(u')$. Using (e), show that $z' \in E_{k,a,b}$, and obtain a contradiction by making t' tend to t and using (d).)

(h) Suppose that E and F are Hilbert spaces and that U is the ball $\|x\| < 1$. Deduce from (g) that if B is the ball $\|x\| < 1/(1 + (k^2 - 1)^{1/2})$, the restriction of f to B is a homeomorphism onto $f(B)$, and that $\|f(x') - f(x)\| \geqq m\|x' - x\|$ for any two points x, x' in B.

(i) With the hypotheses of (h), suppose also that $k < ((1 + \sqrt{5})/2)^{1/2}$. Show that f is then *injective* on U and, more precisely, that for any two points x, x' in U we have $\|f(x') - f(x)\| \geqq \mu\|x' - x\|$, where

$$\mu = \frac{m^2 - M(M^2 - m^2)^{1/2}}{m + (M^2 - m^2)^{1/2}}.$$

(If θ is such that $0 < \theta < 1$, then $\|f(x') - f(x)\| \geqq \|f(\theta x') - f(\theta x)\| - 2M(1-\theta)$, and we can choose θ so that $d(\theta x, E - U) \geqq \frac{1}{2}\sqrt{k^2 - 1}\,\|x' - x\|$,

$$\|f(\theta x') - f(\theta x)\| \geqq m\theta\,\|x' - x\| \quad \text{and} \quad m\theta\,\|x' - x\| - 2M(1-\theta) \geqq \mu\,\|x' - x\|.)$$

8. TOPOLOGICAL GROUPS

If G is a *group*, in which the law of composition is written (for example) multiplicatively, a topology on G is said to be *compatible* with the group structure if the two mappings $(x, y) \mapsto xy$ of $G \times G$ (endowed with the product topology) into G, and $x \mapsto x^{-1}$ of G into G, are *continuous*. A group endowed with a topology compatible with its group structure is called a *topological group*. We leave it to the reader to transcribe this definition (and all the results which follow) into additive notation.

An *isomorphism* of a topological group G onto a topological group G′ is by definition an isomorphism of the group G onto the group G′ which is *bicontinuous*. If $G' = G$, we say *automorphism* instead of *isomorphism*.

If G is any group, the law of composition $(x, y) \mapsto yx$ defines a group structure on the set G (and this structure is different from the given one unless G is commutative). The group so defined is denoted by G^0 and is called the *opposite* of G. Any topology which is compatible with the group structure of G is also compatible with that of G^0, and hence makes G^0 into a topological group. The mapping $x \mapsto x^{-1}$ is then an *isomorphism* of G onto G^0.

Examples

(12.8.1) The discrete topology and the chaotic topology (12.1.1) are compatible with the structure of *any* group. The topology of a normed space (in particular **R** or **C**) is compatible with its additive group structure. On the additive group **Q** of rational numbers, the topology defined by the p-adic distance d (3.2.6) is compatible with the group structure, because by virtue of the definition of this distance and (3.2.6.4) we have

$$d(x_0 + y_0, x + y) \leqq \mathrm{Max}(d(x_0, x), d(y_0, y))$$

and

$$d(-x_0, -x) = d(x_0, x).$$

If E is a (real or complex) Banach space and **GL**(E) the set of linear homeomorphisms of E onto E, then the topology induced on **GL**(E) by that of $\mathscr{L}(E; E)$ is compatible with the group structure ((5.7.5) and (8.3.2)).

In particular, the topology induced on the multiplicative group $\mathbf{R}^*$ (resp. $\mathbf{C}^*$) of real (resp. complex) numbers $\neq 0$ by the topology of $\mathbf{R}$ (resp. $\mathbf{C}$) makes $\mathbf{R}^*$ (resp. $\mathbf{C}^*$) into a topological group.

If G is a topological group (in multiplicative notation), the mappings $(x, y) \mapsto xy^{-1}$ and $(x, y) \mapsto x^{-1}y$ are continuous mappings of $G \times G$ into G (3.11.5). For each $a \in G$, the left and right *translations* $x \mapsto ax$ and $x \mapsto xa$ are *homeomorphisms* of G onto G, because they are bijective and continuous ((3.20.14) and (12.5)) and so are the inverse mappings $x \mapsto a^{-1}x$ and $x \mapsto xa^{-1}$. For any a, b in G, the mapping $x \mapsto axb$ (and in particular the *inner automorphism* $x \mapsto axa^{-1}$) is therefore a homeomorphism of G onto G (3.11.5). Since the mapping $x \mapsto x^{-1}$ is bijective and equal to its inverse, it also is a homeomorphism of G onto G.

(12.8.2) *Let* G *be a topological group.*

(i) *For each open* (resp. *closed*) *subset* A *of* G, *and each* $x \in G$, *the sets* xA, Ax *and* A^{-1} (the set of all y^{-1}, where $y \in A$) *are open* (resp. *closed*) *in* G.

(ii) *For each open set* A *in* G *and each subset* B *of* G, *the sets* AB (the set of all products yz, where $y \in A$ and $z \in B$) *and* BA *are open in* G.

The assertions in (i) follow immediately from the preceding remarks, and (ii) follows from (i), since $AB = \bigcup_{z \in B} Az$ is a union of open sets.

(12.8.2.1) On the other hand, if A and B are *closed* in G, it does not necessarily follow that AB is closed (cf. (12.10.5)). For example, if θ is an irrational number, consider the subgroups $\mathbf{Z}$ and $\theta\mathbf{Z}$ of $\mathbf{R}$. Then the subgroup $\mathbf{Z} + \theta\mathbf{Z}$ *is not closed* in $\mathbf{R}$. To see why, observe that this subgroup is denumerable, and therefore is not the whole of $\mathbf{R}$ (2.2.17). Hence it is enough to prove the following result:

(12.8.2.2) *The only closed subgroups of* $\mathbf{R}$ *are* $\mathbf{R}$ *itself and the subgroups of the form* $\alpha\mathbf{Z}$, *where* $\alpha \in \mathbf{R}$.

Granted this, we cannot have both $1 = n\alpha$ and $\theta = m\alpha$ with m and n integers, because θ is irrational, and the assertion of (12.8.2.1) then follows.

To prove (12.8.2.2), we shall first show that a subgroup H of $\mathbf{R}$ is either *discrete* or *dense in* $\mathbf{R}$. If H is not discrete, then for each $\varepsilon > 0$ there exists $x \neq 0$ in the set $H \cap [-\varepsilon, +\varepsilon]$. Since the integer multiples nx ($n \in \mathbf{Z}$) belong to H, every interval of length greater than ε in $\mathbf{R}$ contains one of these points and therefore H is dense in $\mathbf{R}$.

To complete the proof of (12.8.2.2), it remains to be shown that if H is discrete it is of the form $\alpha\mathbf{Z}$. We may assume that $H \neq \{0\}$. Since $H = -H$, the intersection $H \cap]0, +\infty[$ is not empty. If $b > 0$ belongs to H, the intersection $H \cap [0, b]$ is compact and discrete and therefore *finite* (3.16.3). Let a be the smallest element >0 in this set, and for any $x \in H$ let $m = [x/a]$ be the integral part of x/a. Then $x - ma \in H$, and $0 \leqq x - ma < a$. This implies that $x - ma = 0$ and so we have $H = a\mathbf{Z}$.

(12.8.3) *Let* G *be a topological group.*

(i) *Let* $a \in G$. *If* V *runs through a fundamental system of neighborhoods of the neutral element* e *of* G, *the sets* aV (resp. Va) *form a fundamental system of neighborhoods of* a.

(ii) *For each neighborhood* U *of* e, *there exists a neighborhood* V *of* e *such that* $VV^{-1} \subset U$.

(iii) *For each neighborhood* U *of* e *and each* $a \in G$, *there exists a neighborhood* W *of* e *such that* $aWa^{-1} \subset U$.

(iv) G *is Hausdorff if and only if the set* $\{e\}$ *is closed in* G.

The assertion (i) follows from the fact that translations are homeomorphisms; (ii) expresses that the mapping $(x, y) \mapsto xy^{-1}$ is continuous at (e, e), having regard to the definition of open sets in $G \times G$ (12.5); (iii) expresses the continuity of the mapping $x \mapsto axa^{-1}$ at the point e. As to (iv), it is clear that if G is Hausdorff the set $\{e\}$ is closed (12.3.4). Conversely, if $\{e\}$ is closed and if x, y are distinct points of G, then there exists a neighborhood V of e such that $e \notin x^{-1}yV$, i.e., such that $x \notin yV$. If W is a neighborhood of e such that $WW^{-1} \subset V$ (which is possible by (ii)), then we have $xW \cap yW = \varnothing$, because the relation $xw' = yw''$ with w' and w'' in W would imply

$$x = yw''w'^{-1} \in yWW^{-1} \subset yV.$$

Hence G is Hausdorff.

A neighborhood V of e is said to be *symmetric* if $V^{-1} = V$. The symmetric neighborhoods of e form a *fundamental system of neighborhoods* of e in G, because if U is any neighborhood of e, then so is U^{-1} (12.8.2) and therefore $U \cap U^{-1}$ is a symmetric neighborhood of e contained in U.

For each integer $n > 0$ and each subset V of G we define V^n inductively by the rule $V^n = V^{n-1} \cdot V = V \cdot V^{n-1}$. (The set V^n is *not* the set of x^n with $x \in V$.) If V is a neighborhood of e, then so is $V^n \supset V$ for all $n \geqq 1$, and it follows from (12.8.3(ii)) that if U is any neighborhood of e and n is an integer >1, there exists a symmetric neighborhood V of e such that $V^n \subset U$.

(12.8.4) *For a homomorphism f of a topological group* G *into a topological group* G′ *to be continuous, it is necessary and sufficient that f should be continuous at one point.*

For if f is continuous at $a \in G$ and if V′ is a neighborhood of $f(a)$, then $f^{-1}(V') = V$ is a neighborhood of a. For each $x \in G$, we have

$$f(xa^{-1}V) = f(x)(f(a))^{-1}f(V) \subset f(x)(f(a))^{-1}V',$$

which establishes the continuity of f at the point x, by virtue of (12.8.3(i)).

If H is a subgroup of a topological group G, the induced topology on H is clearly compatible with the group structure of H. Whenever we consider H as a topological group, it is to be understood in this sense unless the contrary is explicitly stated.

(12.8.5) *The closure* $\bar{H}$ *of a subgroup* (resp. *normal subgroup*) H *of a topological group* G *is a subgroup* (resp. *normal subgroup*) *of* G. *If* G *is Hausdorff and* H *is commutative, then* $\bar{H}$ *is commutative.*

The image of $\bar{H} \times \bar{H} = \overline{H \times H}$ under the continuous mapping $(x, y) \mapsto xy^{-1}$ of $G \times G$ into G is contained in $\bar{H}$, because the image of $H \times H$ under this mapping is contained in H (3.11.4). Hence $\bar{H}$ is a subgroup. Likewise, if H is normal and $a \in G$, the image of H under the mapping $x \mapsto axa^{-1}$ is contained in H, hence the image of $\bar{H}$ is contained in $\bar{H}$, and therefore $\bar{H}$ is normal. Finally, if G is Hausdorff and H is commutative, the continuous functions xy and yx are equal on $H \times H$ and therefore also on $\bar{H} \times \bar{H}$ by virtue of the principle of extension of identities ((3.15.2), 12.3, and 12.5).

(12.8.6) (i) *In a topological group* G, *the normalizer* $\mathcal{N}(H)$ *of a closed subgroup* H (i.e., the set of all $x \in G$ such that $xHx^{-1} \subset H$) *is a closed subgroup.*

(ii) *In a Hausdorff topological group* G, *the centralizer* $\mathcal{Z}(M)$ *of any subset* M *of* G (i.e., the set of all $x \in G$ which commute with every element of M) *is a closed subgroup of* G. *In particular, the center of* G *is closed.*

For each $z \in H$, the set of elements $x \in G$ such that $xzx^{-1} \in H$ is the inverse image of H under the continuous mapping $x \mapsto xzx^{-1}$, hence is a closed set (3.11.4). Since $\mathcal{N}(H)$ is the intersection of these sets as z runs through H, it follows that $\mathcal{N}(H)$ is closed (3.8.2). Again, if G is Hausdorff then for each $z \in M$ the set of $x \in G$ such that $zx = xz$ is closed (12.3.5) and hence so is $\mathcal{Z}(M)$ since it is the intersection of these sets.

(12.8.7) (i) *In a topological group* G, *every locally closed subgroup is closed. Every subgroup which has an interior point is both open and closed.*

(ii) *In a Hausdorff group, every discrete subgroup is closed.*

(i) Let H be a locally closed subgroup of G. Then $\bar{H}$ is a subgroup of G, and H is an open subgroup of $\bar{H}$ (12.2.3). Hence it is enough to prove the second assertion. Now if H has an interior point, then by translation it follows that *every* point of H is interior, and therefore H is open. Hence the left cosets xH are also open sets, and therefore $\complement H$ is open in G (because it is a union of cosets xH). Consequently H is closed in G.

(ii) If G is Hausdorff and H is a discrete subgroup of G, then there exists a symmetric open neighborhood V of e such that $V \cap H = \{e\}$. If $x \in \bar{H}$, we have $xV \cap H \neq \varnothing$. Now if $y \in xV \cap H$, then $x \in yV$ and the set $\{y\}$ is closed in the open set yV, because G is Hausdorff. Since $yV \cap H = \{y\}$ (because $y \in H$), we must have $x = y$, and therefore $\bar{H} = H$.

(12.8.8) *If* G *is a connected group and* V *is a symmetric neighborhood of* e, *then* G *is equal to the union* V^∞ *of the sets* V^n *for* $n > 0$.

The set V^∞ is clearly symmetric, and since $V^m V^n = V^{m+n}$ it follows that $V^\infty V^\infty \subset V^\infty$. Hence V^∞ is a subgroup of G. Since e is an interior point of V^∞, this subgroup is both open and closed (12.8.7) and therefore is the whole of G.

(12.8.9) *In a topological group* G, *the connected component* K *of the neutral element* (3.19) *is a closed normal subgroup* (called the *neutral component* or *identity component* of G). *For each* $x \in G$, *the connected component of* x *in* G *is* $xK = Kx$.

If $a \in K$, the set $a^{-1}K$ is connected and contains e, hence $K^{-1}K \subset K$. This shows that K is a subgroup of G. It is invariant under all automorphisms of the topological group G, in particular under all inner automorphisms, hence K is normal in G. Also K is closed in G (3.19). Finally, the last assertion follows from the fact that left and right translations are homeomorphisms of G onto G.

(12.8.10) If G_1, G_2 are two topological groups, then the *product* topology on the product group $G = G_1 \times G_2$ is compatible with the group structure of G. For, identifying canonically the product spaces $G \times G$ and $(G_1 \times G_1) \times (G_2 \times G_2)$ (12.5), the mapping

$$((x_1, x_2), (y_1, y_2)) \mapsto (x_1 y_1, x_2 y_2)$$

is continuous ((3.20.15) and (12.5)), and the mapping $(x_1, x_2) \mapsto (x_1^{-1}, x_2^{-1})$ is continuous for the same reason. The group $G_1 \times G_2$, endowed with the product topology, is called the *product* of the topological groups G_1 and G_2.

If G is a *commutative* topological group, the mapping $(x, y) \mapsto xy$ is a continuous homomorphism of $G \times G$ into G.

PROBLEMS

1. Let G be a group and let $\mathfrak{V}$ be a set of subsets of G, satisfying conditions (V_I) and (V_{II}) of Section 12.3, Problem 3, together with the following:

   (GV_I) For all $U \in \mathfrak{V}$ there exists $V \in \mathfrak{V}$ such that $V \cdot V \subset U$.
   (GV_{II}) For all $U \in \mathfrak{V}$ we have $U^{-1} \in \mathfrak{V}$.
   (GV_{III}) For all $U \in \mathfrak{V}$ and $a \in G$, we have $aUa^{-1} \in \mathfrak{V}$.

   Show that there exists a unique topology on G, compatible with the group structure of G, for which $\mathfrak{V}$ is the set of neighborhoods of the neutral element e.

2. Every topology compatible with the group structure of a *finite* group G is obtained by taking as neighborhoods of the neutral element the sets containing a normal subgroup H of G.

   Give an example of a non-Hausdorff topological group in which the center is not closed and has as its closure a noncommutative subgroup.

3. Let G be a connected topological group. Show that every totally disconnected normal subgroup D of G is contained in the center of G. (If $a \in D$, consider the mapping $x \mapsto xax^{-1}$ of G into D.)

4. Show that the commutator subgroup of a connected topological group is connected (use (3.19.3) and (3.19.7)).

5. Let G be a topological group and let H, K be subgroups of G such that $H \supset K \supset H'$, where H' is the commutator subgroup of H. Show that $\bar{K}$ contains the commutator subgroup of $\bar{H}$. Deduce that if G is Hausdorff, the closure in G of a solvable subgroup is solvable (induction on the length of the derived series).

6. Let G be a topological group and let H be a closed normal subgroup of G, containing the commutator subgroup of G. If the identity component K of H is solvable, show that the identity component L of G is solvable. (Show by using Problem 4 that K contains the commutator subgroup of L.)

7. Let $\varphi : \mathbf{R} \to \mathbf{T}$ be the canonical homomorphism and let θ be an irrational number. On the topological space $G = \mathbf{R}^2 \times \mathbf{T}^2$, a group law is defined by

$$(x_1, x_2, t_1, t_2)(x_1', x_2', t_1', t_2') = (x_1 + x_1', x_2 + x_2', t_1 + t_1' + \varphi(x_2 x_1'), t_2 + t_2' + \varphi(\theta x_2 x_1')).$$

   In this way G becomes a locally compact group (even a Lie group). Show that the commutator subgroup of G is not closed in G.

8. Let $(G_\alpha)_{\alpha \in I}$ be any family of topological groups. Show that the product topology on $G = \prod_{\alpha \in I} G_\alpha$ is compatible with the product group structure (Section 12.5, Problem 4).

   The topological group so defined is called the *product* of the topological groups G_α. Let H be the normal subgroup of G consisting of all $x = (x_\alpha)$ such that for all but a finite number of indices x_α is the identity element of G_α. Show that H is dense in G.

If G is a group, a function f on $G \times G$ is said to be *left* (resp. *right*) *invariant* if $f(xy, xz) = f(y, z)$ (resp. $f(yx, zx) = f(y, z)$) for all x, y, z in G. When G is commutative, these two conditions coincide, and f is then said to be *translation-invariant*. A distance d on G is left (resp. right) invariant if and only if the left (resp. right) translations are *isometries* with respect to d. If f is a left-invariant function on $G \times G$, then the function $(x, y) \mapsto f(x^{-1}, y^{-1})$ is right-invariant, and vice versa.

For example, if E is a normed space, the distance $\|x - y\|$ on E is translation-invariant.

(12.9.1) *In order that the topology of a topological group* G *should be metrizable* (in which case G is said to be a *metrizable group*) *it is necessary and sufficient that there should exist a denumerable fundamental system of neighborhoods of the neutral element e, whose intersection consists of e alone. When this condition is satisfied, the topology of* G *can be defined by a left-invariant distance, or by a right-invariant distance.*

It is enough to show that if there exists a denumerable fundamental system of neighborhoods (U_n) of e in G such that $\bigcap_n U_n = \{e\}$, then the topology of G can be defined by a left-invariant distance. We define inductively a sequence (V_n) of symmetric neighborhoods of e in G such that $V_1 \subset U_1$ and

$$V_{n+1}^3 \subset V_n \cap U_n$$

for all $n \geqq 1$ (12.8.3), so that (V_n) is also a fundamental system of neighborhoods of e. Now define a real-valued function g on $G \times G$ as follows: $g(x, x) = 0$; if $x \neq y$, then either $x^{-1}y \notin V_1$, in which case we take $g(x, y) = 1$; or else there exists a greatest integer k such that $x^{-1}y \in V_k$ (because $x^{-1}y \neq e$ cannot belong to all the V_n), in which case we define $g(x, y) = 2^{-k}$. It is clear from this definition that $g(x, y) = g(y, x)$, that $g(x, y) \geqq 0$, and that $g(zx, zy) = g(x, y)$ for all x, y, z in G.

Now let

$$d(x, y) = \inf \sum_{i=0}^{p-1} g(z_i, z_{i+1}) \tag{12.9.1.1}$$

where the infimum is taken over the set of all finite sequences $(z_0, z_1, \ldots, z_p)$ (with p arbitrary) such that $z_0 = x$ and $z_p = y$. We shall show that d is a left-invariant *distance* on G and satisfies the inequalities

$$\tfrac{1}{2} g(x, y) \leqq d(x, y) \leqq g(x, y). \tag{12.9.1.2}$$

From the definition of d, it follows immediately that d is left-invariant (because g is), satisfies the triangle inequality, is symmetric and positive. Moreover, the right-hand inequality in (12.9.1.2) is obvious, and shows that $d(x, x) = 0$ for all $x \in G$. Hence d is a *pseudo-distance* on G. To prove the left-hand inequality of (12.9.1.2) we shall show by induction on p that, for each finite sequence $(z_i)_{0 \leqq i \leqq p}$ of $p+1$ points in G such that $z_0 = x$ and $z_p = y$, we have

$$\sum_{i=0}^{p-1} g(z_i, z_{i+1}) \geqslant \tfrac{1}{2} g(x, y). \tag{12.9.1.3}$$

This inequality is obvious if $p = 1$. Let us write $\alpha = \sum_{i=0}^{p-1} g(z_i, z_{i+1})$. The inequality (12.9.1.3) is true if $\alpha \geqq \frac{1}{2}$, because $g(x, y) \leqq 1$. If $\alpha = 0$, then $z_i = z_{i+1}$ for $0 \leqq i \leqq p-1$, hence $x = y$ and so (12.9.1.3) is trivially satisfied. So suppose that $0 < \alpha < \frac{1}{2}$, and let h be the greatest index such that $\sum_{i<h} g(z_i, z_{i+1}) \leqq \frac{1}{2}\alpha$. Then we have $\sum_{i<h+1} g(z_i, z_{i+1}) > \frac{1}{2}\alpha$, so that

$$\sum_{i>h} g(z_i, z_{i+1}) \leqq \tfrac{1}{2}\alpha.$$

By the inductive hypothesis, we have $g(x, z_h) \leqq \alpha$ and $g(z_{h+1}, y) \leqq \alpha$; on the other hand, it is clear that $g(z_h, z_{h+1}) \leqq \alpha$. Let k be the smallest integer >0 such that $2^{-k} \leqq \alpha$. Then $k \geqq 2$, and the elements $x^{-1}z_h$, $z_h^{-1}z_{h+1}$, $z_{h+1}^{-1}y$ are all in V_k, by virtue of the definition of g. Hence $x^{-1}y \in V_k^3 \subset V_{k-1}$, which implies that $g(x, y) \leqq 2^{1-k} \leqq 2\alpha$, and proves (12.9.1.3).

Hence (12.9.1.2) is established, and shows first of all that d is a *distance* on G. Also, if $r > 0$, the ball $B'(e; r)$ (with respect to d) contains V_k for all indices such that $2^{-k} < r$, and conversely each V_k contains the ball $B'(e; 2^{-k-1})$. Since d is left-invariant, it now follows from (12.8.3(i)) and (12.2.1) that d defines the topology of G.

In general it is not possible to define the topology of G by means of a distance which is *both* left- and right-invariant (Problem 1 and Section 14.3, Problem 11).

(12.9.2) *Let G be a metrizable group. A necessary and sufficient condition for a sequence (x_n) in G to be a Cauchy sequence with respect to a left-invariant distance defining the topology of G is that, for each neighborhood V of e, there exists an integer n_0 such that $x_n^{-1}x_m \in V$ for all $m \geqq n_0$ and $n \geqq n_0$.*

For if d is a left-invariant distance, we have

$$d(x_n, x_m) = d(e, x_n^{-1}x_m),$$

and the proposition follows from the fact that d defines the topology of G.

Hence the property of being a Cauchy sequence with respect to some left-invariant distance is *independent* of the choice of distance, and depends only on the topology of G. We shall say simply that such a sequence is a *left Cauchy sequence* in G. A *right Cauchy sequence* in G is defined analogously by replacing $x_n^{-1}x_m$ in (12.9.2) by $x_n x_m^{-1}$. It can happen that a left Cauchy sequence is not a right Cauchy sequence (Problem 8); but if (x_n) is a left Cauchy sequence, then (x_n^{-1}) is a right Cauchy sequence. Since the mapping $x \mapsto x^{-1}$ is continuous it follows that, if every left Cauchy sequence in G converges, then so does every right Cauchy sequence. In this case G is said to be a *complete* metrizable group: it is complete with respect to every left-invariant distance and every right-invariant distance defining the topology of G.

(12.9.3) *Let* G, G′ *be two metrizable groups. Then every continuous homomorphism* f *of* G *into* G′ *is uniformly continuous with respect to left-* (resp. *right-*) *invariant distances on* G *and* G′.

Let d, d' be two such distances on G, G′, respectively. By hypothesis, for each $\varepsilon > 0$ there exists $\delta > 0$ such that the relation $d(e, z) \leqq \delta$ implies $d'(e', f(z)) \leqq \varepsilon$. Hence if $d(x, y) = d(e, x^{-1}y) \leqq \delta$, then $d'(f(x), f(y)) \leqq \varepsilon$, because

$$d'(f(x), f(y)) = d'(e', (f(x))^{-1}f(y)) = d'(e', f(x^{-1}y))$$

since f is a homomorphism.

(12.9.4) *Let* G_1, G_2 *be two metrizable groups such that* G_2 *is complete, and let* H_1 (resp. H_2) *be a dense subgroup of* G_1 (resp. G_2). *Then every continuous homomorphism* $u : \mathrm{H}_1 \to \mathrm{H}_2$ *can be extended uniquely to a continuous homomorphism* $\bar{u} : \mathrm{G}_1 \to \mathrm{G}_2$. *If* G_1 *is also complete and if* u *is an isomorphism* (of *topological* groups) *of* H_1 *onto* H_2, *then* $\bar{u}$ *is an isomorphism* (of *topological* groups) *of* G_1 *onto* G_2.

There exist left-invariant distances on G_1 and G_2, and the existence of the continuous extension $\bar{u}$ then follows from (12.9.3) and (3.15.6). The fact that $\bar{u}$ is a *group homomorphism* follows from the principle of extension of identities applied to the two functions $(x, y) \mapsto \bar{u}(xy)$ and $(x, y) \mapsto \bar{u}(x)\bar{u}(y)$ on $\mathrm{G}_1 \times \mathrm{G}_1$ (having regard to (3.20.3)). Finally, if G_1 is complete and if $v : \mathrm{H}_2 \to \mathrm{H}_1$ is the inverse of the isomorphism u, then v can be extended to a continuous homomorphism $\bar{v}$ of G_2 into G_1. Since $\bar{v} \circ \bar{u}$ and $\bar{u} \circ \bar{v}$ agree with the identity mappings on H_1 and H_2 respectively, they are the identity mappings on G_1 and G_2 respectively, by virtue of the principle of extension of identities and (3.11.5). This completes the proof.

It should be carefully noted that, if we suppose only that u is injective (resp. surjective), then $\bar{u}$ is not necessarily injective (resp. surjective) (Problem 9).

(12.9.5) *Let* G *be a metrizable group in which there exists a neighborhood* V *of* e *which is complete* (with respect to some left- or right-invariant distance). *Then* G *is complete. In particular, a locally compact metrizable group is complete.*

Let d be a left-invariant distance on G and let (x_n) be a left Cauchy sequence in G. Let $\varepsilon > 0$ be such that the closed ball $B'(e; \varepsilon)$ is contained in V. By hypothesis there exists an integer n_0 such that $d(x_n, x_m) \leqq \varepsilon$ for all $m \geqq n_0$ and $n \geqq n_0$, hence the sequence $(x_n)_{n \geqq n_0}$ is contained in the closed ball $B'(x_{n_0}; \varepsilon)$. But this closed ball is a complete subspace, because it is obtained by left translation from $B'(e; \varepsilon)$ which is closed in V (3.14.5). Hence the sequence $(x_n)_{n \geqq 1}$ converges in G. The last assertion follows from (3.16.1).

(12.9.6) *In a Hausdorff topological group* G, *every locally compact metrizable subgroup* H *is closed.*

Let $x \in \bar{H}$, let V be a neighborhood of e in G such that $V \cap H$ is compact, and let W be a symmetric neighborhood of e in G such that $W^2 \subset V$. Then $xW \cap H$ is nonempty and relatively compact in H, because if $y_0 \in xW \cap H$ then for each $y \in xW \cap H$ we have $y_0^{-1}y \in W^2 \cap H \subset V \cap H$ and therefore $y \in y_0(V \cap H)$, which is a compact set. It follows (12.3.6) that the closure of $xW \cap H$ in G is contained in H, and therefore $x \in H$.

(12.9.7) Let G be a Hausdorff commutative topological group, written additively. All the material on *series* in Section 5.2 which involves only the topology of G remains valid without any change. The same is true of Cauchy's criterion (5.2.1) if G is *metrizable*, by replacing the norm $\|x\|$ by $d(0, x)$, where d is an invariant distance on G.

PROBLEMS

1. Let $G = \mathbf{GL}(2, \mathbf{R})$ be the multiplicative group of all real 2×2 nonsingular square matrices. For each integer $n > 0$, let V_n be the set of matrices $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G$ such that $|x - 1| \leqq 1/n$, $|y| \leqq 1/n$, $|z| \leqq 1/n$, and $|t - 1| \leqq 1/n$. Show that the family of sets V_n is a fundamental system of neighborhoods of the neutral element I of G for a topology $\mathscr{T}$ compatible with the group structure of G (cf. Section 12.8, Problem 1). The group G is locally compact in the topology $\mathscr{T}$, but $\mathscr{T}$ cannot be defined by a

2. Let S be a compact subset of a metrizable group G, such that $xy \in S$ whenever $x \in S$ and $y \in S$. Show that for each $x \in S$ we have $xS = S$. (Consider a cluster value y of the sequence $(x^n)_{n \geqslant 1}$ in S, and show that yS is the intersection of the sets x^nS; deduce that $yxS = yS$.) Deduce that S is a subgroup of G.

3 Let G be a locally compact and totally disconnected metrizable group.

(a) Show that every neighborhood of e in G contains an open compact subgroup of G. (Every neighborhood V of e contains a neighborhood U of e which is both open and closed (Section 3.19, Problem 9). If $B = \complement U$, show that there exists a symmetric open neighborhood W of e in G such that $W \subset U$ and $UW \cap BW = \varnothing$, and deduce that the subgroup generated by W is contained in U.)

(b) Suppose that the topology of G can be defined by a left- and right-invariant distance. Show that every neighborhood of e in G contains a compact open *normal* subgroup of G (remark that every neighborhood V of e contains a symmetric open neighborhood W such that $xWx^{-1} \subset V$ for all $x \in G$).

4. Let p be a prime number. Consider the family of finite groups $\mathbf{Z}/p^n\mathbf{Z}$ ($n \geqq 1$), each with the discrete topology, and their product G (Section 12.8, Problem 8) which is compact and totally disconnected. For each n, let $\varphi_n : \mathbf{Z}/p^n\mathbf{Z} \to \mathbf{Z}/p^{n-1}\mathbf{Z}$ be the canonical homomorphism.

(a) Show that the set of all $z = (z_n) \in G$ such that $\varphi_n(z_n) = z_{n-1}$ for all n is a closed (hence compact) subgroup $\mathbf{Z}_p$ of G. If ψ_n is the restriction to $\mathbf{Z}_p$ of the projection $\mathrm{pr}_n : G \to \mathbf{Z}/p^n\mathbf{Z}$, then ψ_n is a surjective homomorphism.

(b) For each $z = (z_n) \in \mathbf{Z}_p$, put $|z|_p = 0$ if $z = 0$, and $|z|_p = p^{1-m}$ if m is the smallest integer such that $z_m \neq 0$. Show that $|z - z'|_p$ is a translation-invariant distance which defines the topology of $\mathbf{Z}_p$.

(c) For each $n \geqq 1$, let $f_n : \mathbf{Z} \to \mathbf{Z}/p^n\mathbf{Z}$ be the canonical homomorphism. Show that the homomorphism $f : z \mapsto (f_n(z))_{n \geqslant 1}$ of $\mathbf{Z}$ into $\mathbf{Z}_p$ is injective and that its image $f(\mathbf{Z})$ is a dense subgroup of $\mathbf{Z}_p$. If we put $d(z, z') = |f(z) - f(z')|_p$, show that d is the p-adic distance on $\mathbf{Z}$ (3.2.6). The elements of $\mathbf{Z}_p$ are called p-adic *integers*.

5. Let p be a prime number, and let G be the compact group which is the product of an infinite sequence $(G_n)_{n \geqslant 0}$ of groups G_n all equal to $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. For each n, let φ_n be the homomorphism $x \mapsto px$ of G_n into G_{n-1}. The compact subgroup of G consisting of all $z = (z_n)$ such that $\varphi_n(z_n) = z_{n-1}$ for all n is called the *p-adic solenoid* and is denoted by $\mathbf{T}_p$.

(a) For each n, let $f_n : \mathbf{T}_p \to G_n = \mathbf{T}$ be the restriction of pr_n to $\mathbf{T}_p$. Show that f_n is a surjective homomorphism with kernel isomorphic to the group $\mathbf{Z}_p$ (Problem 4).

(b) Let $\varphi : \mathbf{R} \to \mathbf{T}$ be the canonical homomorphism. For each $x \in \mathbf{R}$, show that the point $\theta(x) = (\varphi(x/p^n))_{n \geqslant 0}$ belongs to $\mathbf{T}_p$. Prove that θ is an injective continuous homomorphism of $\mathbf{R}$ into $\mathbf{T}_p$, and that $\theta(\mathbf{R})$ is a dense subgroup of $\mathbf{T}_p$. Deduce that $\mathbf{T}_p$ is connected.

(c) Let I be an open interval in $\mathbf{R}$, with centre 0 and length <1. Show that the subspace $f_0^{-1}(\varphi(\mathrm{I}))$ of $\mathbf{T}_p$ is homeomorphic to the product $\mathrm{I} \times \mathbf{Z}_p$. Deduce that $\mathbf{T}_p$ is not locally connected.

(d) Show that every closed subgroup H of $\mathbf{T}_p$, other than $\mathbf{T}_p$ and $\{0\}$, is totally disconnected and isomorphic to a group of the form $\mathbf{Z}/n\mathbf{Z}$ or $(\mathbf{Z}/n\mathbf{Z}) \times \mathbf{Z}_p$, where n is an integer prime to p. (Consider the groups $f_n(\mathrm{H})$.)
(e) Show that $\mathbf{T}_p$ is an *indecomposable* compact connected space, i.e., that it cannot be covered by two compact connected sets P, Q, each distinct from $\mathbf{T}_p$. (Observe that there exists an integer n such that $f_n(\mathrm{P}) \neq \mathrm{G}_n$ and $f_n(\mathrm{Q}) \neq \mathrm{G}_n$, and examine the sets $f_{n+1}(\mathrm{P})$ and $f_{n+1}(\mathrm{Q})$ to obtain a contradiction.)

6. A topological group G is said to have *no small subgroups* if there exists a neighborhood V of e such that $\{e\}$ is the only subgroup of G contained in V.

Let G be a locally compact metrizable group with no small subgroups.
(a) Show that there exists a compact neighborhood V of e such that, for all x, y in V, the relation $x^2 = y^2$ implies $x = y$. (We may assume that G is not commutative. If the result were false, there would exist two sequences (x_n), (y_n) of points of G, both tending to e, such that $x_n^2 = y_n^2$ and $x_n y_n^{-1} = a_n \neq e$. Let U be a compact symmetric neighborhood of e not containing any subgroup of G other than $\{e\}$, and let p_n be the smallest integer $p > 0$ such that $a_n^{p+1} \notin \mathrm{U}$. Show, by passing to a subsequence, that we may assume that $a = \lim_{n\to\infty} a_n^{p_n}$ exists, is $\neq e$ and belongs to U. Show that $a^{-1} = a$ and so obtain a contradiction.)
(b) Let U be a compact symmetric neighborhood of e, containing no subgroup other than $\{e\}$, and let V be a neighborhood of e. Show that there exists a number $c(\mathrm{V}) > 0$ such that, whenever p and q are strictly positive integers such that $p \leqq c(\mathrm{V})q$ and $x \in \mathrm{G}$ is such that $x, x^2, \ldots, x^q$ are in U, then $x^p \in \mathrm{V}$. (Argue by contradiction: suppose that there exist two sequences of integers (p_n), (q_n) such that $\lim_{n\to\infty} p_n/q_n = 0$, and for each n an element $a_n \in \mathrm{G}$ such that $a_n^h \in \mathrm{U}$ for $1 \leqq h \leqq q_n$, but $a_n^{p_n} \notin \mathrm{V}$. We may also assume that the sequence $(a_n^{p_n})$ has a limit $a \neq e$ such that $a \in \mathrm{U}$. Show that $a^m \in \mathrm{U}$ for all $m > 0$, and hence get a contradiction.)

7. Let G be a locally compact metrizable group with no small subgroups, and let V be a compact symmetric neighborhood of e containing no subgroup of G other than $\{e\}$, and such that for all $x, y \in \mathrm{V}$ the relation $x^2 = y^2$ implies $x = y$ (Problem 6(a)).
(a) Show that if (a_n) is any sequence of points of V with e as limit, there exists a subsequence (b_n) of (a_n) and a sequence (k_n) of integers >0 such that the sequence $(b_n^{k_n})$ converges to a point $\neq e$. (Consider the smallest of the integers k such that $a_n^{k+1} \notin \mathrm{V}$.)
(b) Show that if r, s are two real numbers such that the sequences $(b_n^{[rk_n]})$, $(b_n^{[sk_n]})$ converge to x, y, respectively, in G, then the sequence $(b_n^{[(r+s)k_n]})$ converges to xy. (Here $[t]$ denotes the integral part of the real number t.)
(c) Using (a), (b), and the uniqueness of the square root, show that for every *dyadic* number $r \in [0, 1]$ the sequence $(b_n^{[rk_n]})$ converges in G.
(d) Let $\mathrm{W} \subset \mathrm{V}$ be a neighborhood of e in G. Show that, for every real number $r \in [0, 1]$, there exists a dyadic number s such that $b_n^{[(r+s)k_n]} \in \mathrm{W}$ for all n (use Problem 6(b)). Deduce that, for each $r \in [0, 1]$, the sequence $(b_n^{[rk_n]})$ has a limit $X(r)$.
(e) If $-1 \leqq r \leqq 0$, put $X(r) = (X(-r))^{-1}$. Show that, if r, s and $r + s$ are all in the interval $[-1, 1]$, we have $X(r)X(s) = X(r + s)$ and that the mapping $r \mapsto X(r)$ of $[-1, 1]$ into G is continuous. Deduce that this mapping can be extended to a non-constant continuous homomorphism of $\mathbf{R}$ into G.

8. Let I be the compact interval $[0, 1]$ in $\mathbf{R}$, and let G be the group of all homeomorphisms of I onto I, which is contained in the Banach space $\mathscr{C}_{\mathbf{R}}(\mathrm{I})$ (7.2.1). Show that the

(metrizable) topology induced on G by that of $\mathscr{C}_{\mathbf{R}}(\mathrm{I})$ is compatible with the group structure of G, and that right Cauchy sequences in G are the same as Cauchy sequences (in G) with respect to the norm on $\mathscr{C}_{\mathbf{R}}(\mathrm{I})$ (7.1). Give an example of a right Cauchy sequence in G which is not a left Cauchy sequence.

9. (a) Let G' be a complete metrizable group and let $G_0 \neq G'$ be a dense subgroup of G'. Let G be the topological group obtained by giving G_0 the discrete topology. The identity mapping $G \to G_0$ is a continuous bijective homomorphism, but its extension by continuity to a homomorphism $G \to G'$ (12.9.4) is not surjective.

(b) Let G be the dense subgroup $\mathbf{Q}^2$ of $\mathbf{R}^2$, let θ be an irrational number, and let u be the continuous homomorphism $(x, y) \mapsto x + \theta y$ of G into $\mathbf{R}$. Let $G' = u(G)$. The mapping u, considered as a homomorphism of G into G', is bijective, but its continuous extension to a homomorphism of $\mathbf{R}^2$ into $\mathbf{R}$ is not injective.

(c) Use (a) and (b) to construct an example of two complete groups G_1, G_2 and a continuous bijective homomorphism u of a dense subgroup H_1 of G_1 onto a dense subgroup H_2 of G_2, such that the continuous extension of u to a homomorphism of G_1 into G_2 is neither injective nor surjective.

10. Let f be a continuous homomorphism of a subgroup $H \neq \{0\}$ of $\mathbf{R}$ into a locally compact metrizable group G. Show that if f is not an isomorphism of H onto the subgroup $f(H)$ of G, then $f(H)$ is relatively compact in G. (Reduce to the case where H is closed in $\mathbf{R}$ and $f(H)$ is dense in G. Begin by showing that, for each neighborhood W of the neutral element e in G, the set $f^{-1}(W)$ is unbounded, and deduce that $f(H \cap \mathbf{R}_+)$ is dense in G. If V is a compact symmetric neighborhood of e in G, show that there exist a finite number of elements $t_i > 0$ in H such that the neighborhoods $f(t_i)V$ cover V. For each $x \in G$, let A_x be the set of all $t \in H$ such that $f(t) \in xV$. Show that, if $t \in A_x$, there exists a t_i such that $t - t_i \in A_x$, and deduce that, if I is the largest compact interval with origin 0 in $\mathbf{R}$ containing all the t_i, then $I \cap A_x$ is not empty. Deduce that $G \subset f(I) \cdot V$ is compact.)

11. Let G be a commutative metrizable topological group, and let d be an invariant distance defining the topology of G. Let h be the Hausdorff distance on $\mathfrak{F}(G)$ corresponding to d (Section 3.16, Problem 3). Show that if M, N, P, Q are four bounded closed subsets of G, then $h(MP, NQ) \leqq h(M, N) + h(P, Q)$.

10. SPACES WITH OPERATORS. ORBIT SPACES

Let G be a group and E a set. An *action* (or *left action*) of G on E is a mapping $(s, x) \mapsto s \cdot x$ of $G \times E$ into E satisfying the following conditions:

(1) If e is the neutral element of G, then $e \cdot x = x$ for all $x \in E$.
(2) For all s, t in G we have $s \cdot (t \cdot x) = (st) \cdot x$ for all $x \in E$.

These conditions imply that $s^{-1} \cdot (s \cdot x) = x$ for all $s \in G$ and all $x \in E$; hence for each $s \in G$ the mapping $x \mapsto s \cdot x$ is a *bijection* of E onto E, and the inverse bijection is $x \mapsto s^{-1} \cdot x$.

For each $x \in E$, the set $G \cdot x$ of elements $s \cdot x$ where $s \in G$ is called the *orbit* of x (for the given action of G on E). The set S_x of elements $s \in G$ such

that $s \cdot x = x$ is a subgroup of G called the *stabilizer* of x. The relation $s \cdot x = t \cdot x$ is equivalent to $t^{-1}s \in S_x$. The mapping $s \mapsto s \cdot x$ of G onto $G \cdot x$ factorizes as follows:

$$G \xrightarrow{p} G/S_x \xrightarrow{\varphi} G \cdot x \tag{12.10.1}$$

where G/S_x is the set of *left cosets* sS_x of S_x in G; p is the canonical mapping $s \mapsto sS_x$; and φ is the bijection $sS_x \mapsto s \cdot x$. The group G is said to act *faithfully* on E if the intersection of the stabilizers S_x, as x runs through E, consists only of e.

The group G is said to act *freely* on E if the stabilizer of *every* $x \in E$ consists only of e, or equivalently if for every $x \in E$ the relation $s \cdot x = t \cdot x$ implies $s = t$.

The relation "y belongs to the orbit of x" is an *equivalence relation* on E, for which the equivalence classes are the orbits of the elements of E. The set of orbits is denoted by E/G (it is a subset of $\mathfrak{P}(E)$). If this set consists of a single element (in other words if, given any two elements x, y of E, there exists $s \in G$ such that $y = s \cdot x$), then G is said to act *transitively* on E. The union $G \cdot A$ of the orbits of the elements of a subset A of E is called the *saturation* of A with respect to G; the restriction to $G \times (G \cdot A)$ of the mapping $(s, x) \mapsto s \cdot x$ is an action of G on $G \cdot A$. If $\pi : E \to E/G$ is the canonical mapping (so that $\pi(x) = G \cdot x$ for all $x \in E$), then $G \cdot A$ is equal to $\pi^{-1}(\pi(A))$. The relation $G \cdot A = A$ is equivalent to $G \cdot A \subset A$.

Now suppose that G is a *topological group* and E a *topological space*. Then G is said to *act continuously* on E if the mapping $(s, x) \mapsto s \cdot x$ of the product space $G \times E$ into E is continuous.

Examples

(12.10.2) Let H be a subgroup of a topological group G. Then each of the actions $(s, x) \mapsto sx$ and $(s, x) \mapsto sxs^{-1}$ of H on G is continuous. For the former action, the stabilizer of each $x \in G$ is the identity subgroup $\{e\}$, and the orbit of x is the right coset Hx; for the latter action, the stabilizer of x is the intersection $H \cap \mathscr{Z}(x)$, where $\mathscr{Z}(x)$ is the centralizer of x in G (12.8.6), and the orbit of x is the set of its conjugates hxh^{-1} by elements $h \in H$.

If H, K are subgroups of G, the product group $H \times K$ acts continuously on G by $((s, t), x) \mapsto sxt^{-1}$. The orbit of x is the *double coset* HxK of x with respect to H and K.

If E is a real (resp. complex) topological vector space (12.13), the group $\mathbf{R}^*$ (resp. $\mathbf{C}^*$) acts continuously on E by $(\lambda, x) \mapsto \lambda x$. The orbits are $\{0\}$ and the sets $D - \{0\}$, where D is a line (i.e. a one-dimensional subspace) in E.

If E is a Banach space, the group $\mathbf{GL}(E)$ of linear homeomorphisms of E onto E (12.8.1) acts continuously on E by $(u, x) \mapsto u(x)$ (5.7.4).

Let G be a topological group and E a topological space. The mapping $(s, x) \mapsto x$ of $G \times E$ into E is a continuous action of G on E, called the *trivial* action.

Let G be a topological group acting continuously on a topological space E. If ρ is a continuous homomorphism of a topological group G' into G, then G' acts continuously on E by $(s', x) \mapsto \rho(s') \cdot x$.

(12.10.3) *If* G *is a topological group acting continuously on a topological space* E, *then for each* $s \in G$ *the mapping* $x \mapsto s \cdot x$ *is a homeomorphism of* E *onto* E.

For it is a continuous bijection, and the inverse bijection $x \mapsto s^{-1} \cdot x$ is also continuous.

(12.10.4) *If* G *is a topological group acting continuously on a Hausdorff topological space* E, *the stabilizer of each point of* E *is a closed subgroup of* G.

This follows immediately from (12.3.5).

(12.10.5) *Let* G *be a metrizable group acting continuously on a metrizable space* E. *Let* A *be a compact subset of* G, *and* B *a closed* (resp. *compact*) *subset of* E. *Then* $A \cdot B$ *is closed* (resp. *compact*) *in* E.

The second assertion follows from the fact that $A \cdot B$ is the image of the compact set $A \times B$ (3.20.16(v)) by the continuous mapping $(s, x) \mapsto s \cdot x$ (3.17.9). As to the first assertion, consider a sequence $(s_n \cdot x_n)$ of points of $A \cdot B$ (where $s_n \in A$ and $x_n \in B$) with a limit $z \in E$. By hypothesis, the sequence (s_n) has a subsequence (s_{n_k}) converging to some point $a \in A$. Since $x_{n_k} = s_{n_k}^{-1} \cdot (s_{n_k} \cdot x_{n_k})$, the sequence (x_{n_k}) converges to $a^{-1} \cdot z$. But B is closed in E and therefore $a^{-1} \cdot z \in B$, hence $z = a \cdot (a^{-1} \cdot z) \in A \cdot B$. By (3.13.13), the proof is complete.

Let G be a topological group acting continuously on a topological space E. Let E/G be the set of orbits and let $\pi : E \to E/G$ be the canonical mapping, so that $\pi(x)$ is the orbit $G \cdot x$ for each $x \in E$. Let $\mathfrak{O}$ be the set of subsets U of E/G such that $\pi^{-1}(U)$ is *open* in E. It follows immediately from the formulas (1.8.5) and (1.8.6) that $\mathfrak{O}$ is a *topology* on E/G. The set E/G, endowed with

this topology, is called the *orbit space* of the action of G on E. The mapping $U \mapsto \pi^{-1}(U)$ is a bijection of the set of open sets of E/G onto the set of *saturated* open sets of E. A subset F of E/G is *closed* in E/G if and only if $\pi^{-1}(F)$ is closed in E, because $\pi^{-1}((E/G) - F) = E - \pi^{-1}(F)$.

(12.10.6) (i) *The canonical mapping* $\pi : E \to E/G$ *is continuous.*

(ii) *The image under* π *of every open set in* E *is open in* E/G.

(iii) *A mapping* f *of* E/G *into a topological space* E′ *is continuous if and only if* $f \circ \pi : E \to E'$ *is continuous.*

The first assertion follows from the definition of the topology of E/G and from (3.11.4(b)). To prove the second, it is enough to show that if V is open in E then its saturation $G \cdot V = \pi^{-1}(\pi(V))$ is open in E; and this is clear because $G \cdot V = \bigcup_{s \in G} s \cdot V$, and each $s \cdot V$ is open in E by virtue of (12.10.3). Finally, if $f : E/G \to E'$ is continuous, then so is $f \circ \pi$ by (i); conversely, if $f \circ \pi$ is continuous, then for every open set U′ in E′ the set $\pi^{-1}(f^{-1}(U'))$ is open in E, hence $f^{-1}(U')$ is open in E/G. This proves (iii).

If x is any point of E and if V runs through a fundamental system of neighborhoods of x in E, the sets $\pi(V)$ form a *fundamental system of neighborhoods* of the point $\pi(x)$ in E/G.

(12.10.7) *Let* A *be a subset of* E *and let* $A' = G \cdot A = \pi^{-1}(\pi(A))$ *be its saturation with respect to* G. *Then the canonical bijection* φ *of the subspace* $\pi(A)$ *of* E/G *onto the orbit space* A′/G *is a homeomorphism.*

The mapping φ is defined as follows: if $x \in A$, the image under φ of the orbit $G \cdot x$ is the same orbit considered as an element of A′/G. As U runs through the set of open subsets of E/G, the mapping φ takes $U \cap \pi(A)$ to the canonical image of $\pi^{-1}(U) \cap A'$ in A′/G. Now $U \cap \pi(A)$ runs through the set of open subsets of the subspace $\pi(A)$ of E/G, and $\varphi(\pi^{-1}(U) \cap A')$ runs through the set of open subsets of A′/G. For if V′ is a saturated open subset of the subspace A′ of E, then V′ is of the form $V \cap A'$, where V is open in E; but also $V' = (G \cdot V) \cap A' = \pi^{-1}(\pi(V)) \cap A'$, and $\pi(V)$ is open in E/G by (12.10.6).

(12.10.8) *Let* G *be a topological group acting continuously on a topological space* E. *In order that* E/G *should be Hausdorff it is necessary and sufficient that, in the product space* $E \times E$, *the set* R *of pairs* (x, y) *belonging to the same orbit be closed. When this is the case, every orbit is closed in* E.

Let $\pi(x)$ and $\pi(y)$ be distinct points of E/G. If E/G is Hausdorff, there exist saturated open sets V, W in E such that $V \cap W = \varnothing$ and $x \in V$ and $y \in W$. It is clear that the open set $V \times W$ in $E \times E$ contains (x, y) and does not meet R. Hence R is closed in $E \times E$. Conversely, if R is closed in $E \times E$, there exists an open neighborhood S of x and an open neighborhood T of y in E such that $(S \times T) \cap R = \varnothing$. By (12.10.6), $\pi(S)$ and $\pi(T)$ are neighborhoods of $\pi(x)$ and $\pi(y)$, respectively. If they intersected, there would exist $s \in S$ and $t \in T$ belonging to the same orbit, which means that $(s, t) \in R$; and this is absurd. Hence E/G is Hausdorff. The last assertion follows from (12.3.4) and the continuity of π.

(12.10.9) *Let* E *be a metrizable space,* G *a topological group acting continuously on* E, *and* $\pi : E \to E/G$ *the canonical mapping. Suppose that* E/G *is metrizable. Then*:

(i) *if* E *is separable,* E/G *is separable.*

(ii) *if* E *is locally compact* (resp. *compact*), E/G *is locally compact* (resp. *compact*);

(iii) *if* E *is locally compact and* K *is any compact subset of* E/G, *there exists a compact subset* L *of* E *such that* $K = \pi(L)$.

If D is a denumerable dense subset of E, then $\pi(D)$ is dense in E/G (3.11.4(d)). This proves (i). If V is a compact neighborhood of $x \in E$, then $\pi(V)$ is a compact neighborhood of $\pi(x)$ in E/G ((3.17.9) and (12.10.6)); hence (ii). Finally, for each $z \in K$, let $V(z)$ be a compact neighborhood of a point of $\pi^{-1}(z)$ in E, so that $\pi(V(z))$ is a compact neighborhood of z. There exist a finite number of points $z_i \in K$ such that the $\pi(V(z_i))$ cover K. Let L_1 be the compact set $\bigcup_i V(z_i)$ in E. We have $K \subset \pi(L_1)$, hence the set $L = L_1 \cap \pi^{-1}(K)$ is compact (because it is closed in L_1 : (3.11.4) and (3.17.3)) and we have $\pi(L) = K$.

(12.10.10) *Let* E *be a locally compact, separable metrizable space and* G *a topological group acting continuously on* E. *Let* $\pi : E \to E/G$ *be the canonical mapping. Suppose that* (1) E/G *is Hausdorff*; (2) *for each* $x \in E$ *there exists a compact subset* $K(x)$ *of* E, *containing* x *and such that the restriction of* π *to* $K(x)$ *is injective and* $\pi(K(x))$ *is a neighborhood of* $\pi(x)$. *Then* E/G *is metrizable* (*and therefore, by* (12.10.9), *locally compact and separable*).

We shall first show that there exists a sequence (K_n) of compact subsets of E such that the restriction of π to K_n is injective and the interiors of the sets $\pi(K_n)$ cover E/G. By (3.18.3) there exists an increasing sequence (A_n) of compact subsets of E, whose union is E. For each $z \in \pi(A_n)$, let x be a point of

$\pi^{-1}(z)$, and let $K(x)$ be a set with the properties enunciated above. Then $V(z) = (\pi(K(x)))^0$ is an open neighborhood of z, and therefore $\pi^{-1}(V(z))$ is an open neighborhood of $\pi^{-1}(z) \cap A_n$. Since A_n is compact, there are a finite number of points $z_i \in \pi(A_n)$ such that the sets $\pi^{-1}(V(z_i))$ $(1 \leqq i \leqq p_n$, say) cover A_n. Let K_{in} be the set $K(x)$ corresponding to $V(z_i)$, so that the interiors of the sets $\pi(K_{in})$ form an open covering of $\pi(A_n)$ in E/G. Then it is clear that the K_{in} $(n \geqq 1,\ 1 \leqq i \leqq p_n$ for each $n)$ satisfy the required conditions.

This being so, the restriction of π to K_n is a homeomorphism of K_n onto the subspace $\pi(K_n)$ of E/G, because E/G is Hausdorff (12.3.6), and this subspace is therefore compact and metrizable. The result therefore follows from (12.4.7).

(12.10.11) *Let* G (resp. G′) *be a topological group acting continuously on a topological space* E (resp. E′). *Then* $G \times G'$ *acts continuously on* $E \times E'$ *by* $(s, s') \cdot (x, x') = (s \cdot x, s' \cdot x')$, *and the mapping* ω *defined by*

$$(G \times G') \cdot (x, x') \mapsto ((G \cdot x), (G' \cdot x'))$$

is a homeomorphism of $(E \times E')/(G \times G')$ *onto* $(E/G) \times (E'/G')$.

It is immediately checked that ω is bijective, and it is continuous by virtue of (12.10.5). Moreover, every open set in $(E \times E')/(G \times G')$ is the image, under the canonical mapping $p : E \times E' \to (E \times E')/(G \times G')$, of an open set U in $E \times E'$, and we have $\omega(p(U)) = \pi(\mathrm{pr}_1 U) \times \pi'(\mathrm{pr}_2 U)$, where $\pi : E \to E/G$ and $\pi' : E' \to E'/G'$ are the canonical mappings. The set $\omega(p(U))$ is therefore open in $(E/G) \times (E'/G')$, and the proof is complete.

(12.10.12) *Let* G *be a connected topological group acting continuously on a topological space* E. *If the orbit space* E/G *is connected, then* E *is connected.*

Since the mapping $s \mapsto s \cdot x$ of G onto $G \cdot x$ is continuous, it follows (3.19.7) that every orbit is connected. Suppose that there exist two non-empty open sets U, V in E such that $U \cap V = \varnothing$ and $U \cup V = E$. For each $x \in E$, the sets $U \cap (G \cdot x)$ and $V \cap (G \cdot x)$ are open in $G \cdot x$; their union is $G \cdot x$ and their intersection is empty. Hence one of them is empty; in other words, U and V are *saturated*. But this implies that $\pi(U)$ and $\pi(V)$ are nonempty open sets in E/G whose intersection is empty and whose union is E/G, and this contradicts the hypothesis that E/G is connected.

Remark

(12.10.13) A *right action* of a group G on a set E is a mapping $(s, x) \mapsto x \cdot s$ of $G \times E$ into E such that $x \cdot e = x$ for all $x \in E$, and $(x \cdot t) \cdot s = x \cdot (ts)$ for all

$x \in E$ and all, s, t in G. Everything we have said can be immediately transposed to this situation; the set of orbits is sometimes denoted by $G\backslash E$. For example, a subgroup H of a topological group G acts continuously on the *right* on G by the action $(s, x) \mapsto xs$.

PROBLEMS

1. Let G be a locally compact metrizable group and E a locally compact metrizable space on which G acts continuously. For each pair of subsets K, L in E let P(K, L) denote the set of all $s \in G$ such that $(s \cdot K) \cap L \neq \varnothing$.

(a) Show that if K is compact and L is closed in G, then the set P(K, L) is closed in G. The group G is said to act *properly* on E if P(K, L) is a compact subset of G whenever K and L are compact subsets of E. (This will always be the case if G is compact.)

(b) Show that, if G acts properly on E, then $F \cdot K$ is closed in E whenever F is closed in G and K is a compact subset of E. In particular, for each $x \in E$, the orbit $G \cdot x$ is closed in E.

(c) Under the same hypotheses, for each $x \in E$ the stabilizer S_x of x is a compact subgroup of G, and the canonical map $G/S_x \to G \cdot x$ is a homeomorphism.

(d) Under the same hypotheses, the orbit space E/G is Hausdorff.

2. For each pair (a, t) of real numbers such that $a \geqq 1$, the point $f_a(t) \in \mathbf{R}^2$ is defined as follows:

$$f_a(t) = \left(t + \frac{a(a+2)}{a+1}, -\frac{a}{a+1}\right) \quad \text{if} \quad t < -\frac{a}{a+1},$$

$$f_a(t) = (a, t) \quad \text{if} \quad -\frac{a}{a+1} \leqq t \leqq \frac{a}{a+1},$$

$$f_a(t) = \left(-t + \frac{a(a+2)}{a+1}, \frac{a}{a+1}\right) \quad \text{if} \quad t > \frac{a}{a+1}.$$

Let C_a denote the set of points $f_a(t)$ $(t \in \mathbf{R})$ and let E be the union of the sets C_a $(a \geqq 1)$ and the lines D′, D″, where D′ (resp. D″) is the set of points $(t, -1)$ (resp. $(t, 1)$) with $t \in \mathbf{R}$.

The additive group $\mathbf{R}$ acts on the locally compact space E as follows: (1) $s \cdot (t, -1) = (s + t, -1)$; (2) $s \cdot f_a(t) = f(s + t)$ for $a \geqq 1$; (3) $s \cdot (t, 1) = (t - s, 1)$. Show that $\mathbf{R}$ acts continuously and freely on E, that the orbits $\mathbf{R} \cdot z$ are closed sets and that the canonical mapping $\mathbf{R} \to \mathbf{R} \cdot z$ is a homeomorphism, but that the orbit space $E/\mathbf{R}$ is not Hausdorff.

3. The group $\mathbf{Z}$ acts continuously and properly on $\mathbf{R}^2$ as follows:

$$n \cdot (x, y) = ((-1)^n x, y + 2n).$$

The orbit space M (the *Möbius strip*) is metrizable and locally compact. If $\pi : \mathbf{R}^2 \to M$ is the canonical mapping, the restriction of π to $E - D''$ (notation of Problem 2) is injective, and $\pi(E - D'') = \pi(E)$ is locally compact. The group $\mathbf{R}$ acts on $\pi(E - D'')$ by the law $s \cdot \pi(z) = \pi(s \cdot z)$ for all $z \in E - D''$. Show that $\mathbf{R}$ acts continuously and

freely on $\pi(E)$, that for each $z \in E$ the orbit $\mathbf{R} \cdot \pi(z)$ is closed in $\pi(E)$, that the canonical mapping $\mathbf{R} \to \mathbf{R} \cdot \pi(z)$ is a homeomorphism, and that the orbit space $\pi(E)/\mathbf{R}$ is separable, metrizable and locally compact; but that $\mathbf{R}$ does not act properly on $\pi(E)$.

4. In $\mathbf{R}^3$, let E be the union of the sets $C_a \times \{z\}$ for $a \geqq 1$ and $z \geqq 0$ (notation of Problem 2) and the line D'_0 and the lines D''_z for $z > 0$, where D'_0 is the set of points $(t, -1, 0)$ with $t \in \mathbf{R}$, and D''_z is the set of points $(t, 1, z)$ with $t \in \mathbf{R}$. The additive group $\mathbf{R}$ acts continuously on E as follows: (1) $s \cdot (t, -1, 0) = (s + t, -1, 0)$; (2) $s \cdot (f_a(t), z) = (f_a(s + t), z)$; (3) $s \cdot (t, 1, z) = (t - s, 1, z)$. Show that the orbits of this action have the same properties as in Problems 2 and 3, and that the orbit space $E/\mathbf{R}$ is Hausdorff but not metrizable.

5. Let E be a locally compact metrizable space and let G be a topological group acting continuously on E. Let $\pi : E \to E/G$ be the canonical map. Suppose that E/G is Hausdorff.

(a) Let K be a compact subset of E and let U be an open neighborhood of K. Show that there exists a continuous mapping of E into $[0, 1]$ which takes the value 1 on $\pi^{-1}(\pi(K))$ and the value 0 on the complement of $\pi^{-1}(\pi(U))$.

(b) Deduce from (a) that there exists a continuous mapping of E/G into $[0, 1]$, taking the value 1 on $\pi(K)$ and the value 0 on the complement of $\pi(U)$. (Show first that there exists a relatively compact open neighborhood U_1 of K in E such that $\bar{U}_1 \subset U$. Deduce that there exist two continuous mappings f_1, f_2 of E into $[0, \frac{1}{2}]$ such that f_1 takes the value $\frac{1}{2}$ on $\pi^{-1}(\pi(K))$ and the value 0 on $E - \pi^{-1}(\pi(U_1))$, and f_2 takes the value $\frac{1}{2}$ on $\pi^{-1}(\pi(\bar{U}_1))$ and the value 0 on $E - \pi^{-1}(\pi(U))$. Consider the function $f_1 + f_2$. Iterate this "interpolation" indefinitely and pass to the limit.)

(c) If E is separable, show that there exists a sequence (U_n) of relatively compact open sets in E such that the $\pi(U_n)$ form a basis for the topology of E/G.

(d) Suppose that E is separable. Show that E/G is metrizable. (For each pair of indices m, n such that $\bar{U}_m \subset U_n$, consider a continuous mapping f_{mn} of E/G into $[0, 1]$ which takes the value 1 on $\pi(U_m)$ and the value 0 on the complement of $\pi(U_n)$. Consider the continuous mapping $x \mapsto (f_{mn}(x))$ of E/G into the product space $\mathbf{R}^{\mathbf{N} \times \mathbf{N}}$).

6. If M is a *monoid* with neutral element e, we define an *action* of M on a set E as at the beginning of (12.10).

Suppose that E is a *compact metric* space and that, for each $s \in M$, the mapping $x \mapsto s \cdot x$ is continuous. The *closed orbit* of x with respect to M is defined to be the set $\overline{M \cdot x}$; it is *stable* under the action of M. Show that for each $x \in E$ there exists in $\overline{M \cdot x}$ a *minimal* closed orbit Z (i.e., such that $\overline{M \cdot z} = Z$ for all $z \in Z$). (For each $y \in \overline{M \cdot x}$ let $\lambda(y)$ be the least upper bound, as t runs through $\overline{M \cdot y}$, of the distances from t to a closed orbit contained in $\overline{M \cdot t}$. Show that the greatest lower bound of the numbers $\lambda(y)$ with $y \in \overline{M \cdot x}$ is 0, by showing that otherwise E would not be precompact. Deduce that there exists a sequence (y_n) of points of $\overline{M \cdot x}$ such that $y_{n+1} \in \overline{M \cdot y_n}$ and the sequence $(\lambda(y_n))$ tends to 0. Now use the compactness of E to complete the proof.)

11. HOMOGENEOUS SPACES

Let G be a group, H a subgroup of G. Recall that the set of left cosets xH (resp. right cosets Hx) of H in G is denoted by G/H (resp. H\G). For each

coset $\dot{x} = xH$ (resp. $\dot{x} = Hx$) in G/H (resp. H\G) and each $s \in G$, write $s \cdot \dot{x} = (sx)H$ (resp. $\dot{x} \cdot s = H(xs)$). Then it is clear that G acts on the left (resp. on the right) on G/H (resp. H\G), and that this action is *transitive*. The set G/H (resp. H\G) together with this action of G is called the *homogeneous space of left* (resp. *right*) *cosets of* H *in* G.

We shall work throughout with G/H. Everything we shall do can be transposed in an obvious way to apply to H\G. Let π denote the canonical mapping $x \mapsto xH$ of G onto G/H.

Clearly G/H is the set of orbits of elements of G for the *right* action $(h, x) \mapsto xh$ (12.10.13) of H on G. If G is a topological group, we can therefore topologize G/H with the topology defined in (12.10), which is called the *quotient* by H of the topology of G. If $x_0 \in G$ and if $\dot{x}_0 = x_0H$ is its image in G/H, we obtain a fundamental system of neighborhoods of $\dot{x}_0$ in G/H by considering the canonical images in G/H of the neighborhoods V of x_0 in G (i.e., for each V, the set of cosets xH of the elements $x \in V$, or equivalently the image of V (or of VH) under the canonical mapping $\pi : G \to G/H$). Whenever we consider G/H as a topological space, it will always be this topology that is meant, unless the contrary is expressly stated.

(12.11.1) *The group* G *acts continuously on* G/H.

Let $(s_0, \dot{x}_0)$ be a point of $G \times (G/H)$ and let x_0 be an element of the coset $\dot{x}_0$. Every neighborhood of $s_0 \cdot \dot{x}_0$ is of the form $\pi(V)$, where V is a neighborhood of $s_0 x_0$ in G. There exists a neighborhood U of s_0 and a neighborhood W of x_0 such that the relations $s \in U$ and $x \in W$ imply $sx \in V$; hence the relations $s \in U$ and $\dot{x} \in \pi(W)$ imply $s \cdot \dot{x} \in \pi(V)$.

(12.11.2) *Let* G *be a topological group,* H *a subgroup of* G.

(i) G/H *is Hausdorff if and only if* H *is closed.*

(ii) G/H *is discrete if and only if* H *is open.*

(iii) *If* H *is discrete, then every* $x \in G$ *has a neighborhood* V *such that the restriction of* π *to* V *is a homeomorphism of* V *onto the neighborhood* $\pi(V)$ *of* $\pi(x) = \dot{x}$ *in* G/H.

(i) If G/H is Hausdorff, then $\{\pi(e)\}$ is closed in G/H (12.3.4) and hence $H = \pi^{-1}(\pi(e))$ is closed in G. Conversely, if H is closed in G, the set of pairs $(x, y) \in G \times G$ belonging to the same orbit under the action of H on the right is the set of (x, y) such that $x^{-1}y \in H$, and is therefore closed, being the inverse image of H under the continuous mapping $(x, y) \mapsto x^{-1}y$. Hence G/H is Hausdorff (12.10.8).

(ii) If G/H is discrete, the set $\{\pi(e)\}$ is open in G/H and therefore $H = \pi^{-1}(\pi(e))$ is open in G. Conversely, if H is open in G, then so is each

coset xH, and therefore $\{\pi(x)\} = \pi(xH)$ is open in G/H (12.10.6). Henc
is discrete.

(iii) Let U_0 be a neighborhood of e in G such that $U_0 \cap H = \{e$
let V_0 be a symmetric open neighborhood of e such that $V_0^2 \subset U_0$ (1
Then, for each $x \in G$, the restriction of π to $V = xV_0$ is *injective*, b
if h, h' in H are such that $xzh = xz'h'$ with z, z' in V_0, we have h'
$z'^{-1}z \in V_0^2 \subset U_0$, so that $h' = h$ and $z' = z$. Since the image under π
open set in V is open in $\pi(V)$ (12.10.6), and π is continuous, it follows th
restriction of π to V is a homeomorphism onto $\pi(V)$.

(12.11.3) *Let* G *be a metrizable group*, H *a closed subgroup of* G, *and*
a right-invariant distance defining the topology of G (12.9.1). *For a*
points $\dot{x}$, $\dot{y}$ *in* G/H, *put* $d_0(\dot{x}, \dot{y}) = d(xH, yH)$ (3.4). *Then* d_0 *is a d*
defining the topology of G/H. *If* G *is complete* (12.9.2), *then* G/H *is co*
with respect to d_0.

We remark first that if $x \in \dot{x}$ and $y \in \dot{y}$, we have $d_0(\dot{x}, \dot{y}) = d(x, yH$
$d(x, yH) = \inf_{h \in H} d(x, yh)$ and therefore, for every $h' \in H$, we have $d(xh',$
$d(x, yH)$ by virtue of the right-invariance of d. Hence, for each $z \in G$, w

$$|d_0(\dot{x}, \dot{z}) - d_0(\dot{y}, \dot{z})| = |d(x, zH) - d(y, zH)| \leqq d(x, y)$$

by (3.4.2). Since this is true for all x, y in the respective cosets $\dot{x}$, $\dot{y}$, it f
that $|d_0(\dot{x}, \dot{z}) - d_0(\dot{y}, \dot{z})| \leqq d_0(\dot{x}, \dot{y})$, which shows that d_0 is a *pseudo-d*
on G/H. The relation $d_0(\dot{x}, \dot{y}) < r$ is equivalent to the existence of a
$y \in \dot{y}$ such that $d(x, y) < r$. Since H is closed in G, this proves that d_0 is
tance, and that if $B(x; r)$ is the open ball with center x and radius
respect to d), then its image under the canonical mapping $\pi : G \to G/H$
open ball with center $\pi(x)$ and radius r with respect to d_0. Hence d_0 defi
topology of G/H.

Now suppose that G is complete. Let $(\dot{x}_n)$ be a Cauchy sequenc
respect to d_0. We shall show that, by passing to a subsequence of
necessary, we can reduce to the case where there exists a right C
sequence (x_n) *in* G such that $\pi(x_n) = \dot{x}_n$. Since by hypothesis the se
(x_n) converges, this will show that $(\dot{x}_n)$ has a cluster value and th
converges (3.14.2), and hence that G/H is complete.

There exists in G a denumerable fundamental system (V_n) of nei
hoods of e such that $V_{n+1}^2 \subset V_n$ for all n (12.9.1 and 12.8.3). Let $(\varepsilon_n$
sequence of real numbers > 0 such that the relation $d(e, z) < \varepsilon_n$ impli
$z \in V_n$. The hypothesis implies that we can define inductively a subse
$(\dot{x}_{n_k})$ of $(\dot{x}_n)$ such that $d_0(\dot{x}_{n_p}, \dot{x}_{n_q}) < \varepsilon_k$ for all $p \geqq k$ and $q \geqq k$. Replac
original sequence $(\dot{x}_n)$ by $(\dot{x}_{n_k})$, we can therefore assume that $d_0(\dot{x}_p, \dot{x}_q) <$

all $p \geqq n$ and $q \geqq n$. Since d is right-invariant, this means that for all $y \in \dot{x}_p$ the intersection of $\dot{x}_q$ and the neighborhood $V_n y$ is not empty. We can then define, by induction on n, a sequence (x_n) in G such that $x_n \in \dot{x}_n$ and $x_{n+1} \in V_n x_n$ for all n. Hence it follows, by induction on p, that for all $p > 0$ we have $x_{n+p} \in V_{n+p-1} V_{n+p-2} \cdots V_{n+1} V_n x_n \subset V_{n-1} x_n$ (because $V_{n+1}^2 \subset V_n$ for all n). Hence the sequence (x_n) is a right Cauchy sequence in G. Q.E.D.

Let G be a topological group, and let E be a topological space on which G acts continuously and *transitively*. Let $x \in E$ and let S_x be the stabilizer of x in G; then (12.10.1) the continuous mapping $h_x : s \mapsto s \cdot x$ of G into E, which by hypothesis is *surjective*, factorizes canonically as

$$h_x : G \xrightarrow{\pi_x} G/S_x \xrightarrow{f_x} E, \tag{12.11.4}$$

where π_x is the canonical mapping of G onto the homogeneous space G/S_x, and f_x is the *bijection* $sS_x \mapsto s \cdot x$.

It follows from (12.10.6) that f_x is a *continuous* bijection, but it is not necessarily a homeomorphism (Section 12.12, Problem 2).

(12.11.5) *With the notation above, the bijection $f_x : G/S_x \to E$ is a homeomorphism for each $x \in E$ if and only if there exists a point $x_0 \in E$ such that the mapping $h_{x_0} : s \mapsto s \cdot x_0$ transforms each neighborhood of e in G into a neighborhood of x_0 in E.*

In view of (12.10.6) we need only prove that the condition is sufficient.

Notice first that each $x \in E$ can by hypothesis be written in the form $x = t \cdot x_0$ for some $t \in G$. If V is a neighborhood of e, it follows from the hypothesis that $V \cdot x = (Vt) \cdot x_0$ is a neighborhood of x, because we can write $(Vt) \cdot x_0 = t \cdot ((t^{-1}Vt) \cdot x_0)$, and $t^{-1}Vt$ is a neighborhood of e in G (12.8.3), and $y \mapsto s \cdot y$ is a homeomorphism of E onto E, for each $s \in G$ (12.10.3). Since each open set in G/S_x is the image under π_x of some open set in G, it is enough to show that for each open set U in G, and each $x \in E$, the set $h_x(U) = f_x(\pi_x(U)) = U \cdot x$ is open in E. Now, for each $t \in U$, the set $t^{-1}U$ is a neighborhood of e, and therefore $(t^{-1}U) \cdot x$ is a neighborhood of x in E, from the first part of the proof. Hence $U \cdot x = t \cdot ((t^{-1}U) \cdot x)$ is a neighborhood of $t \cdot x$; by (3.6.4), the proof is complete.

12. QUOTIENT GROUPS

If H is a *normal* subgroup of a group G, then $xH = Hx$ for each $x \in G$ and therefore the sets G/H and H\G are identical; and G/H has a group structure

(the *quotient* of G by H) for which the canonical mapping $\pi : G \to G/H$ is a homomorphism.

(12.12.1) *Let* G *be a topological group and* H *a normal subgroup of* G. *Then the quotient by* H *of the topology of* G *is compatible with the group structure of* G/H.

Let $(\dot{x}_0, \dot{y}_0)$ be a point of $(G/H) \times (G/H)$, and let x_0 (resp. y_0) be an element of the coset $\dot{x}_0$ (resp. $\dot{y}_0$). Every neighborhood of $\dot{x}_0 \dot{y}_0$ is of the form $\pi(U)$, where U is a neighborhood of $x_0 y_0$. There exists a neighborhood V of x_0 and a neighborhood W of y_0 in G such that the relations $x \in V$ and $y \in W$ imply $xy \in U$. Hence the relations $\dot{x} \in \pi(V)$ and $\dot{y} \in \pi(W)$ imply $\dot{x}\dot{y} \in \pi(U)$, and therefore the mapping $(\dot{x}, \dot{y}) \mapsto \dot{x}\dot{y}$ is continuous at the point $(\dot{x}_0, \dot{y}_0)$. The proof of continuity of the map $\dot{x} \mapsto \dot{x}^{-1}$ is similar.

Whenever we speak of the *quotient* G/H of a topological group G by a subgroup H, it is to be understood that G/H carries the quotient topology.

(12.12.2) *Let* G *be a topological group and let* H, K *be two normal subgroups of* G *such that* $H \supset K$. *Then the canonical bijection* $\varphi : G/K \to (G/H)/(K/H)$ *is an isomorphism of topological groups.*

We know that φ is the unique mapping which makes the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/K \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G/H & \xrightarrow[\pi'']{} & (G/H)/(K/H) \end{array}$$

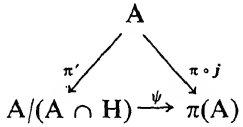
where π, π', and π'' are the canonical homomorphisms. The definition of a quotient topology (12.10) shows that a subset U of $(G/H)/(K/H)$ is open if and only if $\pi'^{-1}(\pi''^{-1}(U))$ is open in G, that is, if and only if $\pi^{-1}(\varphi^{-1}(U))$ is open in G; but this last condition signifies that $\varphi^{-1}(U)$ is open in G/K. Since φ is bijective, the result follows.

(12.12.3) *Let* G *be a topological group,* H *a normal subgroup of* G, $\pi : G \to G/H$ *the canonical mapping, and* A *a subgroup of* G. *Then the canonical bijection of* $\pi(A)$ *onto the quotient group* AH/H *is an isomorphism of topological groups.*

This is a particular case of (12.10.7), applied to the right action $(h, x) \mapsto xh$ of H on G.

There is also a canonical bijection ψ of the quotient group $A/(A \cap H)$ onto the subgroup $\pi(A)$ of G/H; this bijection ψ is the unique mapping which makes the following diagram commutative:

(12.12.4)



where π' is the canonical mapping and $j : A \to G$ the canonical injection. It follows from (12.10.6) that ψ is *continuous*, but it is not necessarily *bicontinuous* (Problem 2). However:

(12.12.5) *Let* G *be a metrizable group and* H *a compact subgroup of* G. *For each closed subgroup* A *of* G, *the canonical bijection*

$$\psi : A/(A \cap H) \to \pi(A)$$

is an isomorphism of topological groups.

It is enough to show that the image under ψ of an arbitrary closed subset F of $A/(A \cap H)$ is closed in $\pi(A)$. Now $\pi'^{-1}(F) = B$ is a closed subset of A, hence closed in G, and $\psi(F) = \pi(j(B)) = \pi(BH)$. But since H is compact, the set BH is closed in G (12.10.5) and saturated, hence $\pi(BH)$ is closed in G/H (12.10) and therefore also closed in $\pi(A)$.

(12.12.6) *Let* G_1, G_2 *be two topological groups and* H_1 (resp. H_2) *a normal subgroup of* G_1 (resp. G_2). *Then the canonical bijection*

$$\omega : (G_1 \times G_2)/(H_1 \times H_2) \to (G_1/H_1) \times (G_2/H_2)$$

is an isomorphism of topological groups.

This is an immediate consequence of (12.10.11).

(12.12.7) Let G, G′ be two topological groups and $u : G \to G'$ a continuous homomorphism. Then u factorizes canonically as

$$G \xrightarrow{p} G/N \xrightarrow{v} u(G) \xrightarrow{j} G'$$

where N is the kernel of u, p is the canonical homomorphism, and j the canonical injection. The mapping v is a *continuous bijection* (12.10.6(iii)), but is not necessarily bicontinuous. When v is *bicontinuous*, the homomorphism u is said to be a *strict morphism* of G into G′; for this to be the case, it is necessary and sufficient that for each neighborhood V of e in G, the set $u(V)$ should be a neighborhood of $e' = u(e)$ *in* $u(G)$ (12.11.5).

PROBLEMS

1. (a) Let G be a topological group, K its identity component, H a subgroup of G contained in K. Show that the connected components of the space G/H are the images of the connected components of G under the canonical mapping $\pi : G \to G/H$. Show that K is the smallest of the subgroups L of G such that G/L is totally disconnected.
(b) Let G be the additive subgroup of the Banach space $\mathscr{B}_{\mathbf{R}}(\mathbf{N})$ (7.1.3) consisting of the mappings $n \mapsto f(n)$ such that $f(n) \in \mathbf{Q}$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} f(n)$ exists in $\mathbf{R}$. The distance induced on G by that on $\mathscr{B}_{\mathbf{R}}(\mathbf{N})$ defines a topology compatible with the group structure of G, and with respect to this topology G is totally disconnected. Let H be the subgroup of G consisting of all $f \in G$ such that $\lim_{n\to\infty} f(n) = 0$. Show that H is closed in G and that G/H is isomorphic to $\mathbf{R}$, and therefore connected.
(c) Let G be a locally compact metrizable group, H a closed subgroup of G, and $\pi : G \to G/H$ the canonical mapping. Show that the connected components of G/H are the closures of the images under π of the connected components of G. (Reduce to the case where G is totally disconnected, and use Problem 3 of Section 12.9.)

2. In the additive group $\mathbf{R}$, let H be the subgroup $\mathbf{Z}$ and let A be the subgroup $\theta\mathbf{Z}$, where θ is an irrational number. Show that the canonical bijection $A/(A \cap H) \to (A + H)/H$ is not an isomorphism of topological groups.

3. Let p be a prime number and let (G_n) be an infinite sequence of topological groups all equal to the discrete group $\mathbf{Z}/p^2\mathbf{Z}$. Let H_n be the subgroup $p\mathbf{Z}/p^2\mathbf{Z}$ of G_n. Let G be the subgroup of the product group $\prod_n G_n$ consisting of all $x = (x_n)$ such that $x_n \in H_n$ for all but a finite set of values of n. Let $\mathfrak{B}$ be the set of neighborhoods of 0 in the product group $\prod_n H_n = H \subset G$. Show that $\mathfrak{B}$ is a fundamental system of neighborhoods of 0 for a topology on G compatible with the group structure, and that in this topology G is metrizable and locally compact and G/H is discrete. Show that the homomorphism $u : x \mapsto px$ of G into G is not a strict morphism of G into G, and that $u(G)$ is not closed in G.

4. Let G be a metrizable group, K a closed normal subgroup of G. If K and G/K are complete, show that G is complete.

5. (a) Let G be a topological group, K a normal subgroup of G. Show that if K and G/K have no small subgroups (Section 12.9, Problem 6), then G has no small subgroups.
(b) Deduce from (a) that if H_1, H_2 are two normal subgroups of a topological group G, such that G/H_1 and G/H_2 have no small subgroups, then $G/(H_1 \cap H_2)$ has no small subgroups.

6. Let G be a connected, locally compact, metrizable group, K a compact normal subgroup of G, and N a closed normal subgroup of K, such that K/N has no small subgroups (Section 12.9, Problem 6). Show that N is a normal subgroup of G. (Observe that the hypothesis on K/N implies the existence of a neighborhood U of e in G such that $xNx^{-1} \subset N$ for all $x \in U$.)

7. Let G be a connected metrizable group and H a compact normal commutative subgroup of G, with no small subgroups. Show that H is contained in the center of G.

(Observe that, if s is close to e in G, then the image of H under the mapping $x \mapsto sxs^{-1}x^{-1}$ is close to e.)

8. Let G be a locally compact metrizable group and H a closed normal subgroup of G, such that G/H is not discrete and has no small subgroups. Show that there exists a continuous homomorphism $f: \mathbf{R} \to G$ such that the composition $\mathbf{R} \xrightarrow{f} G \to G/H$ is nontrivial (see Section 12.9, Problem 7).

13. TOPOLOGICAL VECTOR SPACES

We shall adhere to the conventions of (5.1), so that all vector spaces under consideration have as field of scalars either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.

If E is a vector space over $\mathbf{R}$ (resp. $\mathbf{C}$), a topology on E is said to be *compatible* with the vector space structure if the mappings $(x, y) \mapsto x + y$ of $E \times E$ into E, and $(\lambda, x) \mapsto \lambda x$ of $\mathbf{R} \times E$ (resp. $\mathbf{C} \times E$) into E are *continuous*. A vector space endowed with a topology compatible with its vector space structure is called a *topological vector space*. Since $-x = (-1)x$, the conditions above imply that the topology is *a fortiori* compatible with the *additive group* structure of E, and all the notions and properties established for topological groups in the preceding sections will apply in particular to topological vector spaces (provided of course that these properties are transcribed into *additive* notation).

An *isomorphism* of a topological vector space E onto a topological vector space F is a *linear bijection* of E onto F which is a *homeomorphism*.

If $\|x\|$ is a norm on a vector space E, the topology on E defined by the distance $\|x - y\|$ is compatible with the vector space structure of E (5.1.5) and therefore makes E a topological vector space. Two *equivalent* norms (5.6) define the *same* topology.

In a topological vector space E, the translations $x \mapsto x + a$ and the homotheties $x \mapsto \lambda x$ ($\lambda \neq 0$) are *homeomorphisms* of E onto E, the inverse mappings being, respectively, the translation $x \mapsto x - a$ and the homothety $x \mapsto \lambda^{-1}x$.

In a vector space E over $\mathbf{R}$ (resp. $\mathbf{C}$) a set M is said to be *balanced* if, for each $x \in M$ and each scalar λ such that $|\lambda| \leqq 1$, we have $\lambda x \in M$ (in other words, if $\lambda M \subset M$ whenever $|\lambda| \leqq 1$). The set M is said to be *absorbing* if, for each $x \in E$, there exists a real number $\alpha > 0$ such that $\lambda x \in M$ whenever $|\lambda| \leqq \alpha$. Every absorbing subset of E *generates* the vector space E. For a balanced subset M of E to be absorbing, it is enough that for each $x \in E$ there should exist a scalar $\lambda \neq 0$ such that $\lambda x \in M$.

(12.13.1) *In a topological vector space* E, *the absorbing balanced neighborhoods of* 0 *form a fundamental system of neighborhoods of* 0.

For each $x_0 \in E$, the mapping $\lambda \mapsto \lambda x_0$ of the scalar field into E is continuous at the point 0; hence, for each neighborhood V of 0 in E, there exists $\alpha > 0$ such that the relation $|\lambda| \leqq \alpha$ implies $\lambda x_0 \in V$. This shows that every neighborhood of 0 is absorbing. On the other hand, the continuity of the mapping $(\lambda, x) \mapsto \lambda x$ at the point (0, 0) implies that, for each neighborhood V of 0 in E, there exists $\alpha > 0$ and a neighborhood W of 0 in E such that the relations $|\lambda| \leqq \alpha$ and $x \in W$ imply $\lambda x \in V$. The union U of the sets λW, where $|\lambda| \leqq \alpha$, is clearly a balanced neighborhood of 0 in E, and is contained in V. Hence the result.

Let F be a vector subspace of a topological vector space E. Then the topology induced on F by the topology on E is compatible with the vector space structure of F. Whenever we consider a vector subspace F as a topological vector space, it is always the induced topology that is meant, unless the contrary is expressly stated. The proof of (5.4.1) applies without any change and shows that $\bar{F}$ is a vector subspace of E. The definition of a *total* subset of E is the same as in (5.4).

On the *quotient vector space* E/F, the quotient topology (12.11) is *compatible* with the vector space structure. For if $\pi : E \to E/F$ is the canonical mapping, let $(\lambda_0, \dot{x}_0)$ be any point of $\mathbf{R} \times (E/F)$ (resp. $\mathbf{C} \times (E/F)$) and let x_0 be any point of $\dot{x}_0$. Then every neighborhood of $\lambda_0 \dot{x}_0$ contains a neighborhood of the form $\pi(V)$, where V is a neighborhood of $\lambda_0 x_0$. There exists a neighborhood U of λ_0 in $\mathbf{R}$ (resp. $\mathbf{C}$) and a neighborhood W of x_0 in E such that the relations $\lambda \in U$ and $x \in W$ imply $\lambda x \in V$; consequently the relations $\lambda \in U$ and $\dot{x} \in \pi(W)$ imply $\lambda \dot{x} \in \pi(V)$, and the assertion is proved.

Whenever we consider E/F as a topological vector space, it is always with the quotient topology, unless the contrary is expressly stated.

The criteria for continuity in a product ((3.20.15) and (12.5)) show immediately that if E_1, E_2 are two topological vector spaces, the *product* topology on $E_1 \times E_2$ is *compatible* with the vector space structure of $E_1 \times E_2$. The vector space $E_1 \times E_2$, endowed with this topology, is called the *product of the topological vector spaces* E_1 and E_2. Proposition (5.4.2) and the definition of the topological direct sum of two subspaces are valid without modification for arbitrary topological vector spaces.

If E is the direct sum of two subspaces F_1, F_2, and if $p_1 : E \to F_1$ is the corresponding projection, then the kernel of p_1 is F_2, and p_1 therefore has a

canonical factorization $E \to E/F_2 \xrightarrow{j_1} F_1$, where j_1 is a bijection. To say that E is the topological direct sum of F_1 and F_2 signifies that j_1 is *continuous* (12.10.6), or equivalently that the inverse bijection, which is the restriction to F_1 of the canonical mapping $\pi_2 : E \to E/F_2$ (and which is always continuous) is *bicontinuous.*

(12.13.2) (i) *Let* E *be a topological vector space and let* H *be a hyperplane in* E, *with equation* $f(x) = 0$. *Then* H *is closed in* E *if and only if the linear form* f *is continuous on* E.

(ii) *Let* E *be a Hausdorff topological vector space over* **R** (resp. **C**) *of* finite *dimension* n. *If* $(a_i)_{1 \leqq i \leqq n}$ *is a basis of* E, *the mapping*

$$u : (\xi_1, \ldots, \xi_n) \mapsto \sum_{i=1}^{n} \xi_i a_i$$

of $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) *onto* E *is an isomorphism of topological vector spaces.*

(iii) *Let* E *be a topological vector space,* M *a closed vector subspace of* E, *and* F *a* finite-dimensional *vector subspace of* E. *Then* M + F *is closed in* E. *In particular, if* E *is Hausdorff, every finite-dimensional vector subspace of* E *is closed in* E.

(iv) *Let* E *be a topological vector space and let* M *be a closed vector subspace of* finite codimension *in* E. *Then every algebraic supplement* N *of* M *in* E *is a topological supplement.*

We shall prove (ii) first. Since u is continuous and bijective, we have to show that u is bicontinuous, or equivalently (considering the inverse images under u of open sets in E) that if $\mathfrak{O}$ is a *Hausdorff* topology compatible with the vector space structure of $\mathbf{R}^n$ (resp. $\mathbf{C}^n$), and *coarser* than the product topology $\mathfrak{O}_0$, then $\mathfrak{O} = \mathfrak{O}_0$. Put $\|x\| = \sup|\xi_i|$ for $x = (\xi_i) \in \mathbf{R}^n$ (resp. $\mathbf{C}^n$) and consider a ball $B : \|x\| < r$ with respect to this norm. The sphere $S : \|x\| = r$ of the same radius is compact in $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) for the product topology $\mathfrak{O}_0$ ((3.17.6), (3.20.16), and (3.17.3)). For each $z \in S$, there exists a neighborhood $V(z)$ of z and a neighborhood $W(z)$ of 0, in the topology $\mathfrak{O}$, which do not intersect. Since $V(z)$ is also a neighborhood of z in the topology $\mathfrak{O}_0$, there exists finitely many points $z_i \in S$ such that the union of the $V(z_i)$ contains S. Now let U be a balanced neighborhood of 0 in the topology $\mathfrak{O}$ (12.13.1), contained in the intersection of the neighborhoods $W(z_i)$. If $x \in U$ were such that $\|x\| \geqq r$, then the point $rx/\|x\|$ would belong to both S and U, which is absurd. Hence $U \subset B$, which establishes our assertion (12.2.1).

Part (iv) of the proposition is an immediate consequence of (ii). For the set $\{0\} = M \cap N$ is closed in N, and therefore N is Hausdorff (12.8.3). Since

E/M is also Hausdorff (12.11.2), the canonical mapping of N onto E/M, which is linear and bijective, is bicontinuous by (ii) because both N and E/M are finite-dimensional.

Next, consider (i). If f is continuous it is clear that $H = f^{-1}(0)$ is closed. The converse follows from (iv): if $f(a) = 1$, then the line $D = \mathbf{R}a$ (resp. $D = \mathbf{C}a$) is supplementary to H and therefore a topological supplement; hence the mapping $x \mapsto f(x)a$ is continuous, and by (ii) this implies the continuity of f.

Finally, to prove (iii), assume first that $M = \{0\}$. Then F is a locally compact metrizable subgroup by (ii), and we can apply (12.9.6). In the general case, consider E/M and apply the characterization of closed sets in E/M (12.10).

The result (ii) above may also be expressed as follows: on a finite-dimensional vector space (over **R** or **C**), there is a *unique Hausdorff* topology compatible with the vector space structure (every endomorphism of the vector space $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) being necessarily continuous with respect to the product topology). This topology is said to be *canonical.*

(12.13.3) *Let* E_1, E_2 *be two metrizable topological vector spaces, and assume that* E_2 *is complete. Let* F_1 (resp. F_2) *be a dense vector subspace of* E_1 (resp. E_2). *Then every continuous linear mapping* $u: F_1 \to F_2$ *has a unique extension to a continuous linear mapping* $\bar{u}: E_1 \to E_2$. *If in addition* E_1 *is complete and* u *is an isomorphism (of topological vector spaces) of* F_1 *onto* F_2, *then* $\bar{u}$ *is an isomorphism (of topological vector spaces) of* E_1 *onto* E_2.

In view of (12.9.4), we have only to show that $\bar{u}$ is linear, i.e., that $\bar{u}(\lambda x) = \lambda \bar{u}(x)$ for all $x \in E_1$ and all scalars λ; but this is a consequence of the principle of extension of identities (3.15.2).

14. LOCALLY CONVEX SPACES

Let E be a vector space over **R** or **C**. A *seminorm* on E is a mapping p of E into **R** which satisfies conditions (I), (III), and (IV) of (5.1). A norm is therefore a seminorm p such that $p(x) \neq 0$ for all $x \neq 0$ in E. If p is a seminorm on E, then $d(x, y) = p(x - y)$ is a *pseudo-distance* on E which is translation-invariant and such that $d(\lambda x, \lambda y) = |\lambda|\, d(x, y)$ for all scalars λ.

(12.14.1) *Let* $(p_\alpha)_{\alpha \in I}$ *be a family of seminorms on a vector space* E. *If the upper envelope* $p = \sup_\alpha p_\alpha$ *is finite on* E, *then it is a seminorm.*

For it is clear that $p(\lambda x) = |\lambda| p(x)$; also we have

$$p_\alpha(x + y) \leqq p_\alpha(x) + p_\alpha(y) \leqq p(x) + p(y)$$

for all $\alpha \in I$, hence $p(x + y) \leqq p(x) + p(y)$.

(12.14.2) *Let p be a seminorm on a topological vector space* E. *Then the following conditions are equivalent*:

(a) *p is continuous*;

(b) *there exists a neighborhood* V *of* 0 *on which p is bounded.*

Clearly (a) implies (b). Conversely, if $p(x) \leqq a$ for all $x \in V$, then for all $x_0 \in E$ and all $x \in x_0 + \varepsilon V$ we have $|p(x) - p(x_0)| \leqq p(x - x_0) \leqq a\varepsilon$, and therefore p is continuous on E.

(12.14.3) *Let $(p_\alpha)_{\alpha \in I}$ be any family of seminorms on a vector space* E. *Then the topology defined by the pseudo-distances $d_\alpha(x, y) = p_\alpha(x - y)$* (12.4) *is compatible with the vector space structure of* E.

In view of the definitions, the proofs are the same as in (4.1.1) and (5.1.5).

The topology defined by the pseudo-distances d_α is said to be *defined by the family of seminorms $(p_\alpha)_{\alpha \in I}$*. Two families of seminorms on E are said to be *equivalent* if they define the same topology (cf. (12.14.12)). A topological vector space whose topology can be defined by a family of seminorms is said to be *locally convex*. If $(p_\alpha)_{\alpha \in I}$ is a family of seminorms defining the topology of a locally convex space E, we obtain a fundamental system of neighborhoods of 0 in E by considering the sets

(12.14.3.1) $$W((\alpha_i); r) = \{x \in E \mid p_{\alpha_i}(x) < r \quad \text{for} \quad 1 \leqq i \leqq n\}$$

where $(\alpha_i)_{1 \leqq i \leqq n}$ runs through the set of *finite* families of indices belonging to I, and r is any real number > 0.

(12.14.4) *Let* E *be a vector space and let $(p_\alpha)_{\alpha \in I}$ be a family of seminorms on* E. *For the topology defined by the family (p_α) to be Hausdorff, it is necessary and sufficient that for each $x \neq 0$ in* E *there should exist an index $\alpha \in I$ such that $p_\alpha(x) \neq 0$.*

This follows from (12.4.4).

Hence a seminorm p on a vector space E is a norm if and only if the topology defined by this single seminorm is *Hausdorff*.

(12.14.5) *A locally convex Hausdorff space whose topology can be defined by a denumerable family of seminorms is metrizable.*

This follows from (12.4.6).

A locally convex Hausdorff space E whose topology can be defined by a denumerable family of seminorms, and which is *complete* with respect to every translation-invariant distance on E (12.9.2) is called a *Fréchet space*. A Banach space (5.1) is therefore a Fréchet space.

Example

(12.14.6) Let X be a separable, locally compact, metrizable space and let $\mathscr{C}_{\mathbf{C}}(X)$ denote the set of *continuous* mappings of X into **C**. Clearly $\mathscr{C}_{\mathbf{C}}(X)$ is a vector space over **C**. Let (U_n) be a sequence of relatively compact open sets in X satisfying the conditions of (3.18.3). For each continuous function $f \in \mathscr{C}_{\mathbf{C}}(X)$ and each integer n, let

$$p_n(f) = \sup_{x \in U_n} |f(x)|. \tag{12.14.6.1}$$

It is immediately seen that the p_n are seminorms on $\mathscr{C}_{\mathbf{C}}(X)$ (but they are not norms if X is not compact). Furthermore, for each $f \neq 0$ in $\mathscr{C}_{\mathbf{C}}(X)$, there exists a point $x \in X$ such that $f(x) \neq 0$ and an integer n such that $x \in U_n$, hence $p_n(f) \neq 0$. The locally convex space $\mathscr{C}_{\mathbf{C}}(X)$ so defined is therefore *metrizable*. Moreover, to say that a sequence (f_k) in $\mathscr{C}_{\mathbf{C}}(X)$ is a Cauchy sequence signifies (by virtue of (12.9.2)) that, for each integer n, the sequence of restrictions $f_k \mid \bar{U}_n$ is a Cauchy sequence in the Banach space $\mathscr{C}_{\mathbf{C}}(\bar{U}_n)$ (7.2.1), hence converges *uniformly* in $\bar{U}_n$ to a continuous mapping g_n of $\bar{U}_n$ into **C**. Since it is clear that $g_{n+1} \mid \bar{U}_n = g_n$, there exists a continuous function $f \in \mathscr{C}_{\mathbf{C}}(X)$ whose restriction to each U_n agrees with the restriction of g_n. Evidently $\lim_{m \to \infty} p_n(f - f_m) = 0$ for all n, hence f is the limit of the Cauchy sequence (f_m). Thus we have proved that $\mathscr{C}_{\mathbf{C}}(X)$ is a *Fréchet space*. When X is compact, it is just the Banach space $\mathscr{C}_{\mathbf{C}}(X)$ defined in (7.2). Moreover, (7.4.4) generalizes as follows:

(12.14.6.2) *The Fréchet space $\mathscr{C}_{\mathbf{C}}(X)$ is separable. More precisely, there exists a denumerable dense set in $\mathscr{C}_{\mathbf{C}}(X)$ consisting of continuous functions with compact supports.*

With the notation used above, for each integer n there exists a dense sequence $(h_{mn})_{m \geqq 1}$ in the Banach space $\mathscr{C}_{\mathbf{C}}(\bar{U}_n)$, by (7.4.4). By the Tietze–Urysohn theorem (4.5.1) applied to the closed set $\bar{U}_n \cap \complement U_{n+1}$ in X, there

exists a continuous mapping g_{mn} of X into C, with support contained in $\bar{U}_{n+1}$ (and therefore compact) and agreeing with h_{mn} on $\bar{U}_n$. Consequently, for each $\varepsilon > 0$, each $f \in \mathscr{C}_C(X)$ and each integer n, there exists an integer m such that $p_n(f - g_{mn}) \leqq \varepsilon$. Since the sequence of seminorms (p_n) is increasing, this proves the result, having regard to (12.14.3.1) and the definition of the topology on $\mathscr{C}_C(X)$.

Remark

(12.14.6.3) The proof shows also that if $\mathrm{Supp}(f) \subset U_n$, then f is the limit of a sequence of functions g_{mn} with supports *contained* in U_{n+1}.

(12.14.7) Let $(p_\alpha)_{\alpha \in I}$ be a family of seminorms on a vector space E. For each *finite* subset H of I, the function $p_H = \sup_{\alpha \in H} p_\alpha$ is a seminorm on E (12.14.1). Since the set $W((\alpha_i); r)$ defined in (12.14.3.1) is the set of all $x \in E$ such that $p_H(x) < r$ (where H is the set of the α_i), it follows that the family of seminorms p_H, where H runs through all finite subsets of I, is *equivalent* to the family (p_α). A set Γ of seminorms on E is said to be *directed* if for each pair of seminorms p', p'' belonging to Γ, there exists $p \in \Gamma$ such that $p \geqq \sup(p', p'')$. It follows that the topology of a locally convex space can always be defined by a directed set of seminorms.

Let E be a locally convex space whose topology is defined by a set Γ of seminorms. If F is a vector subspace of E, it is clear that the induced topology on F is defined by the *restrictions* to F of the seminorms belonging to Γ. The corresponding result for quotient spaces is the following:

(12.14.8) *Let* E *be a vector space,* F *a vector subspace of* E, *and* $\pi : E \to E/F$ *the canonical homomorphism.*

(i) *Let* p *be a seminorm on* E. *For each* $\dot{x} \in E/F$ *let*

$$\dot{p}(\dot{x}) = \inf_{\pi(z) = \dot{x}} p(z). \tag{12.14.8.1}$$

Then $\dot{p}$ *is a semi-norm on* E/F.

(ii) *If* Γ *is a directed set of seminorms and if* E *is endowed with the topology defined by* Γ, *then the quotient topology on* E/F (12.11) *is defined by the set of seminorms* $\dot{p}$, *where* $p \in \Gamma$.

(i) We have

$$\dot{p}(\lambda \dot{x}) = \inf_{\pi(z) = \dot{x}} p(\lambda z) = |\lambda| \inf_{\pi(z) = \dot{x}} p(z) = |\lambda| \, \dot{p}(\dot{x}),$$

so that it remains to prove the triangle inequality. If $\dot{x}$, $\dot{y}$ are elements of E/F

and $u \in \dot{x}$, $v \in \dot{y}$, then $p(u+v) \leqq p(u) + p(v)$ and therefore (as $u+v \in \dot{x}+\dot{y}$) $\dot{p}(\dot{x}+\dot{y}) \leqq p(u)+p(v)$. Since (2.3.11)

$$\inf_{u \in \dot{x},\, v \in \dot{y}} (p(u)+p(v)) = \dot{p}(\dot{x}) + \dot{p}(\dot{y}),$$

the proof of (i) is complete.

(ii) Let r be a real number >0 and let U be the set of all $x \in E$ such that $p(x) < r$. Then it follows immediately from the definition (12.14.8.1) that $\pi(U)$ is the set of all $\dot{x} \in E/F$ such that $\dot{p}(\dot{x}) < r$. Since Γ is directed, the sets U, as p runs through Γ and r runs through the set of real numbers >0, form a fundamental system of neighborhoods of 0 in E. Hence (ii) follows from the definition of neighborhoods in E/F (12.11).

Whenever we consider E/F as a locally convex space, it is always the quotient topology that is meant, unless the contrary is expressly stated.

(12.14.9) *Every closed vector subspace of a Fréchet space is a Fréchet space. Every quotient of a Fréchet space by a closed vector subspace is a Fréchet space.*

The first assertion follows from above and from (3.14.5), the second from (12.14.8) and (12.11.3).

Remark

(12.14.10) Let E be a *normed* space and F a closed vector subspace of E. Then

(12.14.10.1) $$\|\dot{x}\| = \inf_{z \in \dot{x}} \|z\| = d(0, \dot{x})$$

(in the last expression, $\dot{x}$ is considered as a linear variety parallel to F in the vector space E) is a *norm* on E/F which defines the quotient topology, by virtue of (12.14.4), (12.14.8), and (12.11.2). Whenever we consider E/F as a normed space, it is always this norm that is meant, unless the contrary is expressly stated.

More particularly, suppose that E is a *Hilbert space*. Then if F is a closed subspace of E, and if $F' = P_F^{-1}(0)$ is the orthogonal supplement of F (6.3.1), the bijective linear mapping $F' \to E/F$, the restriction to F' of the canonical mapping $\pi : E \to E/F$, is an *isometry*. For by Pythagoras' theorem we have $\|x'+z\|^2 = \|x'\|^2 + \|z\|^2$ for all $z \in F$ and all $x' \in F'$; hence $\|x'\|$ is the infimum of the norms of elements of $x' + F$. We may therefore identify E/F with the Hilbert space F'.

(12.14.11) *Let* E, E′ *be two locally convex spaces. Let* Γ *be a* directed *set of seminorms defining the topology of* E, *and* Γ' *a set of seminorms defining the topology of* E′. *Let* $u : E \to E'$ *be a linear mapping. Then the following conditions are equivalent:*

(a) *u is continuous;*

(b) *for each seminorm* $p' \in \Gamma'$, *there exists a neighborhood of* 0 *in* E *on which the function* $x \mapsto p'(u(x))$ *is bounded* (or, equivalently (12.14.2), the seminorm $p' \circ u$ on E *is continuous*);

(c) *for each seminorm* $p' \in \Gamma'$, *there exists a seminorm* $p \in \Gamma$ *and a real number* $c > 0$ *such that* $p'(u(x)) \leqq c \cdot p(x)$ *for all* $x \in E$.

From the definition of the neighborhoods of 0 in E and E′, condition (c) implies that u is continuous at the point 0, hence everywhere (12.8.4). So (c) implies (a). The condition (a) implies that the function $p' \circ u$ is continuous on E (because p' is continuous on E′), hence (b). Finally, suppose that (b) is satisfied. Then by hypothesis there exist two numbers $r > 0$ and $a > 0$ and a seminorm $p \in \Gamma$ such that the relation $p(z) \leqq r$ implies $p'(u(z)) \leqq a$. Let x be any point of E and let λ be a scalar strictly greater than 0 such that $p(\lambda x) = \lambda p(x) \leqq r$. Then $p'(u(\lambda x)) \leqq a$, so that $p'(u(x)) \leqq a/\lambda$. If $p(x) = 0$, we can take λ to be arbitrarily large, and therefore $p'(u(x)) = 0$; if $p(x) > 0$, we take $\lambda = r/p(x)$, and then we have $p'(u(x)) \leqq (a/r) \cdot p(x)$. So in either case the condition (c) is satisfied, with $c = a/r$. Q.E.D.

(12.14.11.1) Likewise, if u is a *multilinear* mapping of a product

$$E_1 \times E_2 \times \cdots \times E_r$$

of locally convex spaces into a locally convex space E′, the continuity of u implies that for each seminorm $p' \in \Gamma'$ there exist neighborhoods V_j of 0 in E_j $(1 \leqq j \leqq n)$ such that the function $(x_1, \ldots, x_n) \mapsto p'(u(x_1, \ldots, x_n))$ is bounded in $V_1 \times \cdots \times V_n$. If Γ_j is a *directed* set of seminorms defining the topology of E_j $(1 \leqq j \leqq n)$, this latter condition implies that there exist numbers $a > 0$, $r_j > 0$ $(1 \leqq j \leqq n)$ and seminorms $p_j \in \Gamma_j$ $(1 \leqq j \leqq n)$ such that the relations $p_j(z_j) \leqq r_j$ $(1 \leqq j \leqq n)$ imply $p'(u(z_1, \ldots, z_n)) \leqq a$. Now let $(x_1, \ldots, x_n)$ be any point of $E_1 \times \cdots \times E_n$, and for each j let $\lambda_j > 0$ be such that $p_j(\lambda_j x_j) = \lambda_j p_j(x_j) \leqq r_j$; then it follows that

$$p'(u(x_1, \ldots, x_n)) \leqq a/\lambda_1 \lambda_2 \ldots \lambda_n .$$

We deduce as above that if $p_j(x_j) = 0$ for some index j, then λ_j may be chosen arbitrarily large, and hence $p'(u(x_1, \ldots, x_n)) = 0$. If on the other hand $p_j(x_j) \neq 0$ for all j, we may take $\lambda_j = r_j/p_j(x_j)$ for $1 \leqq j \leqq n$; putting $c = a/r_1 \ldots r_n$,

it follows that for *each* point $(x_1, \ldots, x_n)$ in $E_1 \times \cdots \times E_n$, we have

$$p'(u(x_1, \ldots, x_n)) \leqq c \cdot p_1(x_1) \cdots p_n(x_n). \tag{12.14.11.2}$$

Conversely, it is clear that if for each seminorm $p' \in \Gamma'$ there exist seminorms $p_j \in \Gamma_j$ and a real number $c > 0$ such that (12.14.11.2) is satisfied throughout $E_1 \times \cdots \times E_n$, then u is continuous at the point $(0, \ldots, 0)$. Now let $(b_1, \ldots, b_n)$ be an arbitrary point of $E_1 \times \cdots \times E_n$, and write

$$u(x_1, \ldots, x_n) - u(b_1, \ldots, b_n) = \sum_{H} u(y_1, \ldots, y_n),$$

where H runs through the $2^n - 1$ proper subsets of $\{1, 2, \ldots, n\}$, and for each such H put $y_j = b_j$ if $j \in H$, and $y_j = x_j - b_j$ if $j \notin H$. Then it follows immediately that u is continuous at the point $(b_1, \ldots, b_n)$, i.e., u is continuous on $E_1 \times \cdots \times E_n$.

(12.14.12) *On a vector space* E, *let* Γ *and* Γ' *be two families of seminorms. Then the topology defined by* Γ *is finer than the topology defined by* Γ' *if and only if, for each seminorm* $p' \in \Gamma'$, *there exists a finite family* $(p_i)_{1 \leqq i \leqq n}$ *of seminorms belonging to* Γ *and a real number* $c > 0$ *such that* $p'(x) \leqq c \cdot \sup_i p_i(x)$ *for all* $x \in E$.

Apply (12.14.11) to the identity map $u = 1_E$ (12.2.1).

(12.14.13) Finally, we remark that the notions of *series* and *convergent series* are defined in a topological vector space E just as in a normed space (5.2); propositions (5.2.2), (5.2.3), (5.2.4) and the fact that the general term of a convergent series tends to 0 remain valid without change. If E is locally convex and its topology is defined by a family (p_λ) of seminorms, then for a series with general term x_n to converge in E it is necessary that, for each $\varepsilon > 0$ and each scalar λ, there should exist an integer n_0 such that

$$p_\lambda(x_{n+1} + \cdots + x_{n+q}) \leqq \varepsilon$$

for all $n \geqq n_0$ and $q > 0$. If E is a Fréchet space, this condition is also sufficient, as follows from Cauchy's criterion (12.9).

PROBLEMS

1. Let E be a vector space over **R** (resp. **C**) and let $\mathfrak{V}$ be a set of subsets of E satisfying conditions (V_I) and (V_{II}) of Section 12.3, Problem 3, together with the following:
   (EV_I) Every set $V \in \mathfrak{V}$ is balanced and absorbing.

(EV_{II}) For all $V \in \mathfrak{V}$ and all $\lambda \neq 0$ in $\mathbf{R}$ (resp. $\mathbf{C}$), we have $\lambda V \in \mathfrak{V}$.
(EV_{III}) For each $V \in \mathfrak{V}$, there exists $W \in \mathfrak{V}$ such that $W + W \subset V$.

Show that there exists a unique topology on E, compatible with the vector space structure, for which $\mathfrak{V}$ is a fundamental system of neighborhoods of 0.

2. Let E be a vector space over $\mathbf{R}$ with a denumerable infinite basis (e_n), and let $\mathfrak{T}$ be the set of *all* balanced absorbing sets in E. Show that $\mathfrak{T}$ does not satisfy axiom (EV_{III}) of Problem 1. (For each $n > 0$, let A_n be the set of points $\sum_{i=1}^{n} t_i e_i$ such that $|t_i| \leqq 1/n$ for $1 \leqq i \leqq n$, and let A be the union of the sets A_n. Show that there exists no set $M \in \mathfrak{T}$ such that $M + M \subset A$.)

3. Let E be the vector space $\mathscr{C}_{\mathbf{R}}(\mathbf{R})$ of continuous mappings of $\mathbf{R}$ into $\mathbf{R}$. For each continuous function m such that $m(t) > 0$ for all $t \in \mathbf{R}$, let V_m denote the set of functions $f \in E$ such that $|f(t)| \leqq m(t)$ for all $t \in \mathbf{R}$.
(a) Show that the V_m form a fundamental system of neighborhoods of 0 in a topology $\mathscr{T}_0$ on E which is compatible with the additive group structure of E, but not with the vector space structure.
(b) Let D be the vector subspace of E consisting of functions with compact support (12.6). Show that the topology $\mathscr{T}$ induced by $\mathscr{T}_0$ on D is compatible with the vector space structure of D, and is not metrizable.

4. Let $(E_\alpha)_{\alpha \in I}$ be any family of topological vector spaces, and $E = \prod_{\alpha \in I} E_\alpha$ the product vector space. Show that the product of the topologies of the E_α (12.5) is compatible with the vector space structure of E. If each E_α is locally convex and its topology is defined by a family of seminorms $(p_{\alpha\lambda})_{\lambda \in L_\alpha}$, then the topology of E is locally convex and is defined by the family of seminorms $p_{\alpha\lambda} \circ \mathrm{pr}_\alpha$ (for all $\alpha \in I$ and all $\lambda \in L_\alpha$). In particular, every (finite or) denumerable product of Fréchet spaces is a Fréchet space.

5. Let p be a seminorm on a (real or complex) vector space E. Show that the set N of $x \in E$ such that $p(x) = 0$ is a vector subspace of E. The seminorm $\dot{p}$ defined by (12.14.8.1) on E/N is a norm.

Let E be a Hausdorff locally convex space whose topology is defined by a family (p_α) of seminorms, and for each α let N_α be the subspace of E consisting of those $x \in E$ for which $p_\alpha(x) = 0$. Show that E is isomorphic to a subspace of the product of the normed spaces E/N_α (the norm on E/N_α being $\dot{p}_\alpha$). In particular, every Fréchet space is isomorphic to a closed subspace of a denumerable product of normed spaces.

6. A subset A of a vector space E over $\mathbf{R}$ is said to *absorb* a subset B of E if there exists $\alpha > 0$ such that $\lambda B \subset A$ whenever $|\lambda| \leqq \alpha$. If E is a topological vector space, a subset B of E is said to be *bounded* (with respect to the topological vector space structure of E) if every neighborhood of 0 absorbs B.
(a) The closure of a bounded set is bounded. Every finite set is bounded. If A, B are bounded, then so are $A \cup B$ and $A + B$. If E is metrizable, every precompact set in E is bounded.
(b) Suppose that E is Hausdorff. Then a subset B of E is bounded if and only if, for every sequence (x_n) in B and every sequence (λ_n) of real numbers tending to zero, the sequence $(\lambda_n x_n)$ tends to 0 in E.
(c) Let (E_α) be a family of topological vector spaces and $E = \prod_\alpha E_\alpha$ their product (Problem 4). Show that a subset B of E is bounded if and only if $\mathrm{pr}_\alpha(B)$ is bounded in E_α for each α.

(d) Let E, F be two topological vector spaces and $f: E \to F$ a continuous linear mapping. Then for each bounded subset B of E, the image set $f(B)$ is bounded in F.

7. (a) Let E be a (real or complex) Hausdorff topological vector space. Show that, if there exists a bounded neighborhood V of 0 in E (Problem 6), the sets $n^{-1}V$ (n an integer >0) form a fundamental system of neighborhoods of 0 in E, and consequently E is metrizable. If furthermore E is locally convex, then its topology can be defined by a single norm.

(b) Let E be a metrizable locally convex space whose topology is defined by an increasing sequence (p_n) of seminorms. Show that the topology of E can be defined by a single norm if and only if there exists an integer N such that, for all $n \geqq N$, there exists a real number $k_n \geqq 0$ for which $p_n \leqq k_n p_N$.

(c) Let E be the real vector space of indefinitely differentiable real-valued functions on the interval $I = [0, 1]$ in $\mathbf{R}$. For each $n \geqq 0$ and each $f \in E$, put

$$p_n(f) = \sup_{0 \leqq k \leqq n} \left(\sup_{t \in I} |D^k f(t)| \right)$$

(where $D^0 f = f$). Show that each p_n is a *norm* on E and that the topology defined by the sequence of norms (p_n) cannot be defined by a single norm (cf. (17.1)).

(d) Let E be a Hausdorff locally convex space whose topology is defined by a sequence of seminorms but cannot be defined by a single norm. If d is a translation-invariant distance defining the topology of E (12.9.2) show that every set in E which is bounded in the sense of the topological vector space structure of E is bounded with respect to the distance d, but there exist subsets of E which are bounded with respect to d but are unbounded with respect to the topological vector space structure of E.

8. Let E be a metrizable topological vector space.

(a) Show that every balanced subset of E which absorbs all sequences in E which converge to 0 is a neighborhood of 0 in E. Deduce that, if u is a linear mapping of E into a topological vector space F, and u transforms every sequence converging to 0 in E into a bounded sequence in F, then u is continuous.

(b) Let (B_n) be any sequence of bounded subsets of E. Show that there exists a sequence (λ_n) of nonzero scalars such that the union of the sets $\lambda_n B_n$ is bounded.

9. Let E be a topological vector space. A set $\mathfrak{B}$ of bounded subsets of E is called a *fundamental system* of bounded subsets if every bounded set in E is contained in a set belonging to $\mathfrak{B}$. Show that, if E is a metrizable locally convex space whose topology cannot be defined by a single norm, no fundamental system of bounded subsets of E is denumerable (use Problem 8).

10. Let $(E_n)_{n \geqq 1}$ be a sequence of normed spaces such that E_1 is not separable. Let F be a vector subspace of $E = \prod_n E_n$, such that $\mathrm{pr}_n(F) = E_n$ for each n. Show that there exists a real number $\delta > 0$ and a *bounded* sequence $(x_m)_{m \geqq 1}$ in F such that

$$\|\mathrm{pr}_i x_i - \mathrm{pr}_i x_j\| \geqq \delta$$

whenever $i \neq j$. (Remark that, for each index n and each nondenumerable subset A of F, there exists a nondenumerable subset B of A such that $\mathrm{pr}_n(B)$ is bounded in E_n.)

Deduce that a Fréchet space is separable if every bounded subset is *relatively compact*.

11. A subset A of a real or complex vector space E is *convex* if, whenever x and y belong to A and $0 \leqq \lambda \leqq 1$, we have $\lambda x + (1 - \lambda)y \in A$ (Section 8.5, Problem 9). It follows

that, if A is convex and $(x_t)_{1 \leqslant t \leqslant n}$ is any finite sequence of points of A and $(\lambda_t)_{1 \leqslant t \leqslant n}$ is any sequence of n real numbers $\geqq 0$ such that $\sum_{t=1}^{n} \lambda_t = 1$, we have $\sum_{t=1}^{n} \lambda_t x_t \in A$.

(a) Let E, F be vector spaces over $\mathbf{R}$ and let f be an affine linear mapping of E into F. Then the image under f of any convex subset of E is convex, and the inverse image under f of any convex subset of F is convex. In particular, if $g \neq 0$ is a linear form on E, then the *half-spaces* defined by one or other of the relations $g(x) > \alpha$, $g(x) \geqq \alpha$ (where α is any real number) are convex sets.

(b) In a vector space E over $\mathbf{R}$, the intersection of any family of convex sets is convex. If A, B are convex subsets of E and α, β are any real numbers, then the set $\alpha A + \beta B$ is convex.

(c) Let $(E_\alpha)_{\alpha \in I}$ be a family of vector spaces over $\mathbf{R}$ and let $E = \prod_\alpha E_\alpha$ be their product. For each α, let A_α be a nonempty subset of E_α. Show that $A = \prod_\alpha A_\alpha$ is convex if and only if each A_α is convex.

(d) In a topological vector space E, the closure $\bar{A}$ and the interior $\mathring{A}$ of a convex set A are convex. If, moreover, $\mathring{A}$ is not empty, then $\bar{A}$ is the closure of $\mathring{A}$, and $\mathring{A}$ the interior of $\bar{A}$.

(e) In a topological vector space E, for a half-space $\{x : g(x) \geqq \alpha\}$ (where g is a non-zero linear form on E) to be either closed or to have an interior point, it is necessary and sufficient that g should be continuous; in which case the hyperplane $g(x) = \alpha$ is the frontier of the half-space.

12. (a) Let E be a vector space over $\mathbf{R}$ and p a seminorm on E. Show that the set of $x \in E$ such that $p(x) < 1$ (resp. $p(x) \leqq 1$) is an absorbing symmetric convex set. Conversely, let A be an absorbing symmetric convex set in E, and for each $x \in E$ let $p(x) = \inf_{\substack{\rho > 0 \\ x \in \rho A}} \rho$. Show that p is a seminorm on E and that A contains the set of points such that $p(x) < 1$ and is contained in the set of points such that $p(x) \leqq 1$.

(b) Let E be a topological vector space over $\mathbf{R}$, and p a seminorm on E. Then the set $A = \{x \in E : p(x) \leqq 1\}$ is closed if and only if p is lower semicontinuous on E. Also A has an interior point if and only if p is continuous at the point 0, and p is then continuous throughout E.

(c) Deduce from (b) that a topological vector space E is locally convex if and only if there exists a fundamental system of convex neighborhoods of 0 in E. The topology of a metrizable locally convex space can be defined by an at most denumerable family of seminorms.

13. Let E be a real vector space. If A is any subset of E, the *convex hull* of A is defined to be the smallest convex set containing A. If E is a topological vector space, the *closed convex hull* of A is defined to be the smallest closed convex set containing A; it is also the closure of the convex hull of A.

(a) Show that the convex hull of A is the set of all points $\sum_\alpha \lambda_\alpha x_\alpha$, where (x_α) is any *finite* family of points of A, and the λ_α are real numbers $\geqq 0$ such that $\sum_\alpha \lambda_\alpha = 1$.

(b) Suppose that E is a metrizable locally convex space. Show that the convex hull in E of a precompact set is precompact. If E is a Fréchet space, the *closed* convex hull of a compact set is compact (cf. Section 13.4, Problem 13).

14. Let E be the real vector space of regulated functions f on $I = [0, 1]$, continuous on the right at all $t < 1$ and such that $f(1) = 0$. For each integer $n > 0$, let V_n be the

set of all functions $f \in E$ such that $\int_0^1 |f(t)|^{1/2}\, dt \leqq 1/n$. Show that the sets V_n form a fundamental system of neighborhoods of 0 for a metrizable topology on E compatible with the vector space structure, that the sets V_n are bounded with respect to this topology, but that the convex hull of each V_n is the whole space E. (Observe that every function $f \in E$ can be written as $f = \frac{1}{2}(g + h)$ with $g, h \in E$ and

$$\int_0^1 |g(t)|^{1/2}\, dt = \int_0^1 |h(t)|^{1/2}\, dt = \frac{1}{\sqrt{2}} \int_0^1 |f(t)|^{1/2}\, dt\,.)$$

Deduce that every continuous seminorm on E is identically zero. In particular, if a linear form on E is continuous, it is zero.

15. (a) Let E be a topological real vector space, H a nonempty convex open subset of E, and f a convex function (Section 8.5, Problem 8) defined on H. Show that f is continuous on H if and only if there exists a nonempty open set $U \subset H$ on which f is bounded above. In particular, every upper semicontinuous convex function on H is continuous on H.

(b) Give an example of a convex function p, which is defined on a compact convex set $K \subset \mathbf{R}^2$, lower semicontinuous and bounded in K, but not continuous at a frontier point of K. (Take K to be a closed disk with 0 as a frontier point, and apply the definition of Problem 12(a).) Hence give an example of a compact convex set $A \supset K$ such that p can be extended to a convex function which is lower semicontinuous on A, but not bounded on A.

16. Let E be a Hausdorff locally convex space, X a locally compact separable metric space, d the distance on X, A a closed subset of X, and f a continuous mapping of A into E.

(a) Show that there exists a denumerable covering (U_n) of $X - A$, consisting of open sets $U_n \subset X - A$, which is locally finite and such that, for every frontier point a of A and every neighborhood V of a in X, there exists a neighborhood $V' \subset V$ of a with the property that each U_n which meets V' is contained in V. (For each $x \in X - A$, consider the open ball with center x and radius $\frac{1}{2}d(x, A)$, and use (12.6.1).)

(b) Let (h_n) be a continuous partition of unity on $X - A$ subordinate to the covering (U_n) (12.6.3). For each n let $b_n \in U_n$ and $a_n \in A$ be such that $d(b_n, a_n) \leqq 2d(b_n, A)$. Let $g(x) = f(x)$ if $x \in A$, and

$$g(x) = \sum_n h_n(x) f(a_n)$$

if $x \in X - A$. Show that g is a continuous mapping of X into E which *extends* f and is such that $g(X)$ is contained in the *convex hull* (Problem 13) of $f(A)$.

17. Let E, F be two locally convex spaces, f a continuous linear mapping of E into F, and L a locally compact separable metrizable subspace of F such that $f(E) \cap L$ is dense in L. For each continuous seminorm p on E and each $\varepsilon > 0$, show that there exists a continuous mapping $s : L \to E$ such that $p(z - f(s(z))) \leqq \varepsilon$ for all $z \in L$. (Consider a locally finite denumerable open covering (U_n) of L such that $p(x' - x) \geqq \frac{1}{2}\varepsilon$ whenever x and x' lie in the same U_n, a continuous partition of unity (h_n) subordinate to (U_n), and for each n a point $a_n \in U_n \cap f(E)$. If $b_n \in f^{-1}(a_n)$, show that the function s defined by $s(z) = \sum_n b_n h_n(z)$ satisfies the required conditions.)

15. WEAK TOPOLOGIES

(12.15.1) *Let* $(E_\alpha)_{\alpha \in I}$ *be a family of locally convex spaces, and for each* α *let* $(p_{\alpha\lambda})_{\lambda \in L_\alpha}$ *be a family of seminorms defining the topology of* E_α. *Let* $E = \prod_{\alpha \in I} E_\alpha$ *be the product vector space, and for each* $a \in I$ *and each* $\lambda \in L_\alpha$, *put*

$$p'_{\alpha\lambda}(x) = p_{\alpha\lambda}(\mathrm{pr}_\alpha x) \qquad \textit{for all} \quad x \in E. \tag{12.15.1.1}$$

Then the $p_{\alpha\lambda}$ *are seminorms on* E *and define the product topology* (12.5) *on* E.

It is straightforward to check that the $p'_{\alpha\lambda}$ are seminorms, and then the second assertion follows from the definition (12.5) of a product of uniformizable spaces.

In particular, when all the E_α are equal to the same locally convex space F, then the *product* topology on the vector space F^I of all mappings of I into F is defined by the seminorms

$$f \mapsto p(f(\alpha))$$

where $\alpha \in I$ and p belongs to a set Γ of seminorms defining the topology of F. This topology is called the *topology of simple* (or *pointwise*) *convergence* on the vector space F^I, or on any of its vector subspaces (cf. (12.5)). With respect to this topology, each mapping $f \mapsto f(\alpha)$ ($\alpha \in I$) is a *continuous* linear mapping on F^I, and maps each open set in F^I to an open set in F (12.5.2). The topology of simple convergence is *Hausdorff* if F is Hausdorff (12.5.7). A mapping $t \mapsto f_t$ of a topological space T into F^I is *continuous* with respect to the topology of simple convergence if and only if the mapping $t \mapsto f_t(\alpha)$ of T into F is *continuous*, for each $\alpha \in I$ (12.5.5). When F is a Banach space over K (= **R** or **C**) and T is an open subset of K^n, a mapping $t \mapsto f_t$ of T into F^I is said to be p *times differentiable* (resp. *indefinitely differentiable*, resp. *analytic*) *with respect to the topology of simple convergence* if, for each index $\alpha \in I$, the mapping $t \mapsto f_t(\alpha)$ of T into F has the corresponding property.

We shall be particularly concerned with the case in which F is the field of scalars K, and I is a *vector space* E over K. Let V be a subspace of K^E whose elements are *linear forms* on E. The topology of simple convergence on such a subspace V is called the *weak topology*; it is defined by seminorms

$$f \mapsto |f(x)|, \tag{12.15.2}$$

where x runs through E. For each $x \in E$, the mapping $f \mapsto f(x)$ is a *continuous linear form* on V with respect to the weak topology. A sequence (f_n) of elements

of V which converges to $f \in V$ with respect to the weak topology (i.e., is such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$) is also said to be *weakly convergent*. On the same principle, we shall speak of subsets of V being *weakly closed* (in V), *weakly compact*, etc. A subset B of K^E is said to be *weakly bounded* if all the seminorms (12.15.2) are bounded on B, that is if $\sup_{f\in B} |f(x)| < +\infty$ for all $x \in E$. We shall also use the phrases *weakly continuous* (resp. *weakly differentiable, weakly analytic*) or *scalarly continuous* (resp. *scalarly differentiable, scalarly analytic*) in place of "continuous (resp. differentiable, analytic) with respect to the weak topology."

If E is a *locally convex space* (over $K = \mathbf{R}$ or $\mathbf{C}$), the vector space of continuous linear forms on E is called the *dual* of E, and is often denoted by E'. If $x' \in E'$ is a continuous linear form on E, its value at a point $x \in E$ is often denoted by the expression $\langle x, x' \rangle$ or $\langle x', x \rangle$ instead of $x'(x)$. Since E' is a vector subspace of K^E, it is endowed with the *weak topology* defined by the seminorms $x' \mapsto |\langle x, x' \rangle|$. For each $x \in E$, the function $x' \mapsto \langle x, x' \rangle$ is a *weakly continuous linear form* on E'.

If E is a Fréchet space and E_0 a *dense* vector subspace of E, then it follows from (12.9.4) and the principle of extension of identities (3.15.2) that the mapping $x' \mapsto x' \mid E_0$ is an (algebraic) *isomorphism* of the dual E' of E onto the dual E_0' of E_0.

(12.15.3) *Let* E, F *be two locally convex spaces,* E' *and* F' *their respective duals, and* $u : E \to F$ *a continuous linear mapping. Then* $y' \mapsto y' \circ u$ *is a linear mapping of* F' *into* E' *which is continuous with respect to the weak topologies.*

If y' is a continuous linear form on F, then $y' \circ u$ is a continuous linear form on E. Given $x_0 \in E$, there exists $y_0 \in F$ such that $\langle x_0, y' \circ u \rangle = \langle y_0, y' \rangle$ for all $y' \in F'$, namely $y_0 = u(x_0)$; the continuity of the mapping $y' \mapsto y' \circ u$ is therefore an immediate consequence of (12.14.11) and the definition (12.15.2) of the seminorms which define the weak topology.

We write $y' \circ u = {}^t u(y')$, and the linear mapping ${}^t u : F' \to E'$ is called the *transpose* of the continuous linear mapping u. Thus we have

$$\langle u(x), y' \rangle = \langle x, {}^t u(y') \rangle \tag{12.15.4}$$

for all $x \in E$ and $y' \in F'$. Clearly

$${}^t(u_1 + u_2) = {}^t u_1 + {}^t u_2, \qquad {}^t(\lambda u) = \lambda \cdot {}^t u \tag{12.15.5}$$

for any scalar λ, and for any continuous linear mappings u_1, u_2 of E into F. If

v is a continuous linear mapping of F into a locally convex space G, then

(12.15.6) $$ {}^t(v \circ u) = {}^t u \circ {}^t v. $$

The weak topology on the dual of a locally convex space *is not metrizable* in general (Problem 2). However, there is the following result:

(12.15.7) *Let* E *be a separable metrizable locally convex space. If* H *is any equicontinuous* (7.5) *subset of the dual* E′ *of* E, *then the weak closure of* H *in* E′ *is a compact metrizable space in the weak topology.*

We shall first prove the following lemma:

(12.15.7.1) *Let* E *be a metrizable vector space and* F *a normed space. In order that a set* H *of linear mappings of* E *into* F *should be equicontinuous, it is necessary and sufficient that there should exist a neighborhood* V *of* 0 *in* E *and a real number* $c > 0$ *such that* $\|u(x)\| \leqq c$ *for all* $x \in V$ *and all* $u \in H$.

(If E is *normed*, this condition is also equivalent to $\sup_{u \in H} \|u\| < +\infty$ by (5.7.1).)

The condition expresses that H is equicontinuous *at the point* 0. If this is the case, then for each $x_0 \in E$ and each $x \in x_0 + \varepsilon V$, we have

$$ \|u(x) - u(x_0)\| = \|u(x - x_0)\| \leqq \varepsilon c $$

for all $u \in H$, and therefore H is equicontinuous at x_0.

In proving (12.15.7), we may first of all restrict ourselves to the case where H is *weakly closed* in E′. For if $\bar{H}$ is the weak closure of H in E′, the continuity of the mapping $x' \mapsto \langle x, x' \rangle$ on E′ implies that if $|\langle x, x' \rangle| \leqq c$ for all $x \in V$ and all $x' \in H$, then also $|\langle x, x' \rangle| \leqq c$ for all $x \in V$ and all $x' \in \bar{H}$ (3.15.4), whence the result by (12.15.7.1).

Now let (a_n) be a sequence which is dense in E. We shall show that the weak topology on H is defined by the pseudo-distances

$$ d_n(x', y') = |\langle a_n, x' - y' \rangle|. $$

Clearly the topology defined by these pseudo-distances is *coarser* than the weak topology; hence it is enough to show that it is also *finer*. By hypothesis, every neighborhood of a point $x'_0 \in H$ in the weak topology contains a subset W of H consisting of elements $x' \in H$ satisfying a *finite* number of inequalities

(12.15.7.2) $$ |\langle x_i, x' - x'_0 \rangle| < r \qquad (1 \leqq i \leqq m) $$

where the x_i are points of E, and $r > 0$. Suppose that H satisfies the condition (12.15.7.1). Choose $\varepsilon > 0$ such that $2\varepsilon c < \frac{1}{2}r$. For each index i, there exists an $a_{n(i)}$ such that $x_i - a_{n(i)} \in \varepsilon V$. By virtue of (12.15.7.1), it follows that for each $x' \in H$ we have

$$|\langle x_i - a_{n(i)}, x' - x'_0\rangle| \leqq 2\varepsilon c < \tfrac{1}{2}r,$$

and the relations $|\langle a_{n(i)}, x' - x'_0\rangle| < \frac{1}{2}r$ therefore imply the inequalities (12.15.7.2). Since the weak topology is Hausdorff, we have shown that H is metrizable (12.4.6). More precisely, the above argument shows that the mapping $x' \mapsto (\langle a_n, x'\rangle)_{n \geqq 0}$ of H into the product space $K^{\mathbf{N}}$ (where $K = \mathbf{R}$ or $\mathbf{C}$) is a *homeomorphism* of H onto a subspace L of $K^{\mathbf{N}}$. So it remains to be shown that L is *compact*.

In the first place, the projections of L are *bounded* subsets of K (and therefore relatively compact ((3.17.6) and (3.20.16))): for each index n there exists $\lambda_n > 0$ such that $\lambda_n a_n \in V$ and therefore $|\langle a_n, x'\rangle| \leqq c/\lambda_n$ for all $x' \in H$. By virtue of (12.5.9), (12.5.4), and (3.17.3), it remains to be shown that L is *closed* in $K^{\mathbf{N}}$. Now if (x'_m) is a sequence of points in H such that for each n the sequence $(\langle a_n, x'_m\rangle)_{m \geqq 1}$ converges in K, then it follows from (7.5.5) that the sequence (x'_m) converges simply in V. Since for each $z \in E$ there exists a scalar γ such that $\gamma z \in V$ (so that $\langle z, x'_m\rangle = \gamma^{-1}\langle \gamma z, x'_m\rangle$), the sequence (x'_m) converges simply *in* E: its limit y' is a linear mapping, by the principle of extension of identities, and is continuous by (7.5.5), and is therefore a point of H. Q.E.D.

We recall that if E is a *normed* space, its dual $E' = \mathscr{L}(E; K)$ is a *Banach space* with respect to the norm $\|x'\| = \sup_{\|x\| \leq 1} |\langle x, x'\rangle|$ (5.7.3); since $|\langle x, x'\rangle| \leqq \|x\| \cdot \|x'\|$, the topology defined by this norm (sometimes called the *strong topology* on E') is *finer* than the weak topology on E'.

(12.15.8) *Let* E *be a normed space. The mapping* $x' \mapsto \|x'\|$ *of* E' *into* $\mathbf{R}$ *is lower semicontinuous with respect to the weak topology.*

For it is the upper envelope of the weakly continuous mappings $x' \mapsto |\langle x, x'\rangle|$, where x runs through the ball $\|x\| \leqq 1$ in E. Hence the result follows from (12.7.7).

If (x'_n) is a sequence in E' which converges weakly to a', then by (12.7.13) we have

(12.15.8.1) $$\liminf_{n \to \infty} \|x'_n\| \geqq \|a'\|.$$

(12.15.9) *Let* E *be a separable normed space. Then every closed ball* B': $\|x'\| \leqq r$ *in* E' *is metrizable and compact with respect to the weak topology.*

By (12.15.7) and (12.15.7.1) it is enough to show that B' is *weakly closed*, and this follows from the fact that the function $x' \mapsto \|x'\|$ is lower semi-continuous on E' (12.7.2).

Now let E be a *Hilbert space* (6.2). For each $x \in E$, let $j(x)$ denote the continuous linear form $y \mapsto (y \mid x)$ on E. It follows from (6.3.2) that $x \mapsto j(x)$ is a *semilinear isometry* (i.e., we have

$$j(x_1 + x_2) = j(x_1) + j(x_2), \qquad j(\lambda x) = \bar{\lambda} j(x)$$

for all scalars λ) of E onto its dual E'. We can therefore "transport" to E the weak topology on E': the *weak topology* on E is therefore the topology defined by the seminorms $x \mapsto |(a \mid x)|$ as a runs through E. This topology is *coarser* than the topology defined by the norm on E (which is called the *strong topology* of the Hilbert space E). From (12.15.9) we have:

(12.15.10) *In a separable Hilbert space, every closed ball is metrizable and compact with respect to the weak topology.*

If u is a continuous endomorphism (with respect to the *strong* topology) of a Hilbert space E, then by virtue of (12.15.4) and (11.5.1)

$$(u(x) \mid y) = \langle u(x), j(y) \rangle = \langle x, {}^t u(j(y)) \rangle = (x \mid j^{-1}({}^t u(j(y)))) = (x \mid u^*(y))$$

for all x, y in E, and therefore (6.3.2)

$$u^* = j^{-1} \circ {}^t u \circ j. \tag{12.15.11}$$

This relation shows immediately that ${}^t u$ is *strongly continuous* and that $\|{}^t u\| = \|u^*\| = \|u\|$. Replacing u by u^* in (12.15.11), we see that every strongly continuous endomorphism of E is also weakly continuous (cf. (12.16.7)).

(12.15.12) *In a Hilbert space* E, *a sequence* (x_n) *converges strongly to a point* a *if and only if it converges weakly to* a *and* $\lim_{n \to \infty} \|x_n\| = \|a\|$.

The conditions are clearly necessary. To see that they are sufficient, consider the formula

$$\|x_n - a\|^2 = \|x_n\|^2 - 2\mathscr{R}(x_n \mid a) + \|a\|^2.$$

Since by hypothesis the sequence $((x_n \mid a))$ converges to $(a \mid a) = \|a\|^2$, it follows that $\|x_n - a\|$ tends to 0.

PROBLEMS

1. Let E be a vector space over K (= **R** or **C**) and let F be a vector subspace of K^E whose elements are linear forms on E. We write $\langle x, x' \rangle$ in place of $x'(x)$ for $x \in E$ and $x' \in F$. Given a finite sequence $(x'_i)_{1 \leqslant i \leqslant n}$ of elements of F, show that for this sequence to be free (A. 4.1) it is necessary and sufficient that there should exist n vectors x_i $(1 \leqq i \leqq n)$ in E such that $\langle x_i, x'_j \rangle = \delta_{ij}$ (Kronecker delta). The vector space E is then the direct sum of the subspace V of dimension n generated by the x_i, and the subspace W of all $x \in E$ such that $\langle x, x'_i \rangle = 0$ for $1 \leqq i \leqq n$. A linear form $x' \in F$ is such that $\langle x, x' \rangle = 0$ for all $x \in W$ if and only if x' is a linear combination of the x'_i. (Prove these assertions simultaneously by induction on n.)

2. The notation and hypotheses are the same as in Problem 1. Suppose also that there exists no vector $x \neq 0$ in E such that $\langle x, x' \rangle = 0$ for *all* $x' \in F$.

(a) Show that the weak topology on F is metrizable if and only if E has an at most denumerable basis (A. 4.1). (Observe that if $(y_i)_{1 \leqslant i \leqslant m}$ and $(z_j)_{1 \leqslant j \leqslant n}$ are two finite sequences of vectors in E such that the relation $\sup_{1 \leqslant i \leqslant m} |\langle y_i, x' \rangle| \leqq 1$ implies the relation $\sup_{1 \leqslant j \leqslant n} |\langle z_j, x' \rangle| \leqq 1$ in F, then the z_j are linear combinations of the y_i. Show that if we have $\langle y_i, x' \rangle = 0$ for $1 \leqq i \leqq m$, then we must also have $\langle z_j, x' \rangle = 0$ for $1 \leqq j \leqq n$, and apply the result of Problem 1 by identifying E with the set of linear forms $x' \mapsto \langle x, x' \rangle$ on F.)

(b) Deduce from (a) that, in an infinite-dimensional separable Hilbert space, the weak topology is not metrizable (cf. Section 5.9, Problem 2).

(c) Show that every linear form on F which is continuous with respect to the weak topology can be written $x' \mapsto \langle x, x' \rangle$ for a uniquely determined $x \in E$ (argue as in (a)).

3. (a) In a Hilbert space E, let A be a closed convex set (Section 12.14, Problem 11) and a a point of E. Show that there exists a unique point $b \in A$ such that $d(a, A) = d(a, b)$ (the *projection* of a on A). For each $z \in A$ show that $(z - b \mid b - a) \geqq 0$ (argue as in (6.3.1)).

(b) Deduce from (a) that every closed convex set A in E is the intersection of closed half-spaces whose frontiers are hyperplanes of support of A (Section 5.8, Problem 3). Deduce that A is weakly closed, and that if C is any convex subset of E, the strong and weak closures of C are the same. Every bounded closed convex subset of E is weakly compact.

(c) Show that the weak closure in E of the sphere $S : \|x\| = 1$ is the ball $\|x\| \leqq 1$. A subset of S is compact in the weak topology if and only if it is compact in the strong topology (use (12.15.12)).

(d) Let U be a nonempty convex open set in E. Show that, if a is any frontier point of U, there exists a closed hyperplane of support of U passing through a (*Hahn–Banach theorem* for a Hilbert space). (Consider a sequence (a_n) of points exterior to U, converging to a, and for each n let b_n be the projection of a_n on U defined in (a). Put $u_n = (b_n - a_n)/\|b_n - a_n\|$, and show that the sequence (u_n) has a subsequence which converges weakly to a point $u \neq 0$, by using the fact that U has an interior point c, and that this implies a relation of the form $(c - b_n \mid u_n) \geqq r > 0$ for all n.)

(e) Suppose that E is separable, and let $(e_n)_{n \geqslant 1}$ be basis (6.5.2) of E. Let A be the closed convex hull (Section 12.14, Problem 13) of the set of points $\pm e_n/n$. Show that A is compact and that the interior of A is empty, and that A has no closed hyperplane of support containing the (frontier) point 0, although there exist lines D passing through 0 and such that $D \cap A = \{0\}$.

4. Let E be a separable real Fréchet space and p a continuous seminorm on E.

(a) Let $(a_n)_{n\geqslant 0}$ be a total (12.13) free sequence in E, and let E_n be the subspace of dimension $n+1$ generated by $a_0, a_1, \ldots, a_n$. Show by induction on n that for each n there exists a linear form f_n on E_n such that f_n is the restriction of f_{n+1} to E_n, $f_0(a_0) = p(a_0)$ and $f_n(x) \leqq p(x)$ for all $x \in E_n$. (Consider the hyperplane H in E_{n-1} given by the equation $f_{n-1}(x) = f_{n-1}(a_0)$, and use Problem 3(d) in the plane which is the quotient of E_n by the hyperplane $H_0 : f_{n-1}(x) = 0$ in E_{n-1}.)

(b) Deduce from (a) that there exists a continuous linear form f on E such that $f(a_0) = p(a_0)$ and $|f(x)| \leqq p(x)$ for all $x \in E$ (*Hahn–Banach theorem* for a separable Fréchet space). In particular, if E is a separable Banach space, then for each $x \in E$ we have

$$\|x\| = \sup_{\|y'\| \leqq 1} |\langle x, y' \rangle|$$

(the norm on the dual E′ being that defined in (5.7.3)). Deduce that if E and F are separable Banach spaces, E′ and F′ their duals, and $u: E \to F$ a continuous linear mapping, then $\|{}^t u\| = \|u\|$.

(c) Deduce from (b) the geometrical form of the Hahn–Banach theorem: given any nonempty convex open set A in E, and a point $x_0 \notin A$, there exists a closed hyperplane with equation $g(x) - \alpha = 0$ such that $g(x) > \alpha$ for all $x \in A$, and $g(x_0) \leqq \alpha$. (Reduce to the case where $0 \in A$ and x_0 is a frontier point of A. We can then assume that A is symmetrical about 0 and therefore is the set of all x such that $p(x) < 1$, where p is a continuous seminorm on E.)

(d) Let A, B be two disjoint closed convex sets in E, one of which (say, A) is *compact*. Show that there exists a closed hyperplane H, with equation $g(x) - \alpha = 0$, which *strictly separates* A and B: that is to say, such that $g(x) > \alpha$ for all $x \in A$ and $g(x) < \alpha$ for all $x \in B$.

(e) Deduce from (d) that every closed convex set in E is an intersection of closed half-spaces, and that every closed linear variety in E is an intersection of closed hyperplanes.

(f) Deduce that for a vector subspace F *not* to be dense in E it is necessary and sufficient that there should exist a continuous linear form $f \neq 0$ on E such that $f(x) = 0$ for all $x \in F$.

5. Let A be a convex set in a real vector space. A point $a \in A$ is said to be an *extremal point* of A if there exists no line-segment containing more than one point, which is contained in A and has a as an interior point: in other words, if the relations $b \in A$, $c \in A$, $a = \lambda b + (1-\lambda)c$, $0 < \lambda < 1$ imply $b = c = a$.

(a) Let E be a Hilbert space, A a bounded closed convex set in E, $\delta > 0$ the diameter of A, and a, b two points of A such that $\|b - a\| \geqq \frac{1}{4}\sqrt{15} \cdot \delta$. Show that there exists a supporting hyperplane H of A orthogonal to the vector $b - a$ and such that the set $H \cap A$ has diameter $\leqq \frac{1}{2}\delta$.

(b) Deduce from (a) that A has at least one extremal point (proof by induction).

(c) Show that A is the closed convex hull of the set of its extremal points (*Krein–Milman theorem* for Hilbert space). (First deduce from (b) that every closed hyperplane of support of A contains at least one extremal point. Then argue by contradiction, assuming that the closed convex hull B of the set of extremal points of A is distinct from A; by using Problem 4(e), show that there would then exist a hyperplane of support of A which did not meet B.) (Cf. Section 13.10, Problem 8.)

(d) If C is a convex set in E, the intersection of C with a closed (resp. open) half-space is called a *closed* (resp. *open*) *cap* in C. If $a \in A$ is an extremal point, show that the open caps in A which contain a form a fundamental system of neighborhoods of a in A. (Remark that the convex hull of the union of two closed caps in A is weakly compact and does not contain a if neither of the two closed caps contains a; then use Problem 4(e), and Section 12.3, Problem 6(e).)

(e) Suppose that there exists a weakly compact set $K \subset A$ such that the closed convex hull of K is A. Show that every extremal point of A belongs to K (use (d)).

(f) In the Banach space (c_0) (Section 5.3, Problem 5) show that the closed ball $\|x\| \leqq 1$ has no extremal points.

6. Let E, F be two locally convex spaces. Let (p_α) be a family of seminorms defining the topology of F, and let $\mathfrak{S}$ be a set of *bounded* subsets of E (Section 12.14, Problem 6). For each continuous linear mapping u of E into F, and each $B \in \mathfrak{S}$, put $p_{\alpha,B}(u) = \sup_{x \in B} p_\alpha(u(x))$. Show that $p_{\alpha,B}$ is a seminorm on the vector space $\mathscr{L}(E; F)$ of continuous linear mappings of E into F. If the set of homothetic images of the sets of $\mathfrak{S}$ covers E, then the topology defined by the seminorms $p_{\alpha,B}$ on $\mathscr{L}(E; F)$ is Hausdorff. If E and F are normed spaces and if $\mathfrak{S}$ consists only of the closed unit ball $\|x\| \leqq 1$ in E, then the corresponding topology on $\mathscr{L}(E; F)$ is that defined by the norm (5.7.1).

7. Let E be an infinite-dimensional separable Hilbert space. Show that, with respect to the topology defined by the norm (5.7.1) on $\mathscr{L}(E; E)$ (this topology is called the *norm* topology), the unit ball $\mathscr{C}$ consisting of the operators U with norm $\|U\| \leqq 1$ (the "contractions" of E) is not separable. (Let $(e_n)_{n \geqq 0}$ be an orthonormal basis of E. For each point $z = (\zeta_n)_{n \geqq 0}$ of the Banach space l^∞ (Section 5.7, Problem 1), let $U_z \in \mathscr{L}(E; E)$ be the mapping defined by $U_z \cdot e_n = \zeta_n e_n$ for all $n \geqq 0$. Show that $z \mapsto U_z$ is an isometric linear mapping of l^∞ onto a subspace of $\mathscr{L}(E; E)$, and use the Problem of Section 5.10.)

8. Let E be a separable Hilbert space of infinite dimension. The *strong topology* on $\mathscr{L}(E; E)$ is that obtained by the procedure of Problem 6 by taking $\mathfrak{S}$ to be the set of all finite subsets of E, so that this topology is defined by the seminorms $p_x(U) = \|U \cdot x\|$ for all $x \in E$. The strong topology is coarser than the norm topology.

(a) Show that the strong topology is not metrizable. (Observe that $\mathscr{L}(E; E)$ contains vector subspaces which, when given the topology induced by the strong topology, are isomorphic to E endowed with the weak topology.)

(b) The unit ball $\mathscr{C}$ of $\mathscr{L}(E; E)$, endowed with the strong topology, is separable and metrizable. (Proceed as in (12.5.7), by embedding $\mathscr{C}$ in $E^{\mathbf{N}}$.)

(c) Show that the mapping $(U, V) \mapsto UV$ is continuous in the strong topology on $\mathscr{C} \times \mathscr{L}(E; E)$ and not continuous on $\mathscr{L}(E; E) \times \mathscr{C}$.

(d) Show that the mapping $U \mapsto U^*$ of $\mathscr{C}$ into $\mathscr{C}$ is not continuous with respect to the strong topology. (Let $(e_n)_{n \geqq 0}$ be an orthonormal basis of E, and consider the sequence of operators U_k such that $U_k \cdot e_n = 0$ if $n < k$ and $U_k \cdot e_n = e_{n-k}$ if $n \geqq k$.)

(e) Let $\mathbf{U}(E) \subset \mathscr{C}$ be the *unitary group* of E, consisting of all linear isometries of E onto E. Show that the topology induced on $\mathbf{U}(E)$ by the strong topology is compatible with the group structure. Give an example in this group of a right Cauchy sequence which is not a left Cauchy sequence. Show that $\mathbf{U}(E)$ is closed in $\mathscr{C}$.

9. Let E be a separable Hilbert space of infinite dimension. The *weak topology* on $\mathscr{L}(E; E)$ is that obtained by the procedure of Problem 6 by taking $\mathfrak{S}$ to be the set of

finite subsets of E, but taking E with the *weak* topology, so that this topology is defined by the seminorms $p_{x,y}(U) = |(U \cdot x|y)|$ for all x, y in E. This topology is coarser than the strong topology.

(a) Show that the weak topology is not metrizable (same method as in Problem 8(a)).

(b) The unit ball $\mathscr{C}$ in $\mathscr{L}(E; E)$ is metrizable and compact in the weak topology (method of (12.5.7)).

(c) The bounded sets in $\mathscr{L}(E; E)$ are the same for the weak topology as for the strong topology.

(d) The mapping $U \mapsto U^*$ of $\mathscr{L}(E; E)$ into itself is continuous with respect to the weak topology.

(e) Show that the mapping $(U, V) \to UV$ of $\mathscr{C} \times \mathscr{C}$ into $\mathscr{C}$ is not continuous for the weak topology. (If $(e_n)_{n \in \mathbf{Z}}$ is a Hilbert basis of E indexed by $\mathbf{Z}$, consider the "shift operator" U defined by $U \cdot e_n = e_{n+1}$ for all $n \in \mathbf{Z}$, and its powers U^n and U^{-n}.) The topology induced on the unitary group $\mathbf{U}(E)$ by the weak topology is identical with that induced by the strong topology, and hence is compatible with the group structure. Also $\mathbf{U}(E)$ is not closed in $\mathscr{C}$ for the weak topology.

(f) Show that, for each $U_0 \in \mathscr{L}(E; E)$, the linear mappings $V \mapsto U_0 V$ and $V \mapsto VU_0$ are continuous on $\mathscr{L}(E; E)$ for the weak topology.

(g) Let $\mathscr{M}$ be a submonoid of $\mathscr{C}$ (i.e., $\mathscr{M}$ is stable under multiplication). Show that the closure of $\mathscr{M}$ with respect to the weak topology on $\mathscr{L}(E; E)$ is a compact monoid, which is commutative if $\mathscr{M}$ is commutative.

10. The notation is the same as in Problem 9.

(a) Let $U \in \mathscr{C}$. Show that the relations $U \cdot x = x$, $(x|U \cdot x) = (x|x)$, and $U^* \cdot x = x$ are equivalent.

(b) Deduce that the idempotent operators in $\mathscr{C}$ are the *orthogonal projections* (Sections 6.3 and 11.5.)

11. Let E be a separable Hilbert space, $\mathscr{M}$ a submonoid containing the identity 1_E which is weakly closed (and therefore weakly compact) *in* $\mathscr{C}$ (notation of Problem 8). For each $x \in E$, the *orbit* of x under $\mathscr{M}$ is the set $\mathscr{M} \cdot x$ of vectors $U \cdot x$ where $U \in \mathscr{M}$; it is *weakly compact* in E. We say that x is a *flight* vector with respect to $\mathscr{M}$ if $0 \in \mathscr{M} \cdot x$, and that x is a *reversible* vector with respect to $\mathscr{M}$ if for each $U \in \mathscr{M}$ there exists $V \in \mathscr{M}$ such that $VU \cdot x = x$. Let $F(\mathscr{M})$ (resp. $R(\mathscr{M})$) denote the set of flight (resp. reversible) vectors with respect to $\mathscr{M}$. If x is reversible, then $\|U \cdot x\| = \|x\|$ for all $U \in \mathscr{M}$.

(a) Every orbit $\mathscr{M} \cdot x$ contains a minimal orbit N (Section 12.10, Problem 6), and every $y \in N$ is reversible with respect to $\mathscr{M}$. Let $U \in \mathscr{M}$ be such that $U \cdot x = y \in N$, and let $\mathscr{A}$ be the weakly closed submonoid generated by 1_E and U: this submonoid $\mathscr{A}$ is *commutative*. Let $P \subset \mathscr{A} \cdot y \subset N$ be a minimal orbit with respect to $\mathscr{A}$, and let $V \in \mathscr{A}$ be such that $V \cdot y = z \in P$. Show that there exists $W \in \mathscr{A}$ such that $VUW \cdot z = z$; deduce that the vector $x - W \cdot z$ is a flight vector, and hence that every $x \in E$ is the sum of a flight vector and a reversible vector belonging to $\mathscr{M} \cdot x$.

(b) Show that if $x \in E$ is reversible with respect to $\mathscr{M}$ then x is also reversible with respect to every weakly closed submonoid $\mathscr{N}$. (Split up x into the sum of a vector $y \in R(\mathscr{N})$ belonging to $\mathscr{N} \cdot x$ and a vector $z \in F(\mathscr{N})$; use the fact that if a, b are two distinct vectors in E with the same norm, then $\|\frac{1}{2}(a + b)\| < \|a\| = \|b\|$, and deduce that $y = x$.) Deduce that if $x \in R(\mathscr{M})$, then given any $U \in \mathscr{M}$ there exists an element V in the weakly closed submonoid of $\mathscr{M}$ generated by 1_E and U, such that $VU \cdot x = x$.

(c) Let $\mathscr{M}^*$ be the monoid consisting of the adjoints U^* of the operators then $\mathscr{M}^*$ is weakly compact (Problem 9(d)). Show that $R(\mathscr{M})$ and $F(\mathscr{M}^*)$ a *gonal supplementary vector subspaces* of E. (Use (b) to reduce to the case whe commutative.) Deduce that $F(\mathscr{M})$ is stable under $\mathscr{M}$.

(d) Let $(x_i)_{1\leqslant i\leqslant n}$ be a finite sequence of vectors in E. For each i, let $x_i =$ where $y_i \in R(\mathscr{M})$ and $z_i \in F(\mathscr{M})$. Use (c) to show that there exists $U \in \mathscr{M}$ s $U \cdot z_i = 0$ $(1 \leqq i \leqq n)$, and then prove that there exists $U \in \mathscr{M}$ such that VU for $1 \leqq i \leqq n$. (Consider the space E^n and the image $\mathscr{M}'$ of $\mathscr{M}$ in $\mathscr{L}(E^n; E^n)$ c of the operators $(t_1, \ldots, t_n) \mapsto (T \cdot t_1, \ldots, T \cdot t_n)$ where $T \in \mathscr{M}$, and use the R$(\mathscr{M})$ is a vector subspace.) Deduce that the set $P(x_1, \ldots, x_n)$ of operators such that $W \cdot x_i = W \cdot y_i = y_i$ for $1 \leqq i \leqq n$ is not empty, and then that t section of all these sets (where (x_i) varies over all finite sequences of vecto is nonempty (Section 12.3, Problem 6(e)). Finally, deduce that the p $P : E \to R(\mathscr{M})$ corresponding to the direct sum decomposition $E = R(M)$ belongs to $\mathscr{M}$ and hence to $\mathscr{G}$, and use Problem 10 to conclude that $R(\mathscr{M})$ *a are orthogonal supplementary vector subspaces* ("Jacobs' theorem").

(e) Show that for each $U \in \mathscr{M}$ there exists $V \in \mathscr{M}$ such that $VU \cdot y = y$ $y \in R(\mathscr{M})$ (same method as in (d)). In other words, the image $\mathscr{G}$ of $\mathscr{M}$ under t ping $U \mapsto U|R(\mathscr{M})$ is a *subgroup of the unitary group* $U(R(\mathscr{M}))$. Show that t of this subgroup are *strongly compact* in $R(\mathscr{M})$ (cf. Problem 3(c)) and deduce *strongly compact* in $\mathscr{L}(R(\mathscr{M}); R(\mathscr{M}))$.

(f) Let $(e_n)_{n\in\mathbf{Z}}$ be an orthonormal basis of E, and let $\mathscr{M}$ be the weakly submonoid of $\mathscr{G}$ generated by the unitary operator U defined by $U \cdot e_n =$ all $n \in \mathbf{Z}$. Show that $F(\mathscr{M}) = E$ and $R(\mathscr{M}) = \{0\}$.

12. In the notation of Problem 9, let $\mathscr{M}$ be any submonoid of $\mathscr{G}$, containing each $x \in E$, let $\Gamma(\mathscr{M} \cdot x)$ be the closed convex hull of the orbit of x under $\mathscr{M}$.

(a) Show that $\Gamma(\mathscr{M} \cdot x)$ contains a $\mathscr{M}$-invariant point (cf. Problem 3(a)). that, if $\Phi(\mathscr{M})$ denotes the set of $\mathscr{M}$-invariant vectors and $\Omega(\mathscr{M})$ the set of $x \in$ that $0 \in \Gamma(\mathscr{M} \cdot x)$, then every vector in E can be written as the sum of belonging to $\Phi(\mathscr{M})$ and a vector belonging to $\Omega(\mathscr{M})$.

(b) Show that $\Phi(\mathscr{M})$ and $\Omega(\mathscr{M}^*)$ are orthogonal supplementary vector su of E. Use Problem 10 to deduce that $\Phi(\mathscr{M})$ and $\Omega(\mathscr{M})$ are *orthogonal supple vector subspaces* of E. Deduce that $\Gamma(\mathscr{M} \cdot x)$ contains only one $\mathscr{M}$-invarian

(c) Deduce from (b) that, for each contraction $U \in \mathscr{C}$ and each vector x limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} U^k \cdot x$$

exists and is invariant under U ("von Neumann's ergodic theorem").

(a) Let E be a vector space over R or C, and let V be a vector space of linea on E, endowed with the weak topology. Let A be a weakly closed convex s such that $0 \notin A$. Show that there exists a weakly closed hyperplane H in V through 0, such that H is the set of all $f \in V$ for which $f(x_0) = 0$ for some $x_0 \neq$ and such that $g(x_0) \geqq 0$ for all $g \in A$. (Use Problem 1 to reduce to the case is finite-dimensional, by observing that there exists a neighborhood of 0 cont closed subspace W of finite codimension in V, and such that the projection of supplement L of W in V does not meet a neighborhood of 0 in L. Then apply t of Problem 3(a).)

(b) Deduce that, if A is a weakly closed convex set in E, and B is a weakly compact convex set disjoint from A, then there exists a weakly closed hyperplane $H = \{f \in V : f(x_0) = \alpha\}$ where $x_0 \neq 0$, such that $g(x_0) \geqq \alpha$ for all $g \in A$, and $h(x_0) \leqq \alpha$ for all $h \in B$. (Consider the convex set $A + (-B)$.)

(c) Deduce from (b) that for a vector subspace F *not* to be dense in V for the weak topology, it is necessary and sufficient that there should exist $x_0 \neq 0$ in E such that $f(x_0) = 0$ for all $f \in F$.

14. In a Hilbert space E, let (x_n) be a sequence which converges weakly to a point a. Show that, for each point $b \neq a$,

$$\liminf_{n\to\infty} \|x_n - b\| > \liminf_{n\to\infty} \|x_n - a\|.$$

(Expand $\|x_n - b\|^2 = \|(x_n - a) + (a - b)\|^2$.)

16. BAIRE'S THEOREM AND ITS CONSEQUENCES

(12.16.1) (Baire's theorem) *Let* E *be a topological space in which every point has a neighborhood homeomorphic to a complete metric space. If* (U_n) *is a sequence of dense open sets in* E, *then the intersection of the* U_n *is dense in* E.

It is enough to prove that, for each $x \in E$ and each neighborhood V of x, the intersection of V and the U_n is not empty. Hence we may assume that E itself is a complete metric space and (bearing in mind (3.14.5)) prove that the intersection G of the U_n is nonempty.

Let d be a distance defining the topology of E, with respect to which E is complete. We shall define by induction on n a sequence (x_n) of points of E and a sequence (r_n) of real numbers >0, as follows: $x_1 \in U_1$; $r_n < 1/n$ for each $n \geqq 1$; the closed ball $B'(x_n; r_n)$ is contained in $U_n \cap B'(x_{n-1}; r_{n-1})$. This is possible because $U_n \cap B(x_{n-1}; r_{n-1})$ is a *nonempty* open set in E, since U_n is dense in E. Clearly we have $d(x_n, x_{n+p}) \leqq r_n < 1/n$ for each $n \geqq 1$ and each $p > 0$, and therefore (x_n) is a Cauchy sequence in E. By hypothesis it converges to a point $a \in E$, and since $x_{n+p} \in B'(x_n; r_n)$ for each $p > 0$, we have $a \in B'(x_n; r_n) \subset U_n$ for each n since $B'(x_n; r_n)$ is closed in E. The point a therefore belongs to G. Q.E.D.

We shall apply Baire's theorem mainly when E is an *open* subspace of a *complete metric space* or a *locally compact metrizable space* (by virtue of (3.16.1)).

In a topological space E, a subset A is said to be *nowhere dense* if the open set $E - \bar{A}$ is dense (or, equivalently, if $\bar{A}$ *contains no nonempty open set*, or if $\bar{A}$ has *empty interior*). For example, in a Hausdorff space E, a set $\{a\}$ consisting

of a single point is nowhere dense unless it is open, i.e., unless a is *isolated* in E (3.10.10).

In a topological group G, every *closed subgroup* H *which is not open* is *nowhere dense* (12.8.7).

In a topological space E, a subset B is said to be *meager* if it is *the union of at most denumerably many nowhere-dense sets.* Since the closure of a nowhere-dense set is nowhere dense by definition, Baire's theorem can be restated in the form that, in a space E satisfying the conditions of (12.16.1), *the complement of a meager set is dense in* E.

For example, in a Fréchet space E, a denumerable union of *closed linear varieties*, each distinct from E, i.e., sets of the form $a_n + V_n$, where $V_n \neq E$ is a closed vector subspace of E, has a *dense complement* in E. For it follows from (12.13.1) that a vector subspace V of a topological vector space E cannot be open in E unless $V = E$. In particular, a *denumerable* set in a Fréchet space is meager, but it should be noted that such a set may also be *dense* (if the space is separable).

(12.16.2) *Let* E *be a topological space in which every point has a neighborhood homeomorphic to a complete metric space, and let u be a lower semicontinuous function on* E. *If* $u(x) < +\infty$ *for each* $x \in E$, *then given any nonempty open set* U *in* E *there exists a nonempty open set* $V \subset U$ *such that* $\sup_{x \in V} u(x) < +\infty$.

It is enough to prove the result when $U = E$. For each integer $n > 0$, let F_n be the set of all $x \in E$ such that $u(x) \leqq n$. By hypothesis, F_n is a *closed* set (12.7.2), and E is the union of the F_n; hence at least one of the F_n is not nowhere dense (12.16.1), and therefore contains a nonempty open set. Q.E.D.

Notice that under the hypotheses of (12.16.2) it can happen that

$$\sup_{x \in E} u(x) = +\infty.$$

Consider for example the real-valued function on $\mathbf{R}$ which is equal to 0 at $x = 0$, and equal to $1/x^2$ when $x \neq 0$.

In particular:

(12.16.3) *In a Fréchet space* E, *every lower semicontinuous seminorm is continuous.*

By definition, a seminorm p on E is *finite* at all points of E. If p is lower semicontinuous, it follows from (12.16.2) that there exists a point $x_0 \in E$, a

neighborhood V of 0 in E, and a real number $c > 0$ such that $p(x) \leqq c$ for all $x \in x_0 + V$. Hence

$$p(z) \leqq p(x_0) + p(x_0 + z) \leqq c + p(x_0)$$

for all $z \in V$, which proves the result (12.14.2).

(12.16.4) (Banach–Steinhaus theorem) *Let* E *be a Fréchet space,* F *a normed space. Let* H *be a set of continuous linear mappings of* E *into* F. *Suppose that* $\sup_{u \in H} \|u(x)\| < +\infty$ *for all* $x \in E$. *Then* H *is equicontinuous.*

The function $p(x) = \sup_{u \in H} \|u(x)\|$, being finite for all $x \in E$, is a *seminorm* on E (12.14.1). Since each of the functions $x \mapsto \|u(x)\|$ is continuous, p is lower semicontinuous (12.7.7). The result therefore follows from (12.16.3) and (12.15.7.1).

As a consequence, we have:

(12.16.5) (i) *Under the hypotheses of* (12.16.4), *let* (u_n) *be a sequence of continuous linear mappings of* E *into* F *which converges* simply *in* E *to a mapping* u *of* E *into* F. *Then* u *is a continuous linear mapping of* E *into* F.

(ii) *More generally, let* Z *be a metric space,* A *a subset of* Z, *and* $z \mapsto u_z$ *a mapping of* A *into the space of continuous linear mappings of* E *into* F. *Let* $z_0 \in Z$ *be a point in the closure of* A, *and suppose that the limit*

$$\lim_{z \to z_0,\, z \in A} u_z(x) = v(x)$$

exists in F, *for each* $x \in E$. *Then* v *is a continuous linear mapping of* E *into* F.

(i) From the hypotheses we have $\sup_n \|u_n(x)\| < +\infty$ for all $x \in E$, hence the Banach–Steinhaus theorem shows that the sequence (u_n) is equicontinuous, and the continuity of u follows from (7.5.5). The fact that u is linear is an immediate consequence of the principle of extension of identities.

(ii) The point z_0 is the limit of a sequence of points (z_n) in A (3.13.13), hence $v(x) = \lim_{n \to \infty} u_{z_n}(x)$ for all $x \in E$. Now apply (i).

(12.16.6) *Let* E *be a Fréchet space,* F *a Banach space,* I *an open interval in* **R**. *Let* $\mathscr{L}(E; F)$ *denote the space of continuous linear mappings of* E *into* F, *endowed with the topology of simple convergence* (12.15). *If a mapping* $t \mapsto f_t$ *of* I *into* $\mathscr{L}(E; F)$ *is differentiable with respect to the topology of simple convergence* (12.15), *then there exists a mapping* $t \mapsto f'_t$ *of* I *into* $\mathscr{L}(E; F)$ *such that* $f'_t(x) = D(f_t(x))$ *for all* $x \in E$.

For $D(f_t(x))$ is the limit, as $h \neq 0$ tends to 0, of the mapping

$$h \mapsto (f_{t+h} - f_t)/h,$$

and the result therefore follows from (12.16.5).

The mapping f'_{t_0} is said to be the *derivative* at the point t_0 of the m
$t \mapsto f_t$ (with respect to the topology of simple convergence), or the
derivative when F is the field of scalars (and therefore $\mathscr{L}(E; F) = E'$).

Remark

(12.16.6.1) Let A be an open set in **C** and let $z \mapsto f_z$ be a *weakly a*
mapping (12.15) of A into the dual E′ of a *complex* Fréchet space F. Th
same argument as in (12.16.6) shows that there exists a *weak derivative*
of $z \mapsto f_z$, with $f'_z(x) = D(f_z(x))$ for all $x \in E$, and this weak deriva
weakly analytic. Moreover, for each $a \in A$ and each circuit γ conta
$A - \{a\}$, we have the Cauchy formula

$$j(a; \gamma) f_a(x) = \frac{1}{2\pi i} \int_\gamma \frac{f_z(x)\, dz}{z - a}. \tag{12.16.6.2}$$

Conversely, let γ be a road in **C** defined in an interval $I = [b, c]$ in
let $z \mapsto g_z$ be a *weakly continuous* mapping of $\gamma(I)$ into E′. Then, for
the mapping

$$x \mapsto \int_\gamma \frac{g_\zeta(x)\, d\zeta}{\zeta - z} \tag{12.16.6.3}$$

is a linear form f_z on E, *belonging* to E′. Indeed, for each $x \in E$ the righ
side of (12.16.6.3) is the limit of a sequence

$$\frac{1}{n} \sum_{k=0}^{n} \frac{g_{\zeta_k}(x)\eta_k}{\zeta_k - z},$$

where

$$\zeta_k = \gamma\left(b + k\frac{c-b}{n}\right) \quad \text{and} \quad \eta_k = \gamma'\left(b + k\frac{c-b}{n}\right)$$

((3.16.5) and (8.7.8)), hence our assertion is a consequence of the B
Steinhaus theorem. By virtue of (9.9.2), f_z is *weakly analytic* in $\mathbf{C} - \gamma$
for $a \notin \gamma(I)$ we can write

$$f_z(x) = \sum_{n=0}^{\infty} c_n(x)(z - a)^n, \tag{12.16.6.4}$$

the series being convergent in any disk with center a not meeting $\gamma($

coefficients $c_n \in E'$ are given by

$$c_n(x) = \int_\gamma \frac{g_\zeta(x)\, d\zeta}{(\zeta - a)^{n+1}}.$$

Applying this result to the situation where γ is a circuit $t \mapsto a + re^{it}$ $(0 \leqq t \leqq 2\pi)$ in $A - \{a\}$ we obtain, for any weakly analytic function $z \mapsto f_z$ on A, the *Taylor expansion*

$$f_z(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f_a^{(n)}(x)(z - a)^n, \tag{12.16.6.5}$$

convergent in the disk $|z - a| < r$ (*independent of* $z \in E$), and the derivatives are given by

$$f_a^{(n)}(x) = \frac{n!}{2\pi i} \int_\gamma \frac{f_z(x)\, dz}{(z - a)^{n+1}}. \tag{12.16.6.6}$$

More generally, suppose that in $A - \{a\}$ we have

$$f_z = \sum_{k=1}^{m} \frac{x_k^*}{(z - a)^k} + g_z, \tag{12.16.6.7}$$

where the x_k^* are linear forms on E (*a priori*, not necessarily continuous), and $z \mapsto g_z$ is a *weakly analytic* function on $A - \{a\}$ which is *weakly bounded* in a neighborhood of a. Then, by (9.15), for each $x \in E$, the function $z \mapsto g_z(x)$ can be extended by continuity to the point a, and the extended function is *analytic* in A. Hence there exists a weakly analytic function $z \mapsto h_z$ *in* A whose restriction to $A - \{a\}$ is $z \mapsto g_z$. Furthermore, the x_k^* are *continuous* linear forms on E (i.e., they are elements of E'). Indeed, for each $x \in E$ we have

$$\langle x_k^*, x \rangle = \frac{1}{2\pi i} \int_\gamma (z - a)^{k-1} f_z(x)\, dz$$

where γ is a circuit $t \mapsto a + re^{it}$ $(0 \leqq t \leqq 2\pi)$ with r sufficiently small and *independent of* x (9.14); our assertion therefore follows from the considerations above (12.16.6.3).

Finally, we remark that the principle of analytic continuation (9.4.2) remains valid for weakly analytic functions. This is an immediate consequence of the definitions.

(12.16.7) *Let* E *be a Hilbert space and* u *an endomorphism of the (non-topological) vector space* E. *Then the following three conditions are equivalent*:

(a) u *is continuous*;

(b) u *is weakly continuous*;

(c) u *has an adjoint* (11.5).

We have already seen (12.15.11) that (a) implies (b). If u is weakly continuous, then for all $y \in E$ the linear form $x \mapsto (u(x) \mid y)$ is weakly continuous, and therefore *a fortiori* continuous with respect to the strong topology, which is finer than the weak topology. Consequently (6.3.2), for each $y \in E$ there exists a unique point $u^*(y)$ such that $(u(x) \mid y) = (x \mid u^*(y))$ for all $x \in E$: in other words (11.5), u has an adjoint, and therefore (b) implies (c). Finally, if u has an adjoint, then we have

$$\|u^*(y)\| = \sup_{\|x\| \leqq 1} |(x \mid u^*(y))| = \sup_{\|x\| \leqq 1} |(u(x) \mid y)|,$$

and each of the linear forms $y \mapsto (y \mid u(x)) = (u(x) \mid y)$ is continuous, hence $y \mapsto \|u^*(y)\|$ is a lower semicontinuous seminorm on E, and is therefore continuous in the strong topology (12.16.3). But this is equivalent to saying that u^* is (strongly) continuous, and therefore so is $u = u^{**}$ (11.5.2).

We remark that it is *essential* for the truth of (12.16.4), (12.16.5), (12.16.6), and (12.16.7) that the space E should be *complete* (cf. Problems 21 and 22).

(12.16.8) (Banach's theorem) *Let* E, F *be two Fréchet spaces and let* u *be a continuous linear mapping of* E *into* F. *Then either* $u(E)$ *is meager in* F, *or else* $u(E) = F$. *In the latter case, if* N *is the kernel of* u *and* $E \to E/N \xrightarrow{v} F$ *the canonical factorization of* u, *the mapping* v *is an isomorphism of the Fréchet space* E/N (12.14.9) *onto* F (in other words (12.12.7), u is a *strict morphism* of E onto F).

Suppose that $u(E)$ is not meager in F. By virtue of (12.13.1) and (12.12.7), the theorem will be proved if we can show that, for each neighborhood V of 0 in E, $u(V)$ is a *neighborhood* of 0 in F. There are two steps to the proof.

(12.16.8.1) *Let* E, F *be two topological vector spaces, and* u *a linear mapping of* E *into* F *such that* $u(E)$ *is not meager in* F. *Then for each neighborhood* V *of* 0 *in* E, *the closure* $\overline{u(V)}$ *of the image of* V *is a neighborhood of* 0 *in* F.

Let W be a balanced neighborhood of 0 in E such that $W + W \subset V$ ((12.13.1) and (12.8.3)). Then for each $x \in E$ there exists an integer $n \geqq 1$ such that $x \in nW$. Consequently $u(E)$ is the union of the sets $u(nW) = n \cdot u(W)$. Since $u(E)$ is not meager, at least one of the sets $n \cdot \overline{u(W)}$ has an interior point, and therefore so does $\overline{u(W)}$. But since $-u(W) = u(W)$, we have $-\overline{u(W)} = \overline{u(W)}$. So if y_0 is an interior point of $\overline{u(W)}$, then so is $-y_0$, and therefore $0 = y_0 + (-y_0)$ is an interior point of $\overline{u(W)} + \overline{u(W)}$ (12.8.2). But $\overline{u(W)} + \overline{u(W)}$ is contained in the closure of $u(W) + u(W) = u(W + W) \subset u(V)$, and so (12.16.8.1) is proved.

Now suppose that E, F are equipped with translation-invariant distances d, d', respectively, compatible with their topologies (12.9.1). By (12.16.8.1), the hypothesis that $u(E)$ is not meager in F implies that, for each $r > 0$, there exists $\rho = \rho(r) > 0$ such that $B'(0; \rho) \subset \overline{u(B'(0; r))}$. By translation it follows that for each $x \in E$ we have $B'(u(x); \rho) \subset \overline{u(B'(x; r))}$. Hence it is enough to prove the following lemma:

(12.16.8.2) *Let* E *be a complete metric space,* F *a metric space, and* u *a continuous mapping of* E *into* F *with the following property: for each* $r > 0$ *there exists* $\rho = \rho(r) > 0$ *such that* $B'(u(x); \rho) \subset \overline{u(B'(x; r))}$ *for all* $x \in E$. *Then* $B'(u(x); \rho) \subset u(B'(x; 2r))$ *for all* $x \in E$.

For each integer $n \geqq 1$, there exists by hypothesis a number $\rho_n > 0$ such that $B'(u(x); \rho_n) \subset \overline{u(B'(x; 2^{-n+1}r))}$ for *all* $x \in E$. We may take $\rho_1 = \rho$ and (replacing ρ_n if necessary by $\inf(\rho_n, 2^{-n})$) assume that $\lim_{n\to\infty} \rho_n = 0$. Let x_0 be any point of E, and let $y \in B'(u(x_0); \rho)$. We shall prove that $y \in u(B'(x_0; 2r))$.

For this we define inductively a sequence $(x_n)_{n\geqq 1}$ of points of E such that, for each $n \geqq 1$, we have $x_n \in B'(x_{n-1}; 2^{-n+1}r)$ and $u(x_n) \in B'(y; \rho_{n+1})$. Suppose that $x_1, \ldots, x_{n-1}$ have been chosen to satisfy these conditions. Then we have $y \in B'(u(x_{n-1}); \rho_n)$, and since $B'(u(x_{n-1}); \rho_n) \subset \overline{u(B'(x_{n-1}; 2^{-n+1}r))}$, there exists a point $x_n \in B'(x_{n-1}; 2^{-n+1}r)$ such that $u(x_n) \in B'(y; \rho_{n+1})$. So the inductive construction can continue.

Now the sequence (x_n) is a *Cauchy sequence* in E, because for each $n \geqq 0$ we have

$$d(x_n, x_{n+p}) \leqq 2^{-n}r + 2^{-n-1}r + \cdots + 2^{-n-p+1}r \leqq 2^{-n+1}r$$

for all $p > 0$. Since E is complete, the sequence converges in E to a point x such that $d(x_0, x) \leqq 2r$. Since u is continuous, we have $u(x) = \lim_{n\to\infty} u(x_n)$; and since $d'(u(x_n), y) \leqq \rho_{n+1}$, it follows that $\lim_{n\to\infty} u(x_n) = y$. Q.E.D.

This theorem has the following corollaries:

(12.16.9) *Let* E, F *be two Fréchet spaces. Then every continuous bijective linear mapping of* E *onto* F *is an isomorphism.*

In this proposition it is essential to assume that both E and F are *complete*. For example, if we take $E = l^2_{\mathbf{R}}$ (6.5) and F to be the canonical image of E under the identity mapping $E \to \mathscr{B}_{\mathbf{R}}(N)$ (7.1), endowed with the topology induced by that of $\mathscr{B}_{\mathbf{R}}(N)$, then this mapping is continuous because

$\sum_n \xi_n^2 \geqq \sup_n \xi_n^2$ (5.5.1), but the inverse bijection of F onto E is not continuous: otherwise F would be complete, and therefore closed in $\mathscr{B}_{\mathbf{R}}(\mathbf{N})$ (3.14.4), which is absurd, because its closure in $\mathscr{B}_{\mathbf{R}}(\mathbf{N})$ is the set of all sequences (ξ_n) such that $\lim_{n\to\infty} \xi_n = 0$.

(12.16.10) *Let* E, F *be two Banach spaces, and* u *a continuous surjective linear mapping of* E *to* F. *Then there exists a number* $m > 0$ *such that for each* $x \in E$ *there exists* $x' \in E$ *for which* $u(x) = u(x')$ *and* $\|u(x)\| \geqq m \cdot \|x'\|$.

Bearing in mind (12.14.8) and (5.5.1), this expresses that the bijection $F \to E/u^{-1}(0)$, the inverse of the bijection $E/u^{-1}(0) \to F$ induced by u, is continuous on F.

(12.16.11) (Closed graph theorem) *Let* E, F *be two Fréchet spaces. For a linear mapping* u *of* E *into* F *to be continuous, it is necessary and sufficient that the graph* (1.4) *of* u *should be closed in the product space* $E \times F$.

In general, if f is a continuous mapping of a topological space X into a Hausdorff topological space Y, the graph of f is closed in $X \times Y$, because it is the set of all $z \in X \times Y$ satisfying the relation $\mathrm{pr}_2 z = f(\mathrm{pr}_1 z)$, and the assertion follows from (12.3.5). To show that the condition stated in (12.16.11) is sufficient, remark that it implies that the graph G of u, being a closed vector subspace of the Fréchet space $E \times F$ (3.20.16(iv)), is a Fréchet space (3.14.5). The projection $z \mapsto \mathrm{pr}_1 z$ of G onto E is therefore a continuous *bijective* linear mapping, hence an isomorphism (12.16.9). Since the inverse mapping is $v : x \mapsto (x, u(x))$, it follows that $x \mapsto u(x) = \mathrm{pr}_2(v(x))$ is continuous on E.

The condition of (12.16.11) may also be expressed by saying that if a sequence $(x_n, u(x_n))$ in $E \times F$ tends to a point (x, y), then $y = u(x)$. Replacing x_n by $x_n - x$, and using the linearity of u, an equivalent formulation is that *if a sequence* (x_n) *in* E *tends to* 0 *and is such that the sequence* $(u(x_n))$ *tends to a limit* y, *then* $y = 0$. It is this criterion that we shall apply in practice to verify the continuity of u.

Finally, the following consequence of Baire's theorem allows us to use the criterion (12.11.5):

(12.16.12) *Let* G *be a separable, metrizable, locally compact group, acting continuously and transitively on a Hausdorff topological space* E *in which every point has a neighborhood homeomorphic to a complete metric space. For each* $x \in E$, *let* S_x *be the stabilizer of* x. *Then the canonical bijection* $f_x : G/S_x \to E$ *is a homeomorphism.*

Let $x_0 \in E$. We have to show that, for each neighborhood V of e in G, the set $V \cdot x_0$ is a neighborhood of x_0 in E (12.11.5). Let W be a *compact* symmetric neighborhood of e in G such that $W^2 \subset V$, and let (s_n) be a sequence which is dense in G, such that $(s_n W)$ is a covering of G. Each of the sets $s_n W \cdot x_0$ is closed in E, because $s \mapsto s \cdot x_0$ is continuous (12.3.6), and E is the union of the denumerable sequence of closed sets $s_n W \cdot x_0$. By Baire's theorem (12.16.1), there exists an index n such that $s_n W \cdot x_0$ has an interior point $s_n s \cdot x_0$, with $s \in W$. It follows (12.10.3) that x_0 is an interior point of the set

$$s^{-1} s_n^{-1}(s_n W \cdot x_0) = s^{-1} W \cdot x_0 \subset V \cdot x_0;$$

in other words, $V \cdot x_0$ is a neighborhood of x_0. Q.E.D.

(12.16.13) *Let* G *be a separable, metrizable, locally compact group, let* G′ *be a metrizable group, and let* $f : G \to G'$ *be a continuous surjective homomorphism. Then* f *is a strict morphism* (12.12.7) (in other words, if H is the kernel of f, then the canonical bijection $g : G/H \to G'$ is an *isomorphism of topological groups*).

For we may consider G′ as a space on which G acts continuously and transitively by the rule $(s, t') \mapsto f(s)t'$, and the stabilizer of the neutral element e' of G′ is H. The result therefore follows from (12.16.12).

PROBLEMS

1. Let E, F be two metric spaces, A a dense subspace of E, and f a continuous mapping of A into F. If F is *complete*, show that the set of points in E at which f has no limit relative to A (3.13) is *meager* in E. (For each n, consider the set of points $x \in E$ at which the oscillation of f with respect to A (3.14) is $> 1/n$.)

2. Let E, F be two complete metric spaces, and f a homeomorphism of a dense subspace A of E onto a dense subspace B of F. Show that there exists a subspace $C \supset A$ in E (resp. $D \supset B$ in F) which is a denumerable intersection of open sets, and an extension of f to a homeomorphism g of C onto D. (Apply Problem 1 to f and its inverse.)

3. Let E be a complete metric space, F a metric space, and (f_n) a sequence of continuous mappings of E into F which converges *simply* (pointwise) in E to a mapping f. Show that the set of points $x \in E$ at which f is not continuous is *meager* in E. (Let $G_{p,n}$ be the set of $x \in E$ for which the distance between $f_p(x)$ and $f_q(x)$ is $\leqq 1/2n$ for all $q \geqq p$. Show that the union of the interiors of the sets $G_{p,n}$ for $p \geqq 1$ is a dense open set, by using Baire's theorem. Deduce that the set of points at which the oscillation of f is $\leqq 1/n$ contains a dense open set.)

   Generalize to the case where it is assumed only that, for each n, the *restriction* of f_n to the complement of a meager set M_n (depending on n) is continuous.

4. Let E be a separable metrizable space. For each subset A of E, let D(A) denote t of points $x \in E$ such that, for each neighborhood V of x, the set $V \cap A$ is not m We have $D(A) \subset \bar{A}$. Show that $D(A) = \varnothing$ if and only if A is meager. Sho D(A) is closed and that $A \cap (E - D(A))$ is meager. Show that D(A) is the c of its interior. (If D'(A) is the closure of the interior of D(A), show $A \cap (E - D'(A))$ is meager.)

5. Let G be a separable metrizable group. Show that if H is a nonmeager subgroup then $\bar{H}$ is an open subgroup of G. (With the notation of Problem 4, show th interior of D(H) is not empty.)

6. If a complete metrizable group is denumerable, then it is discrete.

7. Let G be a separable, metrizable, locally compact group, and let H be a closed n subgroup of G. Show that the canonical bijection $A/(A \cap H) \to AH/H$ is a morphism of topological groups.

8. Let E, F, G be three metric spaces and d, d', d'' the distances on E, F, G, respec Let f be a mapping of $E \times F$ into G such that for each $x_0 \in E$ the mapping $y \mapsto f($ is continuous on F, and for each $y_0 \in F$ the mapping $x \mapsto f(x, y_0)$ is continuous (a) For each $\varepsilon > 0$, each $b \in F$ and each $x \in E$, let $g(x; b, \varepsilon)$ be the least upper b of the set of numbers $\alpha > 0$ such that $d'(b, y) \leqq \alpha$ implies $d''(f(x, b), f(x, y)$ Show that the mapping $x \mapsto g(x; b, \varepsilon)$ is upper semicontinuous.
(b) If E is complete, deduce from (a) that for each $b \in F$ there exists a *meag* $M_b \subset E$ such that f is *continuous* at (a, b) for all $a \notin M_b$.
If F also is complete, deduce that there exists a *meager* set N in $E \times F$ suc the restriction of f to $(E \times F) - N$ is *continuous*. (For each k, consider the set of at which the oscillation of f is $< 1/k$, and use Problem 4 of Section (12.7).
(c) If E, F, G are Fréchet spaces and f is a *bilinear* mapping of $E \times F$ into G that the linear mappings $f(x_0, \cdot)$ and $f(\cdot, y_0)$ are continuous for all $x_0 \in E$ a $y_0 \in F$, show that f is *continuous* on $E \times F$.

9. Let G be a group and d a distance on G defining a topology for which G is a com separable, locally compact, metric space. Suppose that for each $x_0 \in G$ the map $y \mapsto x_0 y$ and $y \mapsto y x_0$ are continuous on G. Show that the topology defined b compatible with the group structure of G. (First prove that the mapping (x, y) is continuous, by using Problem 8. Then consider the set F in $G \times G$ consisting (x, x^{-1}) with $x \in G$. Show that F, with the law of composition $(x, x^{-1})(y, y$ $(xy, y^{-1}x^{-1})$ and the topology induced by the product topology on $G \times G$ topological group and is closed in $G \times G$. Finally show that the restriction to F is bicontinuous, by using (12.16.12), and deduce that the mapping $x \mapsto$ continuous on G.)

10. Let T be a metric space, E a Fréchet space, and M a set of mappings of $E \times T$ metrizable topological vector space F, satisfying the following conditions: (each $t_0 \in T$, the set of mappings $f(\cdot, t_0)$, where $f \in M$, is an equicontinuous *linear* mappings of E into F; (2) for each $x_0 \in E$, the set of mappings $f(x_0, \cdot)$, $f \in M$, is equicontinuous. Show that M is equicontinuous. (Given $t_0 \in T$ balanced closed neighborhood V of 0 in F, for each $x \in E$ let d_x be the least bound of radii of open balls with centre t_0 in T, such that for every point t of s ball we have $f(x, t) - f(x, t_0) \in V$ for all $f \in M$. Show that the mapping $x \mapsto$ upper semicontinuous on E (argue by contradiction); then use (12.16.2).)

11. Let E, F be two Fréchet spaces, G a Banach space, and (f_n) a sequence of continuous bilinear mappings of $E \times F$ into G. Suppose that, for each $(x, y) \in E \times F$, the sequence $(f_n(x, y))$ is bounded in G. If D is a dense subset of $E \times F$ and the sequence $(f_n(x, y))$ converges for all $(x, y) \in D$, show that the sequence (f_n) converges simply in $E \times F$ and that the limit function is continuous. (Use Problem 10 and (7.5.5).)

12. Let E be the set of all sequences $x = (\xi_n)_{n \geq 0}$ of real numbers such that the series with general term ξ_n is convergent, and put $\|x\| = \sup_n \left| \sum_{k=0}^{n} \xi_k \right|$.

(a) Show that $\|x\|$ is a norm on E and that E is complete with respect to this norm.

(b) The Banach space l^1 (Section (5.7), Problem 1) is dense in E with respect to the norm $\|x\|$. If l^1 is endowed with its norm $\|x\|_1 = \sum_{n=0}^{\infty} |\xi_n|$, then the canonical injection $l^1 \to E$ is continuous.

(c) Let (P_n) be an increasing sequence of finite subsets of $N \times N$, forming a covering of $N \times N$. For each $x = (\xi_n) \in E$ and each $y = (\eta_n) \in l^1$, let $f_n(x, y) = \sum_{(i, j) \in P_n} \xi_i \eta_j$. Show that the sequence (f_n) converges simply in $E \times l^1$ if and only if, for each $(x, y) \in E \times l^1$, the sequence $(f_n(x, y))$ is bounded, and that the limit of $(f_n(x, y))$ is then equal to $\left(\sum_{n=0}^{\infty} \xi_n\right)\left(\sum_{n=0}^{\infty} \eta_n\right)$ for all $(x, y) \in E \times l^1$. (Use Problem 11 with $D = l^1 \times l^1$.)

(d) For each $j \in N$, let ρ_{jn} be the smallest number of closed intervals in N whose union is the section $P_n^{-1}(j)$ (the projection of $P_n \cap (N \times \{j\})$). Let $\rho_n = \sup_{j \in N} \rho_{jn}$.

Show that the condition in (c) is equivalent to $\sup_n \rho_n < +\infty$. (If φ_{P_n} is the characteristic function of P_n, show that the norm of the bilinear form f_n (5.7.7) is equal to

$$\sup_{j \in N} \left(\sum_{i=0}^{\infty} |\varphi_{P_n}(i, j) - \varphi_{P_n}(i+1, j)| \right).$$

("Theorem of Mertens–Alexiewicz").

13. The Banach space l^1 (Section 5.7, Problem 1) is algebraically a vector subspace of the space l^2 (6.5). If $\|x\|_1$ and $\|x\|_2$ are the norms on l^1 and l^2 respectively, we have $\|x\|_2 \leqq \|x\|_1$ for all $x \in l^1$. Show that in the space l^2, the unit ball $B : \|x\|_1 \leqq 1$ of l^1 is not closed with respect to the norm $\|x\|_2$, although the intersection of B with every finite-dimensional linear variety is closed (use (12.16.3)).

14. Let E be a Fréchet space and F a normed space. If H is a set of continuous linear mappings of E into F which is not equicontinuous, show that the set of $x \in E$ such that H(x) (the set of all $u(x)$ with $u \in H$) is not bounded in F is the complement of a meager set. Deduce that if (F_n) is a sequence of Banach spaces and if, for each n, H_n is a subset of $\mathscr{L}(E; F_n)$ which is not equicontinuous, then there exists $x_0 \in E$ such that *none* of the sets $H_n(x_0)$ is bounded ("principle of condensation of singularities").

15. Let $(a_{ij})_{i \leqq j}$ be a family of points of the interval $I = [0, 1]$ in R. For each integer $n \geqq 0$, let $\omega_n(x) = \prod_{i=0}^{n} (x - a_{in})$, and suppose that the roots of this polynomial are all distinct.

Let

$$q_{in}(x) = \omega_n(x)/(\omega_n'(a_{in})(x - a_{in}))$$

for $0 \leqq i \leqq n$. For each real-valued function f defined on I, the *Lagrange interpo polynomial* of f corresponding to (a_{ij}) is the polynomial

$$P_n(f) = \sum_{i=0}^{n} f(a_{in}) q_{in}.$$

(a) The mapping $f \mapsto P_n(f)$ is a continuous linear mapping of the Banach $\mathscr{C}(I)$ into itself. The sequence of polynomials $P_n(f)$ converges uniformly to f for *all* $f \in \mathscr{C}(I)$ if and only if the sequence of norms $\|P_n\|$ of these linear mappi bounded (use (12.16.5) and (7.5.5)). Show that

$$\|P_n\| = \sup_{x \in I} \left(\sum_{i=0}^{n} |q_{in}(x)| \right).$$

(b) Take $a_{in} = i/n$ for $0 \leqq i \leqq n$. Show that the sequence $(\|P_n\|)$ tends to Deduce that, for this choice of the a_{in}, there exists a function $f_0 \in \mathscr{C}(I)$ such tha sequence of polynomials $P_n(f_0)$ is not uniformly bounded on I (use Problem 14)

16. Let f be a continuous real-valued function on an open interval I in **R**, having a r hand derivative f'_d (8.4) at every point of I.
(a) Show that the set of points $x \in I$ such that f'_d is bounded in some neighbor of x is a dense open set in I (use (12.16.2)).
(b) Show that the set of points $x \in I$ at which f'_d is continuous is the compleme a meager set in I (cf. Problem 3). Deduce that f has a derivative at all points c complement of a meager set in I (cf. Section 8.6, Problem 2).

17. (a) In the Banach space $\mathscr{C}(I)$, where $I = [0, 1]$, let A_n be the set of functions f that for some $x \in 0, 1 - 1/n[$, depending on f, we have $|f(x') - f(x)| \leqq n|x'$ for all $x' \in]x, x + 1/n[$. Show that A_n is nowhere dense in $\mathscr{C}(I)$. (Notice first th $\mathscr{C}(I)$ every ball contains a function which has a bounded right-hand derivative points of I. Secondly, observe that for each $\varepsilon > 0$ and each integer m, there exi function $g \in \mathscr{C}(I)$ with a right-hand derivative at every point of $[0, 1[$ and such $|g(x)| \leqq \varepsilon$ and $|g'_d(x)| \geqq m$ at all $x \in [0, 1[$.)
(b) Deduce from (a) that the set A of functions $f \in \mathscr{C}(I)$ such that, for at leas point $x \in [0, 1[$ (depending on f), f has a right-hand derivative at x, is *meager* in Consequently, the complement of this set, that is to say the set of all $f \in \mathscr{C}(I)$ ha no right-hand derivative at *any* point of $[0, 1[$, is *dense* in $\mathscr{C}(I)$ (cf. Se 8.4, Problem 1).

18. Let f be a real-valued function defined on an open set $A \subset \mathbf{R}^2$, such that for each $(x_0, y_0) \in A$ the function $f(\cdot, y_0)$ is continuous at x_0 and the function $f(x_0, \cdot)$ continuous and differentiable at the point y_0. Show that there exists a meager s in A such that the derivative $D_2 f$ is *continuous* on $A - M$ (use Problem 8(b) Problem 3).

19. Let E be a complex Banach space and E′ its dual, which is a Banach space with re to the norm (5.7.1). Show that a weakly analytic mapping $z \mapsto f_z$ of an ope $A \subset \mathbf{C}$ into E′ is in fact *analytic* in the sense of (9.3) (with respect to the norm u consideration). (Use the formula (12.16.6.6) and the Banach–Steinhaus theorem.)

20. Let Z be a complete metric space and f a *weakly continuous* mapping of Z into a separable Hilbert space E.
(a) If A is any closed convex set in E, show that $f^{-1}(A)$ is closed in Z (Cf. Section 12.15, Problem 3(b)).
(b) For each integer $n > 1$, let D_n be the set of points $z \in Z$ such that the oscillation of the function f (considered as a mapping of a metric space into a metric space) is $\geqq 1/n$. The set D_n is closed (Section 12.7, Problem 4). Show that D_n is meager in Z. (Cover E by a sequence (S_k) of closed balls of diameter $\leqq 1/2n$, and use (a) and Baire's theorem to show that $D_n \cap f^{-1}(S_k)$ is a nowhere dense closed set in Z, for each k.)
(c) Deduce from (a) that there exists a meager set M in Z such that f is *strongly continuous* at every point of $Z - M$.

21. Let $E = \mathscr{K}(\mathbf{R})$ be the subspace consisting of continuous functions with compact support, in the Banach space $\mathscr{B}(\mathbf{R})$ of real-valued bounded functions on $\mathbf{R}$. Let u_n be the continuous linear form $f \mapsto \sum_{k=1}^{n} f(k)$ on E. Show that the sequence (u_n) converges simply on E but is not bounded in the dual E', and that its limit u is not continuous on E.

22. Let E be the subspace consisting of sequences (ξ_n) with finite support, in the Hilbert space l^2, so that E is the set of all (finite) linear combinations of the vectors e_n in l^2 (6.5). Let u be the endomorphism of E for which $u(e_n) = ne_n$ for all n. Show that, for every $y \in E$, the linear form $x \mapsto (u(x)|y)$ is continuous on E, but that u is not a continuous endomorphism of E.

23. Let E be the subset of $\mathbf{R}^2$ consisting of the line $D = \{0\} \times \mathbf{R}$ and the points $(1/n, k/n^2)$, where n and k belong to $\mathbf{Z}$ and $n > 0$.
(a) For every point $(0, y) \in E$ and every integer $n > 0$, let $T_n(y)$ be the set of points $(u, v) \in E$ such that $u \leqq 1/n$ and $|v - y| \leqq u$. Show that if we take the sets $T_n(y)$ $(n \geqq 1)$ as a fundamental system of neighborhoods of $(0, y)$, for each $y \in \mathbf{R}$, and if for each other point z of E we take the set consisting of that point alone as a fundamental system of neighborhoods of z, then we have a Hausdorff topology $\mathscr{T}$ on E, for which each subspace $T_n(y)$ is metrizable and compact. The topology induced by $\mathscr{T}$ on D is the discrete topology.
(b) Let A be the set $\{0\} \times \mathbf{Q}$, which is closed in D and whose complement B in D is also closed. If a set U which is open with respect to $\mathscr{T}$ contains B, show that there exists an interval $\{0\} \times [a, b]$ in D (with $a < b$) and an integer n such that U contains the union of the sets $T_n(y)$ for $y \in [a, b]$ and $(0, y) \in B$. (Use the fact that $\mathbf{Q}$ is a meager subset of $\mathbf{R}$ in the usual topology.) Deduce that every neighborhood of A in E meets U, and hence that the space E is not metrizable, although it contains a discrete denumerable dense subset (use (4.5.2)).

24. Let (a_n) be a sequence of real numbers. Suppose that, for each sequence (ξ_n) of real numbers belonging to the space l^1 (Section 5.7, Problem 1) (resp. to the space l^2 (6.5)), the series with general term $a_n\xi_n$ is convergent. Prove that the sequence (a_n) belongs to l^∞ (Section 5.6, Problem 1) (resp. belongs to l^2). (For each integer N, consider the continuous linear form $x \mapsto \sum_{n=1}^{N} a_n \xi_n$ on l^1 (resp. l^2), and apply the Banach–Steinhaus theorem.)

25. (a) Give an example of a sequence (f_n) of continuous real-valued functions on an interval $I \subset \mathbf{R}$ which converges simply at each point of I but is such that there exists no nonempty open interval contained in I on which the sequence converges uniformly (cf. (12.7.1)).
(b) Let (f_n) be a sequence of complex *analytic* functions on an open set $D \subset \mathbf{C}$ which converges simply at each point of D. Show that there exists a dense open set U in D such that, for each $z_0 \in U$, there exists a neighborhood of z_0 on which the sequence (f_n) is uniformly convergent. (Consider the sequence of functions $z \mapsto |f_n(z)|$ and use (9.13.2).)

26. Let $t \mapsto A(t)$ be a continuous mapping of the interval $\mathbf{R}_+ = [0, +\infty[$ of $\mathbf{R}$ into the space of real $n \times n$ matrices, and let $t \mapsto C(t)$ be the solution of the homogeneous linear differential equation $U' = A(t)U$ which is equal to the unit matrix when $t = 0$ (10.8.4). Let E_1 denote the vector subspace of $\mathbf{R}^n$ consisting of vectors y such that the function $t \mapsto C(t) \cdot y$ is bounded on $[0, +\infty[$, and let E_2 be a supplement of E_1 in $\mathbf{R}^n$. Let $P_1 : \mathbf{R}^n \to E_1$ and $P_2 : \mathbf{R}^n \to E_2$ be the projections associated with the direct-sum decomposition $\mathbf{R}^n = E_1 \oplus E_2$ (5.4).
(a) Suppose that, for each function $f \in \mathscr{C}^\infty_{\mathbf{R}^n}(\mathbf{R}_+)$, the differential equation $x' = A(t) \cdot x + f(t)$ has at least one solution which is *bounded* on $\mathbf{R}_+$. Show that, for each function $f \in \mathscr{C}^\infty_{\mathbf{R}^n}(\mathbf{R}_+)$, there exists a unique vector $y(f) \in E_2$ such that the unique solution u_f of the equation $x' = A(t) \cdot x + f(t)$ which takes the value $y(f)$ at the point 0 is *bounded* in $\mathbf{R}_+$, and that there exists a constant $c > 0$ such that

$$\|u_f\| \leqq c \cdot \|f\|.$$

(Consider the space F of continuously differentiable functions x on $\mathbf{R}_+$ with values in $\mathbf{R}^n$ such that x and $x' - A \cdot x$ are bounded on $\mathbf{R}_+$ and $x(0) \in E_2$. Show that the function

$$N(x) = \|x\| + \|x' - A \cdot x\|$$

is a norm on F, with respect to which F is a Banach space. Then apply Banach's theorem to the linear mapping $x \mapsto x' - A \cdot x$ of F into $\mathscr{C}^\infty_{\mathbf{R}^n}(\mathbf{R}_+)$, bearing in mind the definition of E_2.)
(b) Deduce from (a) that a necessary and sufficient condition for the equation $x' = A(t) \cdot x + f(t)$ to have a solution which is bounded on $\mathbf{R}_+$, for all functions $f \in \mathscr{C}^\infty_{\mathbf{R}^n}(\mathbf{R}_+)$, is that

$$\sup_{t \geqslant 0} \left(\int_0^t \|C(t)P_1 C^{-1}(s)\| \, ds + \int_t^{+\infty} \|C(t)P_2 C^{-1}(s)\| \, ds \right) < +\infty$$

(cf. (10.8.6) and (13.14.4)).

27. Let $f : [a, b[\to \mathbf{R}$ be an increasing function, continuous on the right. Suppose that the countable set of points of discontinuity of f is dense in $[a, b[$ (Section 3.15, Problem). Show that the set of points $x \in [a, b[$ at which f has a finite derivative on the right is meager. (For each integer $n > 0$, let A_n be the set of points $x \in [a, b[$ such that there exist points $y, z \in [a, b[$ satisfying $x < y < z < x + 1/n$ and

$$\left| \frac{f(y) - f(x)}{y - x} - \frac{f(z) - f(x)}{z - x} \right| \geqq 1.$$

Show that A_n is a dense open subset of $[a, b[$.)

28. Let f be an indefinitely differentiable real-valued function defined on an open interval $]a, b[$ of $\mathbf{R}$. Suppose that for each point $x \in]a, b[$ there exists an integer $N(x)$ such that $D^{N(x)}f(x) = 0$. Show that f is a polynomial. (One may proceed as follows:

(a) Let G be the open set of points $x \in]a, b[$ such that in some neighborhood of x the function f coincides with a polynomial, and let F be the complement of G in $]a, b[$. Show that F has no isolated points (3.10.10).

(b) For each integer n, let E_n be the closed subset of F consisting of the points $x \in F$ such that $D^n f(x) = 0$. If F is not empty, show that there exists a nonempty open interval $I \subset]a, b[$ and an integer N such that $F \cap I$ is nonempty and is contained in E_N (use Baire's theorem). Then deduce from (a) that $F \cap I \subset E_n$ for all $n > N$.

(c) Deduce from (b) that $F \cap I$ is nowhere dense in I, and then that $D^N f(x) = 0$ on every component interval of $G \cap I$. Hence derive a contradiction of the hypothesis $F \neq \varnothing$.)

29. Let E, F be two separable Banach spaces, F′ the dual of F. Suppose that F′ is contained in a Hausdorff locally convex space G, and that the topology on F′ induced by that of G is coarser than the weak topology of F′. Let $u : E \to G$ be a continuous linear mapping.

(a) Show that, for every ball B in the Banach space F′, the inverse image $u^{-1}(B)$ is a closed subset of E. (Use (12.15.9) and (12.15.8.1).)

(b) Suppose that there exists a nonmeager subset A of E such that $u(x) \in F'$ for all $x \in A$. Show that $u(E) \subset F'$ and that u is continuous for the topology defined by the norm on F′.

30. Let E be a Fréchet space, and let (u_α) be a family of continuous linear mappings of E into a normed space F. Suppose that there exists a nonmeager subset A of E such that for each $x \in A$ the set of points $u_\alpha(x)$ is bounded in F. Show that the family (u_α) is equicontinuous. (For each integer $n \geqq 1$, consider the set of points $x \in E$ such that $\|u_\alpha(x)\| \leqq n$ for each index α.)

CHAPTER XIII

INTEGRATION

The theory of integration which we shall develop in this chapter is restricted to *separable metrizable locally compact spaces*, this being sufficient for our purposes in later chapters. We have followed fairly closely the exposition of N. Bourbaki [22], with the simplifications afforded by our more restricted hypotheses.

The key results of the theory of integration are Lebesgue's convergence theorems (13.8), the Fischer–Riesz theorem (13.11.4), the Lebesgue–Nikodym theorem (13.15.5) and the Lebesgue–Fubini theorem (13.21.7). Unfortunately it is necessary to include rather a lot of material on upper integrals, measurable functions and negligible functions, which are indispensable technical tools. The important properties of certain particular measures on locally compact groups or on differential manifolds will be examined in Chapters XIV and XVI.

We have also included, amongst the problems, applications of integration which are not dealt with in the text, especially to ergodic theory and orthogonal systems. The reader who wishes to go further in these directions should consult [21], [26a], [28], [30], and [31].

Nowadays the purposes of a theory of integration are very different from what they were at the beginning of this century. If the aim was only to be able to integrate "very discontinuous" functions, integration would hardly have gone beyond the rather narrow confines of the "fine" theory of functions of one or more real variables. The reasons for the importance that Lebesgue's concept of integral has acquired in modern analysis are of quite a different nature. One is that it leads naturally to the consideration of various new *complete* function spaces, which can be conveniently handled precisely because they are spaces of *functions* (or of classes of "equivalent" functions) and not just abstract objects, as is usually the case when one constructs the completion of a space. Another is that the theorem of Lebesgue–Nikodym

and the properties of measures defined by densities (13.15) give us a method for dealing with denumerable families of *measures* on a locally compact space, by fixing a basic measure and working with the *densities* relative to this basic measure (hence again with *functions*); this again proves to be extremely convenient. Here the modern point of view emerges: given a μ-integrable function f, what is important is not the *values* taken by f so much as the way in which f operates on the space of bounded continuous functions by means of the linear mapping $g \mapsto \int fg\, d\mu$ (this mapping depends only on the equivalence class of f and therefore does not change when we modify f at the points of a set of measure zero). The development of this point of view will lead in Chapter XVII to the theory of distributions, which is a natural generalization of the notion of measure on differential manifolds.

Throughout this chapter, the phrase "locally compact space" will always mean "separable metrizable locally compact space."

1. DEFINITION OF A MEASURE

To begin with, let X be a *compact* (metrizable) space. A *measure* (or *complex measure*) on X is by definition an element of the *dual* of the Banach space $\mathscr{C}_{\mathbf{C}}(X)$ of complex-valued continuous functions on X (7.2), that is to say ((12.15) and (5.5.1)) it is a *linear form* $f \mapsto \mu(f)$ *on* $\mathscr{C}_{\mathbf{C}}(X)$ *which satisfies an inequality of the form*

$$|\mu(f)| \leqq a \|f\| \tag{13.1.1}$$

for all $f \in \mathscr{C}_{\mathbf{C}}(X)$ (recall that $\|f\| = \sup_{x \in X} |f(x)|$).

Now let X be a *locally compact space* (metrizable and separable, in accordance with our conventions). For every *compact* subset K of X, let $\mathscr{K}(X; K)$ (or $\mathscr{K}_{\mathbf{C}}(X; K)$) denote the vector subspace of $\mathscr{C}_{\mathbf{C}}(X)$ consisting of the functions whose *support* (12.6) is *contained in* K (and is therefore compact). We shall denote by $\mathscr{K}_{\mathbf{C}}(X)$ (or $\mathscr{K}(X)$) the union of the $\mathscr{K}_{\mathbf{C}}(X; K)$ as K runs through all compact subsets of X. In other words, $\mathscr{K}_{\mathbf{C}}(X)$ is the vector space of (complex-valued) *continuous functions with compact support*. Clearly $\mathscr{K}_{\mathbf{C}}(X) \subset \mathscr{C}^{\infty}_{\mathbf{C}}(X)$.

A *measure* (or *complex measure*) on X is by definition a *linear form* μ *on* $\mathscr{K}_{\mathbf{C}}(X)$ with the following property: for each compact subset K of X, there exists a real number $a_K \geqq 0$ (in general depending on K) such that

$$|\mu(f)| \leqq a_K \|f\| \tag{13.1.2}$$

for all $f \in \mathscr{K}(X; K)$.

This definition agrees with the preceding one when X is compact. It expresses that the *restriction* of μ to $\mathscr{K}(X; K)$ is *continuous* with respect to the topology induced by that of $\mathscr{C}^0_{\mathbf{C}}(X)$. We remark that $\mathscr{K}(X; K)$ is *closed* in $\mathscr{C}^0_{\mathbf{C}}(X)$, and therefore a *Banach space* (3.14.5).

In general, a measure is *not necessarily continuous on* $\mathscr{K}_{\mathbf{C}}(X)$ with respect to the topology induced by that of $\mathscr{C}^0_{\mathbf{C}}(X)$ (i.e., the topology defined by the norm $\|f\|$). We shall examine this question later (13.20).

Examples of Measures

(13.1.3) Let X be a locally compact space, and let $x \in X$. The mapping $f \mapsto f(x)$ of $\mathscr{K}(X)$ into $\mathbf{C}$ is a *measure*, for it is linear and we have $|f(x)| \leqq \|f\|$ for each compact subset K of X such that $f \in \mathscr{K}(X; K)$. This measure is called the *Dirac measure* at the point x, or the measure defined by the *unit mass at the point* x, and is denoted by ε_x.

More generally, let (a_n) be a sequence of distinct points in X, and (t_n) a sequence of complex numbers, such that for each compact $K \subset X$ *the subsequence formed by the* t_n *for which* $a_n \in K$ *is absolutely summable* (5.3). Let $c_K = \sum_{a_n \in K} |t_n|$. Then, for each function $f \in \mathscr{K}(X; K)$, the series $\sum_n t_n f(a_n)$ is absolutely convergent, because the only nonzero terms are those for which $a_n \in K$, and we have

$$\sum_{a_n \in K} |t_n f(a_n)| \leqq \|f\| \cdot \sum_{a_n \in K} t_n = c_K \cdot \|f\|.$$

This also shows that $f \mapsto \sum_n t_n f(a_n)$ is a *measure* on X. This measure is said to be defined by *the masses* t_n *at the points* a_n for all n (cf. (13.18.8)).

(13.1.4) Let $f \in \mathscr{K}_{\mathbf{C}}(\mathbf{R})$. For each interval $[a, b]$ containing the support of f, the value of the integral $\int_a^b f(t)\,dt$ (8.7) is the same, and we denote it by $\int_{-\infty}^{+\infty} f(t)\,dt$. The mapping $f \mapsto \int_{-\infty}^{+\infty} f(t)\,dt$ is a linear form on $\mathscr{K}_{\mathbf{C}}(\mathbf{R})$, and it is a *measure* because, for each compact interval $K = [a, b]$ in $\mathbf{R}$ and each function $f \in \mathscr{K}(\mathbf{R}; K)$ we have

$$\left| \int_{-\infty}^{+\infty} f(t)\,dt \right| \leqq (b - a)\|f\|$$

by the mean value theorem (8.7.7). This measure is called *Lebesgue measure* on the real line $\mathbf{R}$.

(13.1.5) Let μ be a measure on X and let $g \in \mathscr{C}_{\mathbf{C}}(X)$. Then for each function $f \in \mathscr{K}(X)$ it is clear that $gf \in \mathscr{K}(X)$, and the mapping $f \mapsto \mu(gf)$ is therefore

$x \in K$

and it is called the measure with *density* g relative to μ (cf. (13.13)).

(13.1.6) Let $\pi : X \to X'$ be a *homeomorphism* of X onto a locally compact space X'. For each function $f \in \mathscr{K}(X')$, the function $f \circ \pi$ belongs to $\mathscr{K}(X)$, and we have $\mathrm{Supp}(f \circ \pi) = \pi^{-1}(\mathrm{Supp}\, f)$. It follows immediately that, if μ is any measure on X, then $f \mapsto \mu(f \circ \pi)$ is a *measure* on X', called the *image* of μ under π and denoted by $\pi(\mu)$.

(13.1.7) Let Y be a *closed* subset of X (and therefore a locally compact subspace of X (3.18.4)) and ν a measure on Y. For each $f \in \mathscr{K}(X; K)$, the restriction $f|Y$ belongs to $\mathscr{K}(Y; K \cap Y)$, and hence there exists a constant c_K such that

$$|\nu(f|Y)| \leqq c_K \cdot \sup_{y \in Y} |f(y)| \leqq c_K \|f\|$$

for all $f \in \mathscr{K}(X; K)$. The mapping $f \mapsto \nu(f|Y)$ is therefore a *measure on* X, called the *image* of ν under the canonical injection $Y \to X$, or the *canonical extension of* ν *to* X.

(13.1.8) Let U be an open subset of X (and therefore again a locally compact subspace (3.18.4)). For each compact subset K of U, it is clear that the mapping $f \mapsto f|U$ is an *isometry* of $\mathscr{K}(X; K)$ onto $\mathscr{K}(U; K)$. The image under the inverse isometry of a function $g \in \mathscr{K}(U; K)$ is the function g^U which agrees with g on U and is zero on $X - U$. (By abuse of notation, we shall often write f in place of $f|U$ when $\mathrm{Supp}(f) \subset U$, and g in place of g^U). The mapping $g \mapsto g^U$ of $\mathscr{K}_c(U)$ into $\mathscr{K}_c(X)$ is therefore injective. If μ is any measure on X, the mapping $g \mapsto \mu(g^U)$ is a *measure* on U, said to be *induced by* μ *on* U, or the *restriction* of μ to U, and denoted by μ_U or $\mu|U$. It should be noted that a measure ν on U is not necessarily the restriction of a measure on X (Section (13.4), Problem 1), and that an "extension" of ν to X, if it exists, is not in general unique. However, there is the following result:

(13.1.9) *Let* $(U_\alpha)_{\alpha \in I}$ *be an open covering of* X. *For each* $\alpha \in I$ *let* μ_α *be a measure on* U_α *such that, for each pair of indices* α, β, *the restrictions of* μ_α *and* μ_β *to* $U_\alpha \cap U_\beta$ (13.1.8) *are equal. Then there exists a unique measure* μ *on* X *whose restriction to* U_α *is* μ_α, *for each* $\alpha \in I$.

We shall first show that each function $f \in \mathscr{K}_c(X)$ can be written in the form $f = \sum_{i=1}^{n} f_i$ where, for each index i, there exists $\alpha_i \in I$ such that $f_i \in \mathscr{K}_c(X)$

and $\mathrm{Supp}(f_i) \subset U_{\alpha_i}$. For this purpose we observe that if K is the support of f, then there exist finitely many indices $\alpha_i \in I$ $(1 \leqq i \leqq n)$ such that the U_{α_i} cover K. Hence ((3.18.2) and (12.6.4)) there exist n continuous mappings $h_i : X \to [0, 1]$ such that $\mathrm{Supp}(h_i)$ is compact and is contained in U_{α_i} for $1 \leqq i \leqq n$, and such that $\sum_{i=1}^{n} h_i(x) = 1$ for all $x \in K$. Then the functions $f_i = f h_i$ satisfy the required conditions.

This already proves the *uniqueness* of μ: for by definition we must have

$$\mu(f) = \sum_{i=1}^{n} \mu(f h_i) = \sum_{i=1}^{n} \mu_{\alpha_i}(f h_i).$$

To prove the *existence* of a *linear form* μ on $\mathscr{K}_c(X)$ whose restriction to $\mathscr{K}_c(U_\alpha)$ is μ_α for each $\alpha \in I$, it is enough to establish the following assertion: given two finite sequences $(g_i)_{1 \leqq i \leqq m}$ and $(h_j)_{1 \leqq j \leqq n}$ of functions belonging to $\mathscr{K}_c(X)$, such that $\mathrm{Supp}(g_i) \subset U_{\alpha_i}$ for $1 \leqq i \leqq m$ and $\mathrm{Supp}(h_j) \subset U_{\beta_j}$ for $1 \leqq j \leqq n$, and such that

$$\sum_{i=1}^{m} g_i(x) = \sum_{j=1}^{n} h_j(x) = 1$$

for all $x \in \mathrm{Supp}(f)$, then we have

$$\sum_{i=1}^{m} \mu_{\alpha_i}(f g_i) = \sum_{j=1}^{n} \mu_{\beta_j}(f h_j).$$

Now, we have

$$f g_i = \sum_{j=1}^{n} f g_i h_j$$

and therefore

$$\sum_{i=1}^{m} \mu_{\alpha_i}(f g_i) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \mu_{\alpha_i}(f g_i h_j) \right).$$

Similarly,

$$\sum_{j=1}^{n} \mu_{\beta_j}(f h_j) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \mu_{\beta_j}(f g_i h_j) \right).$$

But since $\mathrm{Supp}(f g_i h_j)$ is contained in $U_{\alpha_i} \cap U_{\beta_j}$, it follows from the hypotheses that $\mu_{\alpha_i}(f g_i h_j) = \mu_{\beta_j}(f g_i h_j)$, and our assertion follows.

It remains to be shown that the linear form μ so defined is a *measure*. Let K be a compact subset of X, and define the U_{α_i} and the h_i as at the beginning of the proof. If $H_i = \mathrm{Supp}(h_i)$, then by hypothesis there exists a

number $a_i \geqq 0$ such that $|\mu_{\alpha_i}(g)| \leqq a_i \|g\|$ for all $g \in \mathscr{K}(X; H_i)$ (13.1.2). Hence, for each function $f \in \mathscr{K}(X; K)$, we have

$$|\mu_{\alpha_i}(fh_i)| \leqq a_i \|fh_i\| \leqq a_i \|f\|,$$

so that

$$|\mu(f)| \leqq \left(\sum_{i=1}^{n} a_i\right)\|f\|. \qquad \text{Q.E.D.}$$

(13.1.10) If λ and μ are two measures on X, then so are $\lambda + \mu$ and $a\lambda$ for any scalar $a \in \mathbf{C}$. The set of all measures on X is therefore a *vector subspace* of $\mathbf{C}^{\mathscr{K}_{\mathbf{C}}(X)}$, which we denote by $M_{\mathbf{C}}(X)$ or $M(X)$.

By analogy with the example (13.1.4), if μ is a measure on a locally compact space X, we write $\int f\,d\mu$ or $\int f(x)\,d\mu(x)$ (or also $\langle f, \mu\rangle$ or $\langle \mu, f\rangle$) in place of $\mu(f)$, for any $f \in \mathscr{K}(X)$, and we call this number the *integral of f with respect to* μ.

2. REAL MEASURES

Let X be a locally compact space. Let $\mathscr{K}_{\mathbf{R}}(X)$ denote the set of all *real-valued* continuous functions on X with compact support, and $\mathscr{K}_{\mathbf{R}}(X; K)$ the set of those whose support is contained in K. Clearly $\mathscr{K}_{\mathbf{R}}(X)$ is a *real* vector subspace of $\mathscr{K}_{\mathbf{C}}(X)$, and we can write

$$\mathscr{K}_{\mathbf{C}}(X) = \mathscr{K}_{\mathbf{R}}(X) \oplus i\mathscr{K}_{\mathbf{R}}(X)$$

(direct sum). For every (complex) measure μ, the restriction of μ to $\mathscr{K}_{\mathbf{R}}(X)$ is an $\mathbf{R}$-linear mapping μ_0 of $\mathscr{K}_{\mathbf{R}}(X)$ into $\mathbf{C}$; moreover μ_0 determines μ uniquely, for if $f = f_1 + if_2$ with f_1, f_2 in $\mathscr{K}_{\mathbf{R}}(X)$, then $\mu(f) = \mu_0(f_1) + i\mu_0(f_2)$. Conversely, if an $\mathbf{R}$-linear mapping $\mu_0 : \mathscr{K}_{\mathbf{R}}(X) \to \mathbf{C}$ is such that, for each compact subset K of X, there exists $a_K > 0$ with the property that $|\mu_0(f)| \leqq a_K \|f\|$ for all $f \in \mathscr{K}_{\mathbf{R}}(X; K)$, then it is immediately obvious that the mapping

$$f_1 + if_2 \mapsto \mu_0(f_1) + i\mu_0(f_2)$$

is a (complex) measure on X. Hence we may identify each measure on X with its restriction to $\mathscr{K}_{\mathbf{R}}(X)$.

Let μ be a (complex) measure on X. It follows immediately from (13.1.2) that the mapping $f \mapsto \overline{\mu(\bar{f})}$ is also a measure on X, called the *conjugate* of μ and denoted by $\bar{\mu}$. We have $\bar{\bar{\mu}} = \mu$, and if λ, μ are measures on X and a, b are any two complex numbers, then $\overline{a\lambda + b\mu} = \bar{a}\bar{\lambda} + \bar{b}\bar{\mu}$. More generally, if g is any function belonging to $\mathscr{C}_{\mathbf{C}}(X)$ and μ is any measure on X, then we have $\overline{g \cdot \mu} = \bar{g} \cdot \bar{\mu}$ (13.1.5).

A measure μ on X is said to be *real* if $\bar{\mu} = \mu$, or equivalently if $\mu(f)$ is *real* for every $f \in \mathscr{K}_{\mathbf{R}}(X)$.

We may therefore identify the set of real measures on X with a vector space of linear forms on the real vector space $\mathscr{K}_{\mathbf{R}}(X)$. This vector space is denoted by $M_{\mathbf{R}}(X)$. Lebesgue measure and all Dirac measures are real. If μ is any complex measure, then the measures $\mu_1 = (\mu + \bar{\mu})/2$ and $\mu_2 = (\mu - \bar{\mu})/2i$ are real. They are called respectively the *real* and *imaginary parts* of μ, and are denoted by $\mathscr{R}\mu$ and $\mathscr{I}\mu$ respectively. For each function $f \in \mathscr{K}_{\mathbf{R}}(X)$, we have

$$(\mathscr{R}\mu)(f) = \mathscr{R}(\mu(f)), \qquad (\mathscr{I}\mu)(f) = \mathscr{I}(\mu(f)) \tag{13.2.1}$$

and by definition

$$\mu = \mathscr{R}\mu + i\mathscr{I}\mu, \qquad \bar{\mu} = \mathscr{R}\mu - i\mathscr{I}\mu. \tag{13.2.2}$$

3. POSITIVE MEASURES: THE ABSOLUTE VALUE OF A MEASURE

A measure μ on a locally compact space X is said to be *positive* if, for each function $f \in \mathscr{K}_{\mathbf{R}}(X)$ such that $f \geqq 0$, we have $\mu(f) \geqq 0$. Consequently, if f and g are two functions belonging to $\mathscr{K}_{\mathbf{R}}(X)$ such that $f \leqq g$, we have $\mu(f) \leqq \mu(g)$. Since each $f \in \mathscr{K}_{\mathbf{R}}(X)$ can be written in the form $f = f^+ - f^-$ (where $f^+(x) = (f(x))^+$ and $f^-(x) = (f(x))^-$ (2.2)), it follows that a positive measure is a *real* measure. We denote by $M_+(X)$ the set of all positive measures on X.

Surprisingly, the property of positivity alone implies the defining property (13.1.2) of a measure:

(13.3.1) *Let μ be a linear form on the real vector space $\mathscr{K}_{\mathbf{R}}(X)$ such that $\mu(f) \geqq 0$ whenever $f \geqq 0$. Then μ is a (positive) measure on* X.

We have to show that (13.1.2) is satisfied. There exists a function $g \in \mathscr{K}_{\mathbf{R}}(X)$ with values in $[0, 1]$ and equal to 1 throughout K ((3.18.2) and (4.5.2)). Hence for all $f \in \mathscr{K}_{\mathbf{R}}(X; K)$, we have

$$0 \leqq f^+ \leqq \|f\| \cdot g, \qquad 0 \leqq f^- \leqq \|f\| \cdot g$$

and therefore

$$0 \leqq \mu(f^+) \leqq \|f\| \cdot \mu(g), \qquad 0 \leqq \mu(f^-) \leqq \|f\| \cdot \mu(g),$$

so that finally $|\mu(f)| \leqq 2\|f\| \cdot \mu(g)$.

The notion of a positive measure enables us to define an *order relation* on the vector space $M_{\mathbf{R}}(X)$ of real measures on X. We write $\mu \leqq \nu$ if the measure $\nu - \mu$ is *positive*. Since the relations $\mu \geqq 0$ and $\mu \leqq 0$ imply that $\mu(f) = \mu(f^+) - \mu(f^-) = 0$ for all $f \in \mathscr{K}_{\mathbf{R}}(X)$, and therefore that $\mu = 0$, it follows that $\mu \leqq \nu$ is indeed an order relation on $M_{\mathbf{R}}(X)$ (not, in general, a total ordering). It is clear that $\mu \leqq \nu$ implies that $\lambda + \mu \leqq \lambda + \nu$ for all real measures λ, and $a\mu \leqq a\nu$ for all real scalars $a \geqq 0$. (For a study of this order relation, see (13.15).)

(13.3.2) *Let μ be a (complex) measure on X. Then there exists a smallest positive measure ρ on X such that $|\mu(f)| \leqq \rho(|f|)$ for all $f \in \mathscr{K}_{\mathbf{C}}(X)$.*

For every positive measure ν such that $|\mu(f)| \leqq \nu(|f|)$ for all $f \in \mathscr{K}_{\mathbf{C}}(X)$, the relations $g \geqq 0$ and $|h| \leqq g$ $(g, h \in \mathscr{K}_{\mathbf{C}}(X))$ imply $|\mu(h)| \leqq \nu(|h|) \leqq \nu(g)$. We shall show that there exists a positive measure ρ on X such that, for each function $f \geqq 0$ in $\mathscr{K}_{\mathbf{R}}(X)$, we have

$$\rho(f) = \sup_{|g| \leqq f,\, g \in \mathscr{K}_{\mathbf{C}}(X)} |\mu(g)|. \tag{13.3.2.1}$$

This ρ will then clearly satisfy the conditions of (13.3.2).

To begin with, we remark that the right-hand side of (13.3.2.1) is *finite*; for if $K = \mathrm{Supp}(f)$, then $\mathrm{Supp}(g) \subset K$, and

$$|\mu(g)| \leqq a_K \cdot \|g\| \leqq a_K \cdot \|f\|$$

whenever $|g| \leqq f$, by virtue of (13.1.2). Also it is clear that, for any real scalar $a \geqq 0$, we have $\rho(af) = a\rho(f)$. We shall show next that, if f_1 and f_2 are two functions $\geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$, then

$$\rho(f_1 + f_2) = \rho(f_1) + \rho(f_2). \tag{13.3.2.2}$$

For each $\varepsilon > 0$, there exists $g_i \in \mathscr{K}_{\mathbf{C}}(X)$ such that

$$|g_i| \leqq f_i \quad \text{and} \quad |\mu(g_i)| \geqq \rho(f_i) - \varepsilon \quad (i = 1, 2).$$

Multiplying g_i by a complex number with absolute value 1, we may assume that $\mu(g_i) = |\mu(g_i)|$, and then we have

$$\mu(g_1 + g_2) = |\mu(g_1)| + |\mu(g_2)| \geqq \rho(f_1) + \rho(f_2) - 2\varepsilon,$$

and since $|g_1 + g_2| \leqq f_1 + f_2$ we have $\rho(f_1 + f_2) \geqq \rho(f_1) + \rho(f_2) - 2\varepsilon$. Since ε is arbitrary, it follows that $\rho(f_1) + \rho(f_2) \geqq \rho(f_1 + f_2)$. On the other hand, let $h \in \mathscr{K}_{\mathbf{C}}(X)$ be such that $|h| \leqq f_1 + f_2$. Let h_i be the function which is equal to $hf_i/(f_1 + f_2)$ at the points x where $f_1(x) + f_2(x) \neq 0$ and is zero

elsewhere ($i = 1, 2$). Then h_i is continuous on X, because $f_i/(f_1 + f_2)$ is continuous at the points x where $f_1(x) + f_2(x) > 0$, and also $|h_i(x)| \leqq |h(x)|$ for all $x \in X$, which proves that h_i is continuous at the points x where $f_1(x) + f_2(x) = 0$, because $h(x)$ also vanishes at these points. It is clear that $|h_i| \leqq f_i$ ($i = 1, 2$) and that $h = h_1 + h_2$; hence

$$|\mu(h)| \leqq |\mu(h_1)| + |\mu(h_2)| \leqq \rho(f_1) + \rho(f_2).$$

Since $|\mu(h)|$ may be taken to be arbitrarily close to $\rho(f_1 + f_2)$, it follows that $\rho(f_1 + f_2) \leqq \rho(f_1) + \rho(f_2)$. Hence (13.3.2.2) is proved.

We shall now extend the definition of $\rho(f)$ to all functions $f \in \mathscr{K}_{\mathbf{R}}(X)$. To do this we write $\rho(f) = \rho(f') - \rho(f'')$, where $f = f' - f''$ is any decomposition of f as the difference of two functions $f', f'' \geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$. The value of $\rho(f)$ so obtained is independent of the decomposition, because if $f = f_1' - f_1'' = f_2' - f_2''$, then $f_1' + f_2'' = f_2'' + f_2'$ and therefore $\rho(f_1') + \rho(f_2'') = \rho(f_1'') + \rho(f_2')$ by virtue of (13.3.2.2).

With this definition, the formula (13.3.2.2) is valid for all f_1, f_2 in $\mathscr{K}_{\mathbf{R}}(X)$. For we can write $f_1 = f_1' - f_1'', f_2 = f_2' - f_2''$ where f_i', f_i'' ($i = 1, 2$) are $\geqq 0$ and belong to $\mathscr{K}_{\mathbf{R}}(X)$; since $f_1 + f_2 = (f_1' + f_2') - (f_1'' + f_2'')$, our assertion follows from the definition above and from (13.3.2.2) for functions $\geqq 0$.

Finally, the above definition shows that, for each scalar $a \geqq 0$, we have $\rho(af) = a\rho(f)$; and if $a < 0$, then we have

$$\begin{aligned}\rho(af) = \rho(af' - af'') &= \rho(-af'') + \rho(af') \\ &= (-a)\rho(f'') - \rho(-af') \\ &= (-a)\rho(f'') - (-a)\rho(f') = a\rho(f).\end{aligned}$$

Hence the relation $\rho(af) = a\rho(f)$ is valid for all real scalars a, and therefore we have proved that ρ is a (real) *linear form* on $\mathscr{K}_{\mathbf{R}}(X)$. Hence by (13.3.1) it is a positive measure. Q.E.D.

The measure ρ defined by (13.3.2) is called the *absolute value* of the complex measure μ, and is denoted by $|\mu|$. Hence by definition we have

(13.3.3) $$|\mu(f)| \leqq |\mu|(|f|) \quad \text{for all} \quad f \in \mathscr{K}_{\mathbf{C}}(X).$$

It is immediately seen that if $a \in \mathbf{C}$ and μ is a measure on X, we have

(13.3.4) $$|a\mu| = |a| \cdot |\mu|.$$

If μ is a *positive* measure on X, then

(13.3.5) $$|\mu| = \mu.$$

By virtue of (13.3.2.1) it is enough to show that $|\mu(f)| \leqq \mu(|f|)$ for all $f \in \mathscr{K}_{\mathbf{C}}(X)$. Given f, there exists $\zeta \in \mathbf{C}$ such that $|\zeta| = 1$ and $|\mu(f)| = \zeta\mu(f) = \mu(\zeta f)$; since μ is real, we have $\mu(\zeta f) = \mu(\mathscr{R}(\zeta f))$; and since $\mu \geqq 0$ and $\mathscr{R}(\zeta f) \leqq |f|$, it follows that $\mu(\mathscr{R}(\zeta f)) \leqq \mu(|f|)$, which establishes the assertion.

If μ is any *real* measure on X, it follows from (13.3.3) that $\mu \leqq |\mu|$, and therefore:

(13.3.6) *Every real measure on* X *is the difference of two positive measures* (for a more precise result, see (13.15)).

If μ is any (complex) measure on X, it follows from (13.3.3) and (13.3.2) that we have

$$|\mathscr{R}\mu| \leqq |\mu|, \qquad |\mathscr{I}\mu| \leqq |\mu|, \tag{13.3.7}$$
$$|\mu| \leqq |\mathscr{R}\mu| + |\mathscr{I}\mu|.$$

Also if μ, ν are any two measures on X we have

$$|\mu + \nu| \leqq |\mu| + |\nu| \tag{13.3.8}$$

by virtue of (13.3.2).

Finally, it follows immediately from the definitions that, if $\pi : X \to X'$ is a *homeomorphism* and μ is any measure on X, then

$$|\pi(\mu)| = \pi(|\mu|). \tag{13.3.9}$$

PROBLEMS

1. Let X be a locally compact space, E a vector subspace of $\mathscr{C}_{\mathbf{R}}(X)$ and P a *convex cone* in $\mathscr{C}_{\mathbf{R}}(X)$ (i.e., a subset of this space such that the relations $f \in P$ and $g \in P$ imply $f + g \in P$ and $af \in P$ for all scalars $a > 0$). Suppose that for each function $h \in \mathscr{K}_{\mathbf{R}}(X)$ there exists $f \in E$ such that $f - h \in P$.

   Let u be a real linear form on E such that the relation $f \in E \cap P$ implies $u(f) \geqq 0$. Let $h \in \mathscr{K}_{\mathbf{R}}(X)$ and let P'_h be the set of all $f \in E$ such that $h - f \in P$, and P''_h the set of all $f \in E$ such that $f - h \in P$. Show that these two sets P'_h, P''_h are nonempty, and that if α' is the least upper bound of the $u(f)$ with $f \in P'_h$ and α'' the greatest lower bound of the $u(f)$ with $f \in P''_h$, then α' and α'' are finite and $\alpha' \leqq \alpha''$. Deduce that there exists a linear form u_1 on the subspace $E_1 = E + \mathbf{R}h$ of $\mathscr{C}_{\mathbf{R}}(X)$ which extends u, such that the relation $f_1 \in E_1 \cap P$ implies $u_1(f_1) \geqq 0$. Show that $\alpha' \leqq u_1(h) \leqq \alpha''$ for any such extension u_1 of u, and that the extension is unique if and only if $\alpha' = \alpha''$.

2. Let X be a compact space and let p be a real-valued function on $\mathscr{C}_{\mathbf{R}}(X)$ satisfying the following two conditions: (i) $p(f+g) \leqq p(f) + p(g)$; (ii) $p(af) = ap(f)$ for all $a > 0$. Then the set P of functions $f \in \mathscr{C}_{\mathbf{R}}(X)$ such that $p(f) \leqq 0$ is a convex cone. Suppose furthermore that (iii) $\inf_{x \in X} f(x) \leqq p(f) \leqq \sup_{x \in X} f(x)$ for all $f \in \mathscr{C}_{\mathbf{R}}(X)$, so that we have $p(1) = 1$. Show that there exists a positive measure μ of mass 1 on X such that $\mu(f) \leqq p(f)$ for all $f \in \mathscr{C}_{\mathbf{R}}(X)$. If moreover we are given a linear form u on a vector subspace E of $\mathscr{C}_{\mathbf{R}}(X)$ such that $u(f) \leqq p(f)$ for all $f \in E$, then there exists a measure μ of the above type which extends the form u. (Consider a denumerable total set $(g_n)_{n \geqslant 0}$ in $\mathscr{C}_{\mathbf{R}}(X)$, with $g_0 = 1$; use the result of Problem 1 inductively to obtain, on the subspace G of $\mathscr{C}_{\mathbf{R}}(X)$ generated by E and the g_n, a linear form v such that $-p(-f) \leqq v(f) \leqq p(f)$ for all $f \in G$, and deduce that v extends by continuity to a measure on X.) Show that the measure μ is unique if and only if $p(f) + p(-f) = 0$ for all $f \in E$.

4. THE VAGUE TOPOLOGY

Since the space $M_{\mathbf{C}}(X)$ is a subspace of $\mathbf{C}^{\mathscr{K}_{\mathbf{C}}(X)}$, we can define the weak topology, i.e., the topology of simple convergence in $\mathscr{K}_{\mathbf{C}}(X)$ (12.15). This topology on $M_{\mathbf{C}}(X)$ is called the *vague topology*. To say that a sequence (μ_n) of measures on X converges *vaguely* to a measure μ therefore means that, for each function $f \in \mathscr{K}_{\mathbf{C}}(X)$, the sequence $(\mu_n(f))$ converges to $\mu(f)$ in $\mathbf{C}$.

(13.4.1) *Let (μ_n) be a sequence of measures on X such that, for each $f \in \mathscr{K}_{\mathbf{C}}(X)$, the sequence $(\mu_n(f))$ tends to a limit $\mu(f)$ in $\mathbf{C}$. Then $f \mapsto \mu(f)$ is a measure on X and is the vague limit of the sequence (μ_n). If the μ_n are all positive, then so is μ.*

We have already remarked (13.1) that, for each compact subset K of X, $\mathscr{K}(X; K)$ is a Banach space and the restrictions of the μ_n to $\mathscr{K}(X; K)$ are continuous linear forms. Hence it follows from the Banach–Steinhaus theorem (12.16.5) that the restriction of μ to $\mathscr{K}(X; K)$ is continuous, and therefore that μ is a measure on X (and clearly a positive measure if the μ_n are positive).

We recall (12.15) that a subset H of $M_{\mathbf{C}}(X)$ is said to be *vaguely bounded* (or just *bounded*, if there is no risk of ambiguity) if, for each $f \in \mathscr{K}_{\mathbf{C}}(X)$, we have $\sup_{\mu \in H} |\mu(f)| < +\infty$. Every vaguely convergent sequence is vaguely bounded.

(13.4.2) *Let* H *be a bounded subset of* $M_{\mathbf{C}}(X)$.

(i) *For each compact subset* K *of* X, *there exists a real number* $c_K > 0$ *such that, for each* $\mu \in H$ *and each* $f \in \mathscr{K}_{\mathbf{C}}(X)$, *we have*

$$|\mu(f)| \leqq c_K \|f\|$$

(*and therefore* (13.3.2.1), $|\mu|(|f|) \leqq c_K \|f\|$).

(ii) *The vague closure of* H *in* $M_c(X)$ *is a compact metrizable space with respect to the vague topology.*

(i) This is an immediate consequence of the Banach–Steinhaus theorem (12.16.4).

(ii) Let (U_n) be a sequence of relatively compact open sets in X which cover X and are such that $\bar{U}_n \subset U_{n+1}$ (3.18.3). Since every compact subset K of X is contained in some U_n (3.16), each space $\mathscr{K}(X; K)$ can be identified with a closed subspace of one of the Banach spaces $\mathscr{C}(\bar{U}_n)$, hence with a closed subspace of $\mathscr{K}(X; \bar{U}_n)$. Now we know (7.4.4) that $\mathscr{C}(\bar{U}_n)$ is separable, hence the same is true of $\mathscr{K}(X; \bar{U}_n)$ (3.10.9). Let $(f_{mn})_{m\geqq 1}$ be a dense sequence in $\mathscr{K}(X; \bar{U}_n)$. To show that H is metrizable it is enough to show that the vague topology on H can be defined by the pseudo-distances $|\langle f_{mn}, \mu - \nu\rangle|$ (12.4.6). This means that if g_i $(1 \leqq i \leqq p)$ is a finite sequence of functions belonging to $\mathscr{K}_c(X)$, if μ_0 is an element of H and r a real number >0, there exists a finite number of functions $f_{m_k n_k}$ $(1 \leqq k \leqq q)$ such that the relations $\mu \in H$ and $|\langle f_{m_k n_k}, \mu - \mu_0\rangle| \leqq \frac{1}{2}r$ $(1 \leqq k \leqq q)$ imply $|\langle g_i, \mu - \mu_0\rangle| \leqq r$ $(1 \leqq i \leqq p)$. But the g_i all belong to $\mathscr{K}(X; \bar{U}_n)$ for some fixed n, and the set of restrictions of the $\mu \in H$ to $\mathscr{K}(X; \bar{U}_n)$ is equicontinuous, by (i) above and (12.15.7.1). Hence the assertion follows from (12.15.7).

It remains to show (by the same reasoning as in (12.15.7)) that, when we identify H with its image L in the product space $\mathbf{C}^{\mathbf{N}\times\mathbf{N}}$ by means of the mapping $\mu \mapsto (\langle f_{mn}, \mu\rangle)$, L is *closed* in $\mathbf{C}^{\mathbf{N}\times\mathbf{N}}$. If (μ_k) is a sequence of points of H such that each of the sequences $(\langle f_{mn}, \mu_k\rangle)_{k\geqq 1}$ is convergent, then it follows from (12.15.7) that, in each $\mathscr{K}(X; \bar{U}_n)$, the restrictions of the μ_k converge to a continuous linear form. Hence the sequence (μ_k) converges vaguely to a measure on X. Q.E.D.

We shall see later (13.20) that condition (i) in (13.4.2) can also be written in the form $|\mu|(K) \leqq c_K$ for each measure $\mu \in H$.

Notice also that (13.4.2(ii)) implies (13.4.1) as a particular case.

In particular:

(13.4.3) *Let* ν *be a positive measure on* X. *Then the set of complex measures* μ *such that* $|\mu| \leqq \nu$ *is metrizable and compact with respect to the vague topology.*

For this set is evidently bounded and *closed* in $M_c(X)$ in the vague topology.

It should be noted that a bounded set of measures does not necessarily satisfy the hypothesis of (13.4.3).

(13.4.4) *Let (μ_n) be an increasing sequence of real measures on* X *such that, for each function $f \geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$, the sequence $(\mu_n(f))$ is bounded above in* **R**. *Then the sequence (μ_n) has a vague limit in $M_{\mathbf{R}}(X)$ which is also its least upper bound for the order-relation on $M_{\mathbf{R}}(X)$.*

For each function $f \geqq 0$ in $\mathscr{K}_{\mathbf{R}}(X)$, the sequence $(\mu_n(f))$ is increasing and bounded above in **R**, hence (4.2.1) has a limit $\mu(f)$ in **R** which is equal to $\sup_n \mu_n(f)$. Since every function belonging to $\mathscr{K}_{\mathbf{C}}(X)$ is a linear combination of four functions $\geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$, it follows that the sequence (μ_n) is vaguely convergent (13.4.1), and it is clear that μ is its least upper bound, by the definition of the order-relation on $M_{\mathbf{R}}(X)$.

(13.4.5) *If a series of* positive *measures on* X, *with general term μ_n, is such that, for each $f \geqq 0$ in $\mathscr{K}_{\mathbf{R}}(X)$, the series with general term $\mu_n(f) \geqq 0$ is convergent in* **R**, *then the series with general term μ_n is vaguely convergent in $M_{\mathbf{R}}(X)$, and its sum $\mu = \sum_n \mu_n$ is such that $\mu(f) = \sum_n \mu_n(f)$ for all $f \in \mathscr{K}_{\mathbf{R}}(X)$.*

Apply (13.4.4) to the partial sums of the series.

PROBLEMS

1. Let λ be Lebesgue measure on **R**, let μ be its restriction to $\mathbf{R}_+^* =]0, +\infty[$, and let g be the function $x \mapsto 1/x$ on $\mathbf{R}_+^*$. Show that the measure $g \cdot \mu$ cannot be extended to a measure on **R** (cf. (17.9)).

2. On the real line **R**, show that the sequence of Dirac measures ε_n (unit mass at the point $+n$) converges vaguely to 0. Give an example of a sequence (f_n) in $\mathscr{K}_{\mathbf{C}}(\mathbf{R})$ which converges to 0 in the Fréchet space $\mathscr{E}_{\mathbf{C}}(\mathbf{R})$ (12.14.6) but is such that the sequence $(\langle \varepsilon_n, f_n \rangle)$ tends to 1.

3. Let X be a compact space.
   (a) Show that if (f_n) is a sequence of functions in the Banach space $\mathscr{E}_{\mathbf{C}}(X)$ which tends to 0 and if (μ_n) is a sequence of measures on X which tends vaguely to 0, then the sequence $(\langle \mu_n, f_n \rangle)$ tends to 0 (use (13.4.2)).
   (b) Suppose that X is infinite. Let V be a neighborhood of 0 in M(X) for the vague topology, defined by a finite number of inequalities $|\langle \mu, f_j \rangle| \leqq 1$ $(1 \leqq j \leqq m)$, and let f be any function belonging to $\mathscr{E}_{\mathbf{C}}(X)$ which is not a linear combination of the f_j. Show that there exists $\mu \in V$ such that $|\langle \mu, f \rangle|$ is arbitrarily large (cf. Section 12.15, Problem 1). Deduce that the mapping $(\mu, f) \mapsto \langle \mu, f \rangle$ of $\mathscr{E}_{\mathbf{C}}(X) \times M(X)$ into **C** is not continuous, and hence (using (a)) that the space M(X) is not metrizable with respect to the vague topology.

4. Let X be a compact space. Show that the topology induced on $M_+(X)$ by the vague topology is separable, metrizable and locally compact. (Observe that for each $\mu_0 \in M_+(X)$ there exists a neighborhood V of μ_0 in M(X) such that $V \cap M_+(X)$ is bounded.) Deduce that the mapping $(\mu, f) \mapsto \langle \mu, f \rangle$ of $\mathscr{C}_C(X) \times M_+(X)$ into **C** is continuous.

Deduce that if X is locally compact but not compact, then $M_+(X)$ is still metrizable with respect to the topology induced by the vague topology, but is not separable.

5. (a) Let X be a locally compact space. Show that if (g_n) is a sequence of functions in the Fréchet space $\mathscr{C}_C(X)$ tending to 0 and (μ_n) is a sequence of measures on X tending vaguely to 0, then the sequence of measures $(g_n \cdot \mu_n)$ tends vaguely to 0.
(b) Show that if $M_+(X)$ is endowed with the topology induced by the vague topology, the mapping $(\mu, g) \mapsto g \cdot \mu$ of $M_+(X) \times \mathscr{C}_C(X)$ into M(X) is continuous.
(c) Suppose that X is compact and infinite. Show that the mapping $(\mu, g) \mapsto g \cdot \mu$ of $M(X) \times \mathscr{C}_C(X)$ into M(X) is not continuous when M(X) is endowed with the vague topology (argue as in Problem 3(b)).

6. Let X be the unit interval [0, 1] in **R**.
(a) Let μ_n be the measure $\varepsilon_0 - \varepsilon_{1/n}$. Show that the sequence (μ_n) tends vaguely to 0 but that the sequence $(|\mu_n|)$ tends vaguely to $2\varepsilon_0$.
(b) Let λ be Lebesgue measure on X, and let $g_n(x) = \sin nx$. Show that the sequence of positive measures $\mu_n = (1 - g_n) \cdot \lambda$ converges vaguely to λ as $n \to +\infty$, but does not converge to λ with respect to the topology defined by the norm (5.7.1) on M(X).

7. Let X be a compact space and μ a positive measure of mass 1 on X. A sequence (x_n) of points of X is said to be *equirepartitioned* with respect to μ if the sequence of measures $(n^{-1}(\varepsilon_{x_1} + \cdots + \varepsilon_{x_n}))$ converges vaguely to μ. Show that a necessary and sufficient condition for this to be the case is that, if (f_k) is any total sequence (5.4) in $\mathscr{C}_C(X)$, then for each k the sequence $(n^{-1}(f_k(x_1) + \cdots + f_k(x_n)))$ converges to $\int f_k \, d\mu$. Consider in particular the case where $X = [0, 1]$ and $x_n = n\theta - [n\theta]$ (where $[t]$ is the integral part of a real number t). Show that if θ is irrational, then the sequence (x_n) is equirepartitioned with respect to Lebesgue measure ("Bohl's theorem"). (Use the sequence of functions $f_k(x) = e^{2\pi i k x}$.)

8. Let X, Y be two locally compact spaces and $\pi : X \to Y$ a *proper* continuous mapping (Section 12.7, Problem 2). For each function $g \in \mathscr{K}(Y)$, we have $g \circ \pi \in \mathscr{K}(X)$. If μ is any measure on X, then the *image* of μ under π is defined to be the measure $g \mapsto \mu(g \circ \pi)$ on Y, and is denoted by $\pi(\mu)$. The mapping $\mu \mapsto \pi(\mu)$ of M(X) into M(Y) is linear and vaguely continuous.

Suppose that X is *compact*. Let Γ be a set of continuous mappings of X into X, *each pair of which commute*. Let Γ' be the set of linear mappings of the form $\mu \mapsto n^{-1}(\mu + u(\mu) + \cdots + u^{n-1}(\mu))$ of M(X) into M(X), where $u \in \Gamma$ and n is any positive integer, and let Γ'' be the set of all compositions of any finite number of elements of Γ'. If K is the convex and vaguely compact set of positive measures of total mass 1 on X, then $u(K) \subset K$ for all $u \in \Gamma''$. Show that the sets $u(K)$, where $u \in \Gamma''$, form a filter base in K consisting of vaguely compact sets, and deduce that the intersection I of these sets is nonempty. Show finally that each measure $\mu \in I$ is *invariant* under Γ, i.e., that $u(\mu) = \mu$ for all $u \in \Gamma$ ("Markoff–Kakutani theorem"). (Observe that, by definition, there exists for each n a measure $\nu \in K$ such that $\mu = n^{-1}(\nu + u(\nu) + \cdots + u^{n-1}(\nu))$, and evaluate $\|u(\mu) - \mu\|$.) Consider the case of a compact commutative group operating on itself by translation (cf. (14.1)).

9. If f is any real-valued function on $I = [0, 1]$, the nth *Bernstein polynomial of* f is defined to be

$$B_{n,f}(t) = \sum_{p=0}^{n} \binom{n}{p} t^p (1-t)^{n-p} f\left(\frac{p}{n}\right).$$

(a) If $f \geqq 0$ on I, then $B_{n,f} \geqq 0$ on I, and $B_{n,1}$ is equal to the constant 1. Deduce that $\|B_{n,f}\| \leqq \|f\|$ in $\mathscr{B}(I)$.

(b) Show by induction on k that, if $f_k(t) = t^k$, we have

$$(*_k) \qquad B_{n,f_k}(t) = a_{k,n} t^k + P_{k,n}(t),$$

where $a_{k,n} = (1 - (k-1)/n) a_{k-1,n}$ and $a_{0,n} = 1$, and $P_{k,n}$ is a polynomial of degree $\leqq k-1$ whose coefficients are less than C_k/n in absolute value, where C_k is a constant independent of n. (Differentiate the equation $(*_k)$ with respect to t, then multiply through by t.)

(c) Deduce from (b) and Weierstrass' theorem that, for every continuous function $f \in \mathscr{C}(I)$, the sequence $(B_{n,f})$ converges uniformly to f on I.

10. Let X be a metrizable compact space, let $(f_n)_{n \geqq 0}$ be a sequence of complex-valued continuous functions on X, and let $(c_n)_{n \geqq 0}$ be a sequence of complex numbers.

(a) In order that there should exist a complex measure μ on X such that $\mu(f_n) = c_n$ for all n, it is necessary and sufficient that there should exist a number $A > 0$ with the following property: for every finite sequence $(\lambda_k)_{0 \leqq k \leqq n}$ of complex numbers, we have

$$\left| \sum_{k=0}^{n} \lambda_k c_k \right| \leqq A \left\| \sum_{k=0}^{n} \lambda_k f_k \right\|.$$

(Use the Hahn–Banach theorem.)

(b) Suppose that the f_k are real-valued, the numbers c_k real, and $f_0 = 1$. Then there exists a *positive* measure μ on X such that $\mu(f_n) = c_n$ for all n if and only if, for each sequence $(\lambda_k)_{0 \leqq k \leqq n}$ of real numbers such that $\sum_{k=0}^{n} \lambda_k f_k(x) \geqq 0$ for all $x \in X$, we have $\sum_{k=0}^{n} \lambda_k c_k \geqq 0$ (cf. Section 13.3, Problem 2).

11. In Problem 10, take $X = [0, 1]$ and $f_n(t) = t^n$ ("Hausdorff's moment problem"). For each sequence (c_n) of scalars, put

$$\Delta^k c_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} c_{n+j}.$$

(a) Show that there exists a complex measure μ on X such that $\mu(f_n) = c_n$ for all n if and only if there exists a number $A > 0$ such that

$$\sum_{p=0}^{n} \binom{n}{p} |\Delta^{n-p} c_p| \leqq A \qquad \text{for all } n.$$

(Observe that $\Delta^k c_n$ must be the value of μ for the polynomial $t^n(1-t)^k$, and use Problem 9(b) by remarking that, for each real polynomial P, there exists a constant C_P such that

$$B_{n,P} - \frac{1}{n} C_P \leqq P \leqq B_{n,P} + \frac{1}{n} C_P$$

for all n.)

(b) There exists a *positive* measure μ on X such that $\mu(f_n) = c_n$ for all n if and only if $\Delta^k c_n \geqq 0$ for all $k \geqq 0$ and all $n \geqq 0$ (same method).

12. Let X be a compact space. Show that in M(X) the set of positive measures with finite support and total mass 1 is *dense* (with respect to the vague topology) in the set P of all positive measures with total mass 1. (Let U be a neighborhood of $\mu \in P$ with respect to the vague topology, consisting of measures $\nu \in P$ such that $|\mu(f_i) - \nu(f_i)| \leqq \delta$, with $f_i \in \mathscr{C}_C(X)$. Consider a continuous partition (g_j) of unity and points $a_j \in X$ such that $|f_i(x) - \sum_j f_i(a_j)g_j(x)| \leqq \delta$ for all i.)

13. Let X be the unit interval [0, 1] in R. Show that the set K of Dirac measures ε_x ($x \in X$) is vaguely compact, and that in M(X) the Lebesgue measure lies in the vague closure of the convex hull of K, but does not lie in the convex hull of K.

5. UPPER AND LOWER INTEGRALS WITH RESPECT TO A POSITIVE MEASURE

In Sections (13.5) *to* (13.14) (including the problems), μ *denotes a* positive *measure on a locally compact space* X.

We shall show (13.7.3) that μ can be *extended* from $\mathscr{K}_{\mathbf{R}}(X)$ to a vector subspace $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$ of $\mathbf{R}^X$, depending on μ and containing (and in general distinct from) $\mathscr{K}_{\mathbf{R}}(X)$, in such a way that this extension (also denoted by μ) is a *positive linear form* on $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$ (i.e., μ takes values $\geqq 0$ on functions $\geqq 0$ belonging to $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$) and possesses the fundamental property of *passage to the limit for increasing sequences*: that is to say, if (f_n) is an increasing sequence of functions in $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$ whose upper envelope f (12.7.5) also belongs to $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$, then we have $\lim_{n\to\infty} \mu(f_n) = \mu(f)$.

Let $\mathscr{I}$ (or $\mathscr{I}(X)$) be the set of all functions $f: X \to \overline{\mathbf{R}}$ which are *lower semicontinuous on* X and *bounded below by a function belonging to* $\mathscr{K}_{\mathbf{R}}(X)$ (this implies that $f(x) > -\infty$ for all $x \in X$; but we can have $f(x) = +\infty$ at some points $x \in X$: indeed, the constant function equal to $+\infty$ belongs to $\mathscr{I}$). Every function $f \geqq 0$ which is lower semicontinuous on X belongs to $\mathscr{I}$. For every $f \in \mathscr{I}$, we put

$$\mu^*(f) = \sup_{g \leqq f,\, g \in \mathscr{K}_{\mathbf{R}}(X)} \mu(g); \tag{13.5.1}$$

this is a real number, or $+\infty$. Clearly if $f \in \mathscr{K}_{\mathbf{R}}(X)$ we have $\mu^*(f) = \mu(f)$; if $f, g \in \mathscr{I}$ and $f \leqq g$, we have $\mu^*(f) \leqq \mu^*(g)$; and for any real number $a > 0$, we have $\mu^*(af) = a\mu^*(f)$ for all $f \in \mathscr{I}$.

(13.5.2) *Let* (f_n) *be an increasing sequence of functions belonging to* $\mathscr{I}$, *and let* $f = \sup_n f_n$ (*so that* $f \in \mathscr{I}$ (12.7.6)). *Then*

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(13.5.2.1) $$\mu^*(f) = \sup_n \mu^*(f_n) = \lim_{n\to\infty} \mu^*(f_n).$$

Suppose first of all that $f \in \mathscr{K}_{\mathbf{R}}(X)$ and that $f_n \in \mathscr{K}_{\mathbf{R}}(X)$ for clear that the supports of all the f_n are contained in the same c $K = \mathrm{Supp}(f) \cup \mathrm{Supp}(f_1)$. By Dini's theorem (7.2.2), the sequenc verges *uniformly* to f on X. The formula (13.5.2.1) then follows fr that the restriction of μ to $\mathscr{K}(X; K)$ is a *continuous* linear fo Banach space.

Now pass to the general case. Clearly we have $\mu^*(f_n) \leqq \mu^*(f$ Hence it is enough to show that, for each function $u \in \mathscr{K}_{\mathbf{R}}(X)$ $u \leqq f$, we have $\mu^*(u) \leqq \sup \mu^*(f_n)$. Now by (12.7.8) we know tha function f_n, there exists an increasing sequence $(g_{mn})_{m\geqq 1}$ of functio ing to $\mathscr{K}_{\mathbf{R}}(X)$ such that $f_n = \sup_m g_{mn}$. We have $f = \sup_{m,n} g_{mn}$, $f = \sup_n h_n$, where $h_n = \sup_{p\leqq n,\, q\leqq n} g_{pq}$. The functions h_n clearly belong and form an increasing sequence. Since $u \leqq f$, we have $u = \sup_n(\inf($ sequence of functions $\inf(u, h_n)$ is increasing and belongs to $\mathscr{K}_{\mathbf{R}}(X)$ $u \in \mathscr{K}_{\mathbf{R}}(X)$, the first part of the proof shows that $\mu^*(u) = \sup_n \mu^*$ but since $h_n \leqq f_n$, we have $\mu^*(\inf(u, h_n)) \leqq \mu^*(f_n)$, and theref $\mu^*(u) \leqq \sup_n \mu^*(f_n)$.

We remark that because the functions in $\mathscr{I}$ never take the v the sum $f_1 + f_2$ of two such functions is defined at every point belongs to $\mathscr{I}$ (12.7.5).

(13.5.3) *If $f_1, f_2 \in \mathscr{I}$, then $\mu^*(f_1 + f_2) = \mu^*(f_1) + \mu^*(f_2)$.*

We may write $f_1 = \lim_{n\to\infty} g_n$ and $f_2 = \lim_{n\to\infty} h_n$, where (g_n) and $(h$ increasing sequences of functions belonging to $\mathscr{K}_{\mathbf{R}}(X)$ (12.7.8). have $f_1 + f_2 = \lim_{n\to\infty}(g_n + h_n)$ (4.1.8). Since $\mu(g_n + h_n) = \mu(g_n) +$ result follows from (13.5.2) and (4.1.8).

Let (t_n) be any sequence whose terms are real numbers $\geqq 0$ Since the partial sums $s_n = t_1 + \cdots + t_n$ are defined (4.1.8) an increasing sequence, this sequence has a limit *in* $\overline{\mathbf{R}}$ (4.2.1), denot and called the *sum* of the series with general term t_n. For ever (f_n) of *positive* functions belonging to I, the function $x \mapsto \sum_{n=1}^{\infty} f_n(x)$ i

defined, and is denoted by $\sum_{n=1}^{\infty} f_n$; also it belongs to $\mathscr{F}$ (12.7.6). If we apply (13.5.2) to the sequence of partial sums $\sum_{n=1}^{N} f_n$, and (13.5.3) to each of the terms in this partial sum, we obtain the following corollary:

(13.5.4) *If $(f_n)_{n\geq 1}$ is a sequence of functions $\geqq 0$ belonging to $\mathscr{F}$, then*

$$\mu^*\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} \mu^*(f_n).$$

Now consider an *arbitrary* mapping f of X into $\overline{\mathbf{R}}$. There always exist functions $h \in \mathscr{F}$ such that $h \geqq f$, for example the constant function equal to $+\infty$. Put

$$\mu^*(f) = \inf_{h \geqq f,\, h \in \mathscr{F}} \mu^*(h); \tag{13.5.5}$$

this number $\mu^*(f)$ is called the *upper integral* of f with respect to the measure μ.

If $f \in \mathscr{F}$ it is clear that this definition agrees with the preceding one. Here the value of $\mu^*(f)$ can be *any* element of $\overline{\mathbf{R}}$. The relation $f \leqq g$ implies $\mu^*(f) \leqq \mu^*(g)$. For any scalar $a > 0$ we have $\mu^*(af) = a\mu^*(f)$.

(13.5.6) *If the sum $f_1 + f_2$ of two mappings of* X *into* $\overline{\mathbf{R}}$ *is defined at all points of* X, *and if $\mu^*(f_1) > -\infty$ and $\mu^*(f_2) > -\infty$, then*

$$\mu^*(f_1 + f_2) \leqq \mu^*(f_1) + \mu^*(f_2).$$

This is obvious if one of the numbers $\mu^*(f_1)$, $\mu^*(f_2)$ is $+\infty$. If not, given any $a > \mu^*(f_1)$ and $b > \mu^*(f_2)$, there exist h_1, h_2 in $\mathscr{F}$ such that $f_1 \leqq h_1$, $f_2 \leqq h_2$ and $\mu^*(h_1) \leqq a$, $\mu^*(h_2) \leqq b$. It follows that $h_1 + h_2 \geqq f_1 + f_2$ and $\mu^*(h_1 + h_2) \leqq a + b$ by (13.5.3). Hence the result.

(13.5.7) *If (f_n) is any increasing sequence of mappings of* X *into* $\overline{\mathbf{R}}$ *such that $\mu^*(f_n) > -\infty$ for all sufficiently large n, then*

$$\mu^*\left(\sup_n f_n\right) = \sup_n \mu^*(f_n) = \lim_{n\to\infty} \mu^*(f_n).$$

The inequality $\mu^*\left(\sup_n f_n\right) \geqq \sup_n \mu^*(f_n)$ is clear. Let us prove the reverse inequality. We may assume that $\sup_n \mu^*(f_n) < +\infty$, otherwise there is nothing to prove. By hypothesis, we may therefore assume that $\sup_n \mu^*(f_n)$ and all the $\mu^*(f_n)$ are *finite*. For each $\varepsilon > 0$, we shall show that there exists an

increasing sequence (g_n) of functions belonging to $\mathscr{S}$ such that for each n we have $f_n \leqq g_n$ and $\mu^*(g_n) \leqq \mu^*(f_n) + \varepsilon$. Then, if we put $f = \sup_n f_n$ and $g = \sup_n g_n$, we shall have $g \in \mathscr{S}$, $f \leqq g$, $\mu^*(g) = \sup_n \mu^*(g_n) \leqq \sup_n \mu^*(f_n) + \varepsilon$ by virtue of (13.5.2), and finally $\mu^*(f) \leqq \mu^*(g) \leqq \sup_n \mu^*(f_n) + \varepsilon$. Since ε was arbitrary, this will complete the proof.

By definition, for each n there exists $h_n \in \mathscr{S}$ such that $f_n \leqq h_n$ and $\mu^*(f_n) \leqq \mu^*(h_n) \leqq \mu^*(f_n) + 2^{-n}\varepsilon$. Put $g_n = \sup(h_1, \ldots, h_n)$, so that $g_n \in \mathscr{S}$ (12.7.5). Clearly the sequence (g_n) is increasing, and $f_n \leqq g_n$. We shall show by induction on n that

$$\mu^*(g_n) \leqq \mu^*(f_n) + \varepsilon(1 - 2^{-n}). \tag{13.5.7.1}$$

This is evident from the definitions when $n = 1$. Suppose that (13.5.7.1) is true, and remark that $g_{n+1} = \sup(h_{n+1}, g_n)$ and $f_n \leqq \inf(h_{n+1}, g_n)$. Since the functions h_{n+1} and g_n do not take the value $-\infty$, we have

$$\sup(h_{n+1}, g_n) + \inf(h_{n+1}, g_n) = h_{n+1} + g_n$$

and therefore, by virtue of (13.5.3) and the fact that $\mu^*(g_n)$ and $\mu^*(h_{n+1})$ are finite,

$$\begin{aligned}
\mu^*(g_{n+1}) &= \mu^*(g_n) + \mu^*(h_{n+1}) - \mu^*(\inf(h_{n+1}, g_n)) \\
&\leqq \mu^*(g_n) + \mu^*(h_{n+1}) - \mu^*(f_n) \\
&\leqq \mu^*(f_{n+1}) + \frac{\varepsilon}{2^{n+1}} + \varepsilon\left(1 - \frac{1}{2^n}\right) \\
&= \mu^*(f_{n+1}) + \varepsilon\left(1 - \frac{1}{2^{n+1}}\right),
\end{aligned}$$

by the inductive hypothesis (13.5.7.1). Q.E.D.

It should be noted that for a *decreasing* sequence (f_n), it is not necessarily true that $\mu^*\left(\inf_n f_n\right) = \inf \mu^*(f_n)$, even if all the numbers $\mu^*(f_n)$ are *finite* (Section 13.8, Problem 13).

(13.5.8) *If (f_n) is any sequence of functions $\geqq 0$, then*

$$\mu^*\left(\sum_{n=1}^{\infty} f_n\right) \leqq \sum_{n=1}^{\infty} \mu^*(f_n).$$

For each integer $N \geqq 1$ it follows from (13.5.7) that

$$\mu^*\Big(\sum_{n=1}^{N} f_n\Big) \leqq \sum_{n=1}^{N} \mu^*(f_n);$$

ow apply (12.5.7) to the increasing sequence of partial sums $\sum_{n=1}^{N} f_n$.

If (f_n) is any sequence of mappings of X into $\overline{\mathbf{R}}$, the mapping

$$x \mapsto \liminf_{n\to\infty} f_n(x) \qquad (\text{resp. } x \mapsto \limsup_{n\to\infty} f_n(x))$$

s defined for all $x \in X$. It is denoted by $\liminf_{n\to\infty} f_n$ (resp. $\limsup_{n\to\infty} f_n$).

13.5.9) (Fatou's lemma) *If (f_n) is any sequence of functions $\geqq 0$, then*

$$\mu^*\Big(\liminf_{n\to\infty} f_n\Big) \leqq \liminf_{n\to\infty} \mu^*(f_n).$$

For each $n \geqq 1$, let $g_n = \inf_{p\geqq 0}(f_{n+p})$. Clearly we have $\mu^*(g_n) \leqq \inf_{p\geqq 0} \mu^*(f_{n+p})$, nd $g_n \geqq 0$; since the sequence (g_n) is increasing and $\liminf_{n\to\infty} f_n = \sup_n g_n$, it ollows from (13.5.7) that

$$\begin{aligned}\mu^*\Big(\liminf_{n\to\infty} f_n\Big) = \sup_n \mu^*(g_n) &\leqq \sup_n \Big(\inf_{p\geqq 0} \mu^*(f_{n+p})\Big) \\ &= \liminf_{n\to\infty} \mu^*(f_n).\end{aligned}$$

If $f: X \to \overline{\mathbf{R}}$ is any mapping, we define $\mu_*(f) = -\mu^*(-f)$; this number is alled the *lower integral* of f with respect to the measure μ. All the properties bove which were proved for the upper integral can be immediately translated nto properties of the lower integral. In particular, if we put $-\mathscr{I} = \mathscr{S}$ (or $\mathscr{S}(X)$), then $\mathscr{S}$ is the set of all *upper semicontinuous functions* on X which are *ounded above* by a function belonging to $\mathscr{K}_{\mathbf{R}}(X)$ (which implies that they do ot take the value $+\infty$). For all functions $f \in \mathscr{S}$, we have

$$\mu_*(f) = \inf_{g\geqq f,\, g\in\mathscr{K}_{\mathbf{R}}(X)} \mu(g)$$

nd for all $f: X \to \overline{\mathbf{R}}$ we have

$$\mu_*(f) = \sup_{h\leqq f,\, h\in\mathscr{S}} \mu_*(h).$$

13.5.10) *If f is any mapping of X into $\overline{\mathbf{R}}$, then $\mu_*(f) \leqq \mu^*(f)$.*

By virtue of the definitions, it is enough to show that if $u \in \mathscr{S}$, $v \in$ $u \leqq v$, then $\mu_*(u) \leqq \mu^*(v)$. Since $-u \in \mathscr{S}$ it follows that $v - u = v +$ defined throughout X, belongs to $\mathscr{S}$ and is $\geqq 0$; hence

$$0 \leqq \mu^*(v-u) = \mu^*(v+(-u)) = \mu^*(v) + \mu^*(-u)$$
$$= \mu^*(v) - \mu_*(u)$$

by virtue of (13.5.3).

We shall often write $\int^* f\,d\mu$ or $\int^* f(x)\,d\mu(x)$ (resp. $\int_* f\,d\mu$ or $\int_* f(x)$ instead of $\mu^*(f)$ (resp. $\mu_*(f)$).

For any subset A of X, we put $\mu^*(A) = \mu^*(\varphi_A)$ and $\mu_*(A) =$ where φ_A is the characteristic function of A (12.7). These numbers and $\mu_*(A)$ are $\geqq 0$ (possibly $+\infty$); they are called, respectively, th *measure* and *inner measure* of A.

Example

(13.5.11) Let λ be Lebesgue measure on $\mathbf{R}$ and let $I =]a, b[$ be a interval in $\overline{\mathbf{R}}$. We shall show that $\lambda^*(I) = b - a$ (which is equal to $b = +\infty$ or $a = -\infty$). If $a < a' < b' < b$, then there exists a con mapping f of $\mathbf{R}$ into the unit interval [0, 1], with support contained in and equal to 1 throughout $[a', b']$ (4.5.2). We have $\int_{-\infty}^{+\infty} f(t)\,dt \geqq$ Conversely, for each function $g \in \mathscr{K}_{\mathbf{R}}(\mathbf{R})$ such that $0 \leqq g \leqq \varphi_I$, w $\int_{-\infty}^{+\infty} g(t)\,dt \leqq b - a$. Hence it follows that $\lambda^*(I) = b - a$.

Now let U be any *open set* in $\mathbf{R}$. Then its *connected components* (3 open intervals ((3.19.1) and (3.19.5)), and the set of connected comp is at most denumerable, because each component contains a point denumerable set $\mathbf{Q}$, and the components are pairwise disjoint. Hence are $I_k =]a_k, b_k[$, then $\varphi_U = \sum_k \varphi_{I_k}$, and therefore ((12.7.4) and (13.5.

$$\lambda^*(U) = \sum_k (b_k - a_k). \tag{13.5.11.1}$$

6. NEGLIGIBLE FUNCTIONS AND SETS

A mapping $f: X \to \overline{\mathbf{R}}$ is said to be *negligible* (with respect to the m μ), or *μ-negligible*, if $\mu^*(|f|) = 0$. Then af is also negligible for all $a \neq$ and if $|g| \leqq |f|$, then g is negligible.

(13.6.1) *If (f_n) is a sequence of negligible functions $\geqq 0$, then $\sum_{n=1}^{\infty} f_n$ is negligible.*

This follows immediately from (13.5.8).

A subset N of X is said to be *negligible* (with respect to μ) or *μ-negligible* if its characteristic function φ_N (12.7) is μ-negligible. Clearly any subset of a negligible set is negligible.

(13.6.2) *A denumerable union of negligible sets is negligible.*

For if (N_k) is a sequence of negligible sets and $N = \bigcup_k N_k$, we have $\varphi_N = \sup_k \varphi_{N_k} \leqq \sum_k \varphi_{N_k}$, and the result follows from (13.6.1).

For example, with respect to Lebesgue measure λ, every set $\{t\}$ consisting of a single real number is negligible. For if $\varepsilon > 0$ there exists a function $f \in \mathscr{K}_{\mathbf{R}}(\mathbf{R})$ with values in $[0, 1]$ which is equal to 1 at the point t and vanishes on the complement of the interval $[t - \varepsilon, t + \varepsilon]$, and therefore

$$\lambda^*(\varphi_{\{t\}}) \leqq \lambda(f) \leqq 2\varepsilon.$$

Hence, by (13.6.2), it follows that every *denumerable* subset of $\mathbf{R}$ (for example, the set $\mathbf{Q}$ of rational numbers) is negligible with respect to Lebesgue measure.

One can also give examples of nondenumerable sets in $\mathbf{R}$ which are negligible with respect to Lebesgue measure (Section 13.8, Problem 4).

A property $P(x)$ of the points of X is said to be true *almost everywhere* (with respect to μ) if the complement of the set of points for which $P(x)$ is true is μ-negligible.

(13.6.3) *A mapping $f: X \to \overline{\mathbf{R}}$ is negligible if and only if it is zero almost everywhere.*

Let N be the set of all $x \in X$ such that $|f(x)| > 0$. Then we have $\varphi_N \leqq \sup_n n|f|$, and $|f| \leqq \sup_n n\varphi_N$, hence the result follows from (13.5.7).

(13.6.4) *If a mapping $f: X \to \overline{\mathbf{R}}$ is such that $\mu^*(f) < +\infty$ (resp. $\mu_*(f) > -\infty$), then $f(x) < +\infty$ (resp. $f(x) > -\infty$) almost everywhere.*

It is enough to prove the assertion relating to the upper integral. By hypothesis, there exists a function $h \in \mathscr{I}$ such that $f \leqq h$ and $\mu^*(h) < +\infty$. We may therefore limit ourselves to the case where $f \in \mathscr{I}$, and since there is

then a function $u \in \mathscr{K}_{\mathbf{R}}(X)$ such that $f - u \geqq 0$, we may also assume t
Now let N be the set of points $x \in X$ such that $f(x) = +\infty$; we hav
for all integers $n > 0$, hence $n\mu^*(\varphi_N) \leqq \mu^*(f)$, and the hypothesis
implies that $\mu^*(\varphi_N) = 0$.

The relation "$f(x) = g(x)$ almost everywhere in X" is an eq
relation between mappings of X into a set E, because the unio
negligible sets is negligible. We say then that f and g are *equival*
respect to μ) or *μ-equivalent*, and we shall denote by $\tilde{f}$ the equivale
of a mapping $f: X \to E$. A mapping $f: A \to E$, where A is a subse
said (by abuse of language) to be *defined almost everywhere* in X if
negligible. The equivalence class of such a mapping f is then defined
equivalence class of *any* mapping of X into E which extends f; a
we denote the equivalence class by $\tilde{f}$. Two functions f, g define
everywhere are said to be *equivalent* if $\tilde{f} = \tilde{g}$; this means that the set
$x \in X$ at which f and g are both defined and $f(x) = g(x)$ has a negl
for its complement.

(13.6.5) *Let f, g be two equivalent mappings of* X *into* $\overline{\mathbf{R}}$. *Then* $\mu^*(f)$

Let N be the negligible set of points $x \in X$ such that $f(x) \neq g($
the functions f, g and $\sup(f, g)$ are equal on $X - N$, we may ass
$f \leqq g$. Let h be the negligible function which is equal to $+\infty$ at the
N, and to 0 elsewhere. If $v \in \mathscr{S}$ is such that $f \leqq v$, the function $v + h$
at all points of X, and we have $g \leqq v + h$, so that

$$\mu^*(g) \leqq \mu^*(v + h) \leqq \mu^*(v) + \mu^*(h) = \mu^*(v)$$

by virtue of (13.5.6), because $\mu^*(v) > -\infty$. From the definition of
follows that $\mu^*(g) \leqq \mu^*(f)$ and hence that $\mu^*(g) = \mu^*(f)$.

If f is a mapping into $\overline{\mathbf{R}}$ which is defined and finite almost every
X, then f is equivalent to a *finite* function defined on the whole of
are functions defined almost everywhere in X, with values *in* $\overline{\mathbf{R}}$,
almost everywhere, then the same is true of $f + g$ and fg, and the eq
classes of these functions depend only on $\tilde{f}$ and $\tilde{g}$. They are denoted
and $\tilde{f}\tilde{g}$, respectively.

If $f(x) \leqq g(x)$ almost everywhere, then $f_1(x) \leqq g_1(x)$ almost every
any functions f_1, g_1 equivalent to f, g, respectively. In this case
$\tilde{f} \leqq \tilde{g}$, and this defines an *order relation* on the set of equivalence clas
respect to μ) of mappings of X into $\overline{\mathbf{R}}$.

7. INTEGRABLE FUNCTIONS AND SETS

We have seen (13.5.10) that every function $f: X \to \overline{\mathbf{R}}$ satisfies $\mu_*(f) \leqq \mu^*(f)$. The function f is said to be *integrable* (with respect to μ) or *μ-integrable* if $\mu_*(f)$ and $\mu^*(f)$ are *finite and equal.* Their common value is then called the *integral of f with respect to μ*, and is written $\mu(f)$, or $\langle f, \mu \rangle$, or $\int f \, d\mu$, or $\int f(x) \, d\mu(x)$. Clearly every function $f \in \mathscr{K}_{\mathbf{R}}(X)$ is integrable and its integral is the value of μ at f, so that our notation is consistent.

An integrable function is therefore *finite almost everywhere* (13.6.4), but a *bounded* function is not necessarily integrable. For example, a constant non-zero function on $\mathbf{R}$, or the continuous function $1/(1 + |x|)$, is not integrable (with respect to Lebesgue measure) (cf. (13.20)).

In view of the definitions in (13.5), we have the following criterion for integrability:

(13.7.1) (i) *For a function $f: X \to \overline{\mathbf{R}}$ to be integrable, it is necessary and sufficient that, given any $\varepsilon > 0$, there should exist $g \in \mathscr{S}$ and $h \in \mathscr{I}$ such that $g \leqq f \leqq h$ and $\mu^*(h) - \mu_*(g) \leqq \varepsilon$* (or, equivalently (13.5.3), $\mu^*(h - g) \leqq \varepsilon$).

(ii) *If f is integrable, there exists a decreasing sequence (h_n) of functions belonging to $\mathscr{I}$ and an increasing sequence (g_n) of functions belonging to $\mathscr{S}$ such that $g_n \leqq f \leqq h_n$ for all n and*

$$\lim_{n \to \infty} \mu^*(h_n) = \lim_{n \to \infty} \mu_*(g_n) = \mu(f).$$

(i) The condition implies that $\mu^*(h)$ and $\mu_*(g)$ are necessarily *finite* (and therefore so are $\mu^*(f)$ and $\mu_*(f)$), and that $\mu^*(f) = \mu_*(f)$.

(ii) For each n there exists $h_n' \in \mathscr{I}$ and $g_n' \in \mathscr{S}$ such that $g_n' \leqq f \leqq h_n'$ and $\mu(f) - n^{-1} \leqq \mu_*(g_n') \leqq \mu(f) \leqq \mu^*(h_n') \leqq \mu(f) + n^{-1}$. The required conditions are then satisfied by taking $h_n = \inf(h_1', \ldots, h_n')$ and $g_n = \sup(g_1', \ldots, g_n')$.

If f is integrable, then clearly so is any function f_1 *equivalent* to f (13.6), and we have $\int f_1 \, d\mu = \int f \, d\mu$ (which we denote also by $\mu(\tilde{f})$). It is now clear how integrability should be defined for functions defined *almost everywhere* in X: such a function f is *integrable* if the functions equivalent to f and defined on the whole of X are integrable. The number $\mu(\tilde{f})$ is also denoted by $\int f \, d\mu$, or $\int f(x) \, d\mu(x)$, or $\mu(f)$, or $\langle f, \mu \rangle$.

(13.7.2) *For a mapping $f: X \to \overline{\mathbf{R}}$ to be integrable it is necessary and sufficient that, given any $\varepsilon > 0$, there exists a function $u \in \mathscr{K}_{\mathbf{R}}(X)$ such that $\mu^*(|f - u|) \leqq \varepsilon$.*

Necessity. Suppose that f is integrable, then there exist functions $g \in \mathscr{S}$ and $h \in \mathscr{I}$ such that $g \leqq f \leqq h$ and $\mu^*(h-g) \leqq \frac{1}{2}\varepsilon$. Also ((13.5.1) and (13.5.3)) there exists $u \in \mathscr{K}_{\mathbf{R}}(X)$ such that $u \leqq h$ and $\mu^*(h-u) \leqq \frac{1}{2}\varepsilon$. Since $|f-u| \leqq |h-u| + |h-g|$, it follows by (13.5.3) that

$$\mu^*(|f-u|) \leqq \mu^*(|h-u|) + \mu^*(|h-g|) \leqq \varepsilon.$$

Sufficiency. If $u \in \mathscr{K}_{\mathbf{R}}(X)$ is such that $\mu^*(|f-u|) \leqq \varepsilon$, then by definition (13.5.5) there exists a function $v \in \mathscr{I}$ such that $|f-u| \leqq v$ and $\mu^*(v) \leqq 2\varepsilon$. But the relation $-v \leqq f-u \leqq v$ can be written in the form

$$-v+u \leqq f \leqq v+u$$

(u being finite); also we have $-v+u \in \mathscr{S}$ and $v+u \in \mathscr{I}$ (12.7.5), and $\mu^*(v+u-(-v+u)) = 2\mu^*(v) \leqq 4\varepsilon$; hence f is integrable.

(13.7.3) *The set* $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$ (also denoted by $\mathscr{L}^1_{\mathbf{R}}(\mu)$ or $\mathscr{L}^1_{\mathbf{R}}$) *of finite* μ*-integrable functions on* X *is a vector space over* **R**, *and the mapping* $f \mapsto \int f\,d\mu$ *is a positive linear form on* $\mathscr{L}^1_{\mathbf{R}}$ (i.e., *the relation* $f \geqq 0$ *implies* $\int f\,d\mu \geqq 0$).

If f is finite and integrable, then clearly so is af for every scalar $a \in \mathbf{R}$, and we have $\int af\,d\mu = a\int f\,d\mu$. If f and g are finite and integrable, it follows from (13.5.6) applied to f and g and to $-f$ and $-g$ that

$$\int f\,d\mu + \int g\,d\mu \leqq \int_*(f+g)\,d\mu \leqq \int^*(f+g)\,d\mu \leqq \int f\,d\mu + \int g\,d\mu,$$

which completes the proof.

It follows that if f and g are integrable functions (finite or not) on X with values in $\overline{\mathbf{R}}$, then $f+g$ (which is defined almost everywhere) is integrable, and that $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$.

(13.7.4) *If* f *is integrable, then so are* $|f|$, f^+ *and* f^-, *and we have*

$$\left|\int f\,d\mu\right| \leqq \int |f|\,d\mu. \tag{13.7.4.1}$$

If f *and* g *are integrable, then so are* $\sup(f, g)$ *and* $\inf(f, g)$.

If $u \in \mathscr{K}_{\mathbf{R}}(X)$ is such that $\mu^*(|f-u|) \leqq \varepsilon$, then since

$$\big||f| - |u|\big| \leqq |f-u|,$$

it follows that $\mu^*(\big||f| - |u|\big|) \leqq \varepsilon$, which by (13.7.2) shows that $|f|$ is integrable. Moreover, we have $-|f| \leqq f \leqq |f|$ and hence

$$\mu^*(-|f|) = -\mu(|f|) \leqq \mu^*(f) = \mu(f) \leqq \mu^*(|f|) = \mu(|f|),$$

which establishes (13.7.4.1). Since $f^+ = \frac{1}{2}(f + |f|)$ and $f^- = \frac{1}{2}(|f| - f)$, it follows that f^+ and f^- are integrable (13.7.3). If f and g are integrable, then $f - g$ is defined almost everywhere and integrable, and the functions (everywhere defined) $\sup(f, g)$ and $\inf(f, g)$ are equivalent respectively to the functions (defined almost everywhere) $\frac{1}{2}(f + g + |f - g|)$ and $\frac{1}{2}(f + g - |f - g|)$. Hence they are integrable.

(13.7.5) *For a function $h \in \mathscr{I}$* (resp. $g \in \mathscr{S}$) *to be integrable, it is necessary and sufficient that* $\mu^*(h) < +\infty$ (resp. $\mu_*(g) > -\infty$).

For if $\mu^*(h) < +\infty$, then for any $\varepsilon > 0$ there exists $u \in \mathscr{K}_{\mathbf{R}}(X)$ such that $u \leqq h$ and $\mu^*(h - u) = \mu^*(h) - \mu(u) \leqq \varepsilon$, by (13.5.1) and (13.5.3). The result now follows from the definition of integrable functions.

A subset A of X is said to be *integrable* if its characteristic function φ_A is integrable, or equivalently if $\mu^*(A)$ and $\mu_*(A)$ are *finite* and *equal*. Their common value $\int \varphi_A \, d\mu$ is then denoted by $\mu(A)$ and is called the *measure* of A. We have $\mu(\varnothing) = 0$. The negligible sets are the same as the integrable sets with *measure zero*. If A is integrable, then so is every set B for which $A \cap \complement B$ and $B \cap \complement A$ are *negligible*, and $\mu(B) = \mu(A)$.

(13.7.6) *If* A *and* B *are integrable sets, then* $A \cup B$, $A \cap B$ *and* $A \cap \complement B$ *are integrable.*

This follows from the formulas (12.7.3).

(13.7.7) *Every compact set is integrable. For an open set* U *to be integrable it is necessary and sufficient that* $\mu^*(U) < +\infty$. *In particular, every relatively compact open set is integrable.*

This follows immediately from (13.7.5) and (12.7.4).

Example

(13.7.8) With respect to Lebesgue measure on $\mathbf{R}$, every bounded interval with endpoints a, b is integrable and its measure is $|b - a|$, by virtue of

(13.5.11) and the fact that all finite sets are negligible (for Lebesgue measure) (13.6).

If $I = [a, b]$ is a bounded interval, the "Dirichlet function" (3.11) $\varphi_I - \varphi_I \varphi_Q$ (which is 0 at the rational points of I and on the complement of I, and 1 at the irrational points of I) is integrable, and its integral is $b - a$, since the set Q is negligible (13.6).

(13.7.9) *A subset* A *of* X *is integrable if and only if, given any* $\varepsilon > 0$, *there exists a compact set* K *and an open set* G *such that* $K \subset A \subset G$ *and* $\mu^*(G - K) \leqq \varepsilon$.

The condition is sufficient by virtue of the definition of integrable functions and (12.7.4). Suppose conversely that A is integrable, and let ε be such that $0 < \varepsilon < 1$. By hypothesis, there exists $h \in \mathscr{S}$ such that $\varphi_A \leqq h$ and

$$\int (h - \varphi_A)\, d\mu \leqq \varepsilon.$$

Let G be the set of points $x \in X$ such that $h(x) > 1 - \varepsilon$; then G is open (12.7.2) and contains A. Clearly $h \geqq (1 - \varepsilon)\varphi_G$, hence

$$\mu(G) \leqq \frac{1}{1-\varepsilon} \int h\, d\mu \leqq \frac{1}{1-\varepsilon} (\mu(A) + \varepsilon).$$

Also there exists $g \in \mathscr{S}$ such that $g \leqq \varphi_A$ and $\int (\varphi_A - g)\, d\mu \leqq \varepsilon$, and by the definition of $\mathscr{S}$ the set of points $x \in X$ such that $g(x) > 0$ is relatively compact. Choose $\delta > 0$ so that $\delta\mu(A) \leqq \varepsilon$, and let K be the set of all $x \in X$ such that $g(x) \geqq \delta$; then K is closed in X (12.7.2) and therefore compact (3.17.3), since it is contained in a relatively compact set. Clearly we have $K \subset A$ and $g \leqq \varphi_K + \delta\varphi_B$, where $B = A - K$; hence

$$\int g\, d\mu \leqq \mu(K) + \delta\mu(B) \leqq \mu(K) + \delta\mu(A) \leqq \mu(K) + \varepsilon,$$

and finally $\mu(A) \leqq \int g\, d\mu + \varepsilon \leqq \mu(K) + 2\varepsilon$. Q.E.D.

There exist *bounded* subsets of **R** which are *not integrable* with respect to Lebesgue measure (Section 13.21, Problem 6).

(13.7.10) Let $\pi : X \to X'$ be a *homeomorphism*. It follows directly from the definitions that if f' is any function on X' with values in $\overline{\mathbf{R}}$, we have

$$\int^* f' d(\pi(\mu)) = \int^* (f' \circ \pi)\, d\mu, \qquad \int_* f' d(\pi(\mu)) = \int_* (f' \circ \pi)\, d\mu.$$

The function f' is $\pi(\mu)$-integrable if and only if $f' \circ \pi$ is μ-integrable, and in that case we have $\int f' \, d(\pi(\mu)) = \int (f' \circ \pi) \, d\mu$.

8. LEBESGUE'S CONVERGENCE THEOREMS

(13.8.1) *Let (f_n) be an increasing sequence of integrable functions. For $\sup_n f_n$ to be integrable, it is necessary and sufficient that $\sup_n \int f_n \, d\mu < +\infty$, and if this condition is satisfied we have*

$$\int \left(\sup_n f_n\right) d\mu = \sup_n \int f_n \, d\mu. \tag{13.8.1.1}$$

Since $\int^* f_n \, d\mu > -\infty$ for all n, we have $\int^* \left(\sup_n f_n\right) d\mu = \sup_n \int f_n \, d\mu$ (13.5.7). This already shows that the condition is necessary. Conversely, if the condition is satisfied, let $f = \sup_n f_n$. Then, given any $\varepsilon > 0$, there exists n such that the function $f - f_n$ (which is defined almost everywhere because f and f_n are finite almost everywhere (13.6.4)) satisfies

$$\int^* (f - f_n) \, d\mu \leqq \int^* f \, d\mu + \int^* (-f_n) \, d\mu = \int^* f \, d\mu - \int f_n \, d\mu \leqq \varepsilon$$

(13.5.6). But there exists a function $u \in \mathscr{K}_{\mathbf{R}}(X)$ such that $\int |f_n - u| \, d\mu \leqq \varepsilon$ (13.7.2), hence by (13.5.6) we have

$$\int^* |f - u| \, d\mu \leqq \int^* |f - f_n| \, d\mu + \int^* |f_n - u| \, d\mu \leqq 2\varepsilon,$$

and the result follows by (13.7.2).

There is of course a corresponding theorem for *decreasing* sequences of integrable functions, obtained by replacing f_n by $-f_n$ in (13.8.1).

(13.8.2) *Let (f_n) be any sequence of integrable functions. In order that $f = \sup_n f_n$ should be integrable, it is necessary and sufficient that there should exist a function $g \geqq 0$ such that $\int^* g \, d\mu < +\infty$ and $f_n \leqq g$ almost everywhere.*

The condition is obviously necessary (take $g = f^+$). Conversely, if it is satisfied, let $g_n = \sup_{1 \leqq k \leqq n} f_k$. Then g_n is integrable (13.7.4) and $f = \sup_n g_n$.

Since the sequence (g_n) is increasing and $\int g_n\,d\mu \leqq \int^* g_n\,d\mu < +\infty$, the result follows from (13.8.1).

(13.8.3) *Let (f_n) be a sequence of integrable functions, and suppose that there exists a function $h \geqq 0$ such that $\int^* h\,d\mu < +\infty$ and $-h \leqq f_n \leqq h$ almost everywhere, for all n. Then the functions $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are integrable, and we have*

$$\text{(13.8.3.1)}\quad \int \left(\liminf_{n\to\infty} f_n\right) d\mu \leqq \liminf_{n\to\infty} \int f_n\,d\mu \leqq \limsup_{n\to\infty} \int f_n\,d\mu \leqq \int \left(\limsup_{n\to\infty} f_n\right) d\mu.$$

It is enough to prove that $\liminf_{n\to\infty} f_n$ is integrable and that the left-hand inequality in (13.8.3.1) is valid; the other assertions then follow by replacing f_n by $-f_n$. Let $g_n = \inf_{p\geqq 0} f_{n+p}$. It follows from (13.8.2) applied to the sequence $(-f_{n+p})_{p\geqq 0}$ and from the hypothesis $f_{n+p} \geqq -h$, that g_n is integrable, and clearly $\int g_n\,d\mu \leqq \int f_{n+p}\,d\mu$ for all $p \geqq 0$, so that $\int g_n\,d\mu \leqq \inf_{p\geqq 0} \int f_{n+p}\,d\mu$. The sequence (g_n) is increasing; we have $g_n \leqq h$ for all n, and by definition (12.7.10), $\liminf_{n\to\infty} f_n = \sup_n g_n$. Hence it follows from (13.8.1) that $\liminf_{n\to\infty} f_n$ is integrable and that

$$\int \left(\liminf_{n\to\infty} f_n\right) d\mu = \sup_n \int g_n\,d\mu \leqq \sup_n \left(\inf_{p\geqq 0} \int f_{n+p}\,d\mu\right) = \liminf_{n\to\infty} \int f_n\,d\mu.$$

(13.8.4) (Dominated convergence theorem) *Let (f_n) be a sequence of integrable functions; suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ exists almost everywhere and that there exists a function $g \geqq 0$ such that $\int^* g\,d\mu < +\infty$ and $|f_n| \leqq g$ almost everywhere, for all n. Then f is integrable and*

$$\text{(13.8.4.1)}\quad \int f(x)\,d\mu(x) = \lim_{n\to\infty} \int f_n(x)\,d\mu(x).$$

This follows from (13.8.3). In this situation the two extreme terms of (13.8.3.1) are equal, whence the result.

The usefulness of this theorem is that *it does not require* the convergence to be *uniform* (not even on compact subsets). In particular:

(13.8.5) *Let (f_n) be a sequence of integrable functions, such that the series with general term $\int |f_n|\, d\mu$ is convergent. Then the series with general term $f_n(x)$ is almost everywhere absolutely convergent in* $\mathbf{R}$, *and if we put* $f(x) = \sum_{n=1}^{\infty} f_n(x)$, *the function f (defined almost everywhere) is integrable and we have*

$$\int f(x)\, d\mu(x) = \sum_{n=1}^{\infty} \int f_n(x)\, d\mu(x) \tag{13.8.5.1}$$

("term-by-term integration of a series of integrable functions").

For each integer $N \geqq 0$, let $g_N = \sum_{n=1}^{N} |f_n|$. The sequence (g_N) is an increasing sequence of integrable functions, and

$$\int g_N\, d\mu = \sum_{n=1}^{N} \int |f_n|\, d\mu$$

((13.7.4) and (13.7.3)). By (13.8.1), the function $g = \lim_{N\to\infty} g_N = \sum_{n=1}^{\infty} |f_n|$ is integrable. This implies first of all that g is finite almost everywhere (13.6.4), in other words that the series with general term $|f_n(x)|$ is convergent almost everywhere, which proves the first assertion ((5.3.1) and (5.3.2)). Moreover, for all N we have $\sum_{n=1}^{N} |f_n| \leqq g$, from which by (13.8.4) follow the integrability of f and the relation (13.8.5.1).

(13.8.6) (i) *Let* E *be a metric space,* $(x, z) \mapsto f(x, z)$ *a mapping of* $X \times E$ *into* $\overline{\mathbf{R}}$, *and* z_0 *a point of* E. *Suppose that*

(1) *for each* $z \in E$, *the function* $x \mapsto f(x, z)$ *is integrable*;

(2) *for almost all* $x \in X$, *the function* $z \mapsto f(x, z)$ *is continuous at* z_0;

(3) *there exists an integrable function* $g \geqq 0$ *such that, for all* $z \in E$, *we have* $|f(x, z)| \leqq g(x)$ *almost everywhere in* X.

Then $h(z) = \int f(x, z)\, d\mu(x)$ *is continuous at* z_0.

(ii) *Suppose furthermore that* E *is an open interval in* $\mathbf{R}$ *and that f satisfies the following conditions:*

(4) *for almost all* $x \in X$, *the function* $z \mapsto f(x, z)$ *is finite and admits a derivative* $D_2 f(x, z)$;

(5) *there exists an integrable function* $g_1 \geqq 0$ *such that, for all* $z \in$ *have* $|D_2 f(x, z)| \leqq g_1(x)$ *almost everywhere in* X.

Then h is differentiable at every point of E, *and we have*

$$h'(z) = \int D_2 f(x, z)\, d\mu(x) \tag{13.8.6.1}$$

("differentiation under the integral sign").

(iii) *Now suppose that* E *is an open set in* C *and that f is a mapp* $X \times E$ *into* C, *satisfying the following conditions:*

(6) *for almost all* $x \in X$, *the function* $z \mapsto f(x, z)$ *is analytic in* E;

(7) *for all* $z \in E$, *the function* $x \mapsto f(x, z)$ *is integrable*;

(8) *there exists an integrable function* $g \geqq 0$ *such that for all* $z \in$ *have* $|f(x, z)| \leqq g(x)$ *almost everywhere in* X.

Then h is analytic in E, *and for each integer* $k > 0$ *and each point the function* $x \mapsto D_2^k f(x, z)$ *is integrable, and we have*

$$D^k h(z) = \int D_2^k f(x, z)\, d\mu(x).$$

(i) By (3.13.14) it is enough to show that, for every sequence (z_n) t to z_0 in E, we have $\lim_{n \to \infty} h(z_n) = h(z_0)$; and this follows from the hypo and from (13.8.4).

(ii) Again by virtue of (3.13.14), it is enough to show that, fo sequence (t_n) of real numbers $\neq 0$ and tending to 0, the sequence

$$\int \frac{1}{t_n} (f(x, z + t_n) - f(x, z))\, d\mu(x)$$

has as limit the right-hand side of (13.8.6.1). But it follows from the hypo and from (8.5.4) that

$$\left| \frac{1}{t_n} (f(x, z + t_n) - f(x, z)) \right| \leqq g_1(x)$$

for all n, so that the result again follows from (13.8.4).

(iii) Let $\gamma : t \mapsto a + re^{it}$ $(0 \leqq t \leqq 2\pi)$ be a circuit contained in each z in the interior of the disc $|z - a| \leqq r$, we have

$$f(x, z) = \frac{1}{2\pi i} \int_\gamma \frac{f(x, \zeta)\, d\zeta}{\zeta - z} \tag{13.8.6.2}$$

from which it follows ((3.16.5) and (8.7.8)) that, for almost all x, the function $f(x, z)$ is the limit of the sequence $\left(n^{-1}\sum_{k=1}^{\infty} f_{kn}(x)\right)$, where

$$f_{kn}(x) = \frac{f(x, a + re^{2k\pi i/n})re^{2k\pi i/n}}{a + re^{2k\pi i/n} - z}.$$

Applying (13.8.4), it follows that $h(z)$ is the limit of the sequence

$$\frac{1}{n}\sum_{k=1}^{n}\int f_{kn}(x)\, d\mu(x)$$

and therefore that

$$h(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{h(\xi)\, d\xi}{\xi - z},$$

which shows that h is analytic in E (9.9.4). By replacing the term $1/(\zeta - z)$ in the right-hand side of (13.8.6.2) by $k!/(\zeta - z)^{k+1}$, we obtain the formula stated for $D^k h(z)$.

(13.8.7) (i) *If* (A_n) *is a sequence of integrable sets, then* $A = \bigcap_n A_n$ *is integrable. If also the sequence* (A_n) *is decreasing, we have*

$$\mu(A) = \inf_n \mu(A_n). \tag{13.8.7.1}$$

(ii) *If* (A_n) *is a sequence of integrable sets contained in an integrable set* B, *then* $A = \bigcup_n A_n$ *is integrable.*

(iii) *If* (A_n) *is a sequence of integrable sets such that*

$$\sum_{n=1}^{\infty}\mu(A_n) < +\infty,$$

then the union $A = \bigcup_n A_n$ *is integrable, and we have*

$$\mu(A) \leqq \sum_{n=1}^{\infty}\mu(A_n). \tag{13.8.7.2}$$

If moreover the sets A_n *are pairwise disjoint, then*

$$\mu(A) = \sum_{n=1}^{\infty}\mu(A_n) \tag{13.8.7.3}$$

("complete additivity of measure").

Assertion (i) is a consequence of (13.8.1) and (13.8.2), applied to the sequence $(-\varphi_{A_n})$. Assertion (ii) follows from (13.8.2). The first part of (iii) and the formula (13.8.7.2) follow from (13.8.2), the inequality $\varphi_A \leqq \sum_{n=1}^{\infty} \varphi_{A_n}$ and (13.5.8). Finally, if the sets A_n are pairwise disjoint, then we have $\varphi_A = \sum_{n=1}^{\infty} \varphi_{A_n}$, and (13.8.7.3) follows from (13.8.5), applied to the functions φ_{A_n}.

Example

(13.8.8) Let f be a *regulated* function on $\mathbf{R}$ with compact support (and therefore *bounded*, by (7.6.1)). Then f is integrable with respect to Lebesgue measure λ. If $J = [a, b]$ is an interval containing $\mathrm{Supp}(f)$, we have

$$\int f \, d\lambda = \int_a^b f(t)\, dt \tag{13.8.8.1}$$

in the sense of (8.7). For we have $|f| \leqq \|f\| \varphi_J$, and by (7.6.1), (8.7.8) and (13.8.4) it is enough to prove (13.8.8.1) for a step function with compact support (7.6). Such a function is a linear combination of characteristic functions of intervals, so that finally we are reduced to the case where $f = \varphi_I$, with I an interval; and in this case our assertion follows from (13.7.8).

PROBLEMS

1. On the discrete space $\mathbf{N}$, let μ be the measure for which $\mu(\{n\}) = 1$ for all n. Show that the functions $n \mapsto u_n$ which are μ-integrable are those for which the series with general term u_n is absolutely convergent. They are also those for which the upper (or the lower) integral is finite. We have $\int u_n \, d\mu(n) = \sum_{n=0}^{\infty} u_n$, and the space $\mathscr{L}^1_{\mathbf{R}}(\mathbf{N}, \mu)$ is the Banach space l^1 (Section 5.7, Problem 1).

2. (a) Show that if f and g are two real-valued functions $\geqq 0$ defined on X, then $\mu^*(\sup(f, g)) + \mu^*(\inf(f, g)) \leqq \mu^*(f) + \mu^*(g)$.
(b) Let f be a real-valued function on X such that $\mu^*(f)$ is finite. Show that there exists an integrable function $f_1 \geqq f$ such that $\mu(f_1) = \mu^*(f)$. If f_2 is another integrable function such that $f_2 \geqq f$ and $\mu(f_2) = \mu^*(f)$, then f_1 and f_2 are equivalent.
(c) Let f be a real-valued function on X such that both $\mu^*(f)$ and $\mu_*(f)$ are finite. Let g, h be integrable functions such that $g \leqq f \leqq h$ and $\mu(g) = \mu_*(f)$, $\mu(h) = \mu^*(f)$. Show that $\mu_*(f - g) = \mu_*(h - f) = 0$ and that

$$\mu^*(f - g) = \mu^*(h - f) = \mu^*(f) - \mu_*(f).$$

(d) Let f_1, f_2 be real-valued functions on X such that $\mu^*(f_1)$, $\mu^*(f_2)$, $\mu_*(f_1)$, $\mu_*(f_2)$ are all finite. Show that

$$\mu_*(f_1+f_2) \leqq \mu_*(f_1) + \mu^*(f_2) \leqq \mu^*(f_1+f_2).$$

(If g_2 is an integrable function such that $f_2 \leqq g_2$ and $\mu^*(f_2) = \mu(g_2)$, observe that we have $h - g_2 \leqq f_1$ for every integrable function h such that $h \leqq f_1 + f_2$.) Deduce that, if f_1 is integrable, then

$$\mu^*(f_1+f_2) = \mu(f_1) + \mu^*(f_2),$$
$$\mu_*(f_1+f_2) = \mu(f_1) + \mu_*(f_2),$$

and

$$\mu^*(f_1+f_2) = \mu^*(\sup(f_1,f_2)) + \mu^*(\inf(f_1,f_2))$$

(use (a)).

(e) Let f be a finite integrable function, and let g be a function such that $\mu^*(g)$ is finite. Show that g is integrable if and only if $\mu(f) = \mu^*(g) + \mu^*(f-g)$. (If g_1 is an integrable function such that $g \leqq g_1$ and $\mu^*(g_1) = \mu(g_1)$, remark that $f - g_1 \leqq f - g$.)

3. (a) For each pair of subsets A, B of X, show that

$$\mu^*(A \cup B) + \mu^*(A \cap B) \leqq \mu^*(A) + \mu^*(B).$$

(b) For each subset A of X, show that $\mu_*(A)$ is the least upper bound of measures of compact sets contained in A.

(c) For each subset A of X with *finite* outer measure, show that there exist two integrable sets A_1, A_2 such that $A_1 \subset A \subset A_2$ and $\mu_*(A) = \mu(A_1)$ and $\mu^*(A) = \mu(A_2)$. Show that $\mu_*(A - A_1) = \mu_*(A_2 - A) = 0$ and that $\mu^*(A - A_1) = \mu^*(A_2 - A) = \mu^*(A) - \mu_*(A)$.

(d) Let A be an integrable set. Show that if $B \subset A$ then

$$\mu(A) = \mu^*(B) + \mu_*(A - B).$$

(e) Let A, B, be disjoint sets of finite outer measure. If $C = A \cup B$, show that

$$\mu_*(A) + \mu_*(B) \leqq \mu_*(C) \leqq \mu_*(A) + \mu^*(B) \leqq \mu^*(C) \leqq \mu^*(A) + \mu^*(B)$$

and that

$$\mu_*(C) - \mu_*(A) - \mu_*(B) \leqq \mu^*(A) + \mu^*(B) - \mu^*(C).$$

(To prove the latter inequality, reduce to the case where $\mu_*(A) = \mu_*(B) = 0$ with the help of (c): if A_2, B_2 are integrable sets such that $A \subset A_2$, $B \subset B_2$ and $\mu^*(A) = \mu(A_2)$, $\mu^*(B) = \mu(B_2)$, show that $\mu_*(C) \leqq \mu(A_2 \cap B_2)$.)

4. Let λ be Lebesgue measure on the unit interval $I = [0, 1]$.

(a) Let α be a real number such that $0 \leqq \alpha < 1$. Construct a nowhere-dense (12.16) compact set in I whose Lebesgue measure is α. (Adapt the method of construction of the Cantor set (Section 4.2, Problem 2), for which $\alpha = 0$.)

(b) Construct a sequence (A_n) of mutually disjoint, nowhere-dense compact sets, such that $\lambda(A_n) = 2^{-n}$ and such that every interval which is a connected component of $I - \bigcup_{i \leqq n} A_i$ contains a subset of A_{n+1} of measure >0. If $A = \bigcup_n A_n$, show that A is a *meager* set of measure 1, and consequently that $I - A$, which is the complement of a meager set, is λ-*negligible*.

5. Let λ be Lebesgue measure on $\mathbf{R}$, and let E be the topological vector space obtained by endowing the space $\mathscr{L}^1(\lambda)$ with the topology of simple convergence (12.15). Show that the linear form $f \mapsto \int f\,d\lambda$ on E is not continuous.

6. Let (A_n) be a sequence of μ-integrable subsets of X, such that $\sum_n \mu(A_n) < +\infty$. For each integer k, let G_k be the set of all $x \in X$ such that $x \in A_n$ for at least k values of n. Show that G_k is integrable and that $k \cdot \mu(G_k) \leqq \sum_n \mu(A_n)$.

7. Let X be a compact space, μ a positive measure on X, and (A_n) a sequence of integrable subsets of X such that $\inf_n \mu(A_n) = m > 0$. Show that the set B of points of X which belong to infinitely many of the sets A_n is integrable, and that $\mu(B) \geqq m$.

8. Let λ be Lebesgue measure on the interval $I = [0, 1]$ and let A_n be a sequence of integrable sets in I such that $\inf_n \lambda(A_n) = m > 0$. For each integer k, let $\mathfrak{D}_k$ denote the set of 2^k intervals of the form $[j \cdot 2^{-k}, (j+1) \cdot 2^{-k}]$ in I $(0 \leqq j < 2^k)$, and $\mathfrak{C}_k$ the set of all the unions of sets of $\mathfrak{D}_k$. Show that there exists a decreasing sequence $(I_k)_{k \geqq 0}$ of subsets of I such that $I_0 = I$, $I_k \in \mathfrak{C}_k$ for all k, and a subsequence (A_{n_k}) of the sequence (A_n) with the following properties: (i) $\lambda(A_{n_r} \cap (I - I_k)) \leqq \frac{1}{2}m\lambda(I - I_k)$ for all $r \geqq k$); (ii) $\lambda(A_{n_r} \cap J) \geqq \frac{1}{2}m\lambda(J)$ for all $r \geqq k$ and all $J \in \mathfrak{D}_k$ contained in I_k. (Proceed by induction on k, using the diagonal trick.) Deduce that there exists a subsequence (B_n) of (A_n) such that, for all k and all $J \in \mathfrak{D}_k$ contained in I_k, the intersection of J with the sequence (B_n) is not empty. (Use Problem 7 and the diagonal trick.) Deduce that $\bigcap_n B_n$ contains a non-empty compact set with no isolated points (and therefore nondenumerable (Section 4.2, Problem 3(c))).

9. Let f be a real-valued function $\geqq 0$ on X. Show that the mapping $\mu \mapsto \mu^*(f)$ of $M_+(X)$ into $\bar{\mathbf{R}}$ is continuous with respect to the vague topology (13.4) if and only if f is continuous and compactly supported. The mapping $\mu \mapsto \mu^*(f)$ is lower semicontinuous with respect to the vague topology if and only if f is lower semicontinuous.

10. Let (μ_n) be an increasing sequence of positive measures on X. Suppose that the sequence is bounded above in $M_+(X)$, and let μ be its least upper bound (13.4.4). Show that, for every function $f \geqq 0$ on X, we have $\mu^*(f) = \lim_{n \to \infty} \mu_n^*(f)$.

11. (a) Let (μ_n) be a decreasing sequence of positive measures on X, and let μ be the greatest lower bound of the sequence in $M_+(X)$ (13.4.4). If f is a function $\geqq 0$ on X such that $\mu_n^*(f) < +\infty$ for all sufficiently large n, show that $\mu^*(f) = \lim_{n \to \infty} \mu_n^*(f)$. (If $g \geqq 0$ is lower semicontinuous and $\mu^*(g) < +\infty$, observe that there exists a sequence (h_m) of continuous functions $\geqq 0$ with compact supports, such that $\sum_m h_m \leqq g$ and $\mu_n^*(g) = \sum_m \mu_n^*(h_m)$ for *all* n.)

(b) On the discrete space N, let μ_n be the measure such that $\mu_n(\{m\}) = 0$ for $m < n$ and 1 for $m \geqq n$. Show that the greatest lower bound of the decreasing sequence (μ_n) in $M_+(N)$ is 0, but that $\mu_n^*(N) = +\infty$ for all n.

12. Let X, Y be two locally compact spaces and $\pi : X \to Y$ a *proper* continuous mapping (Section 12.7, Problem 2). Let μ be a positive measure on X, and let $\nu = \pi(\mu)$ (Section 13.4, Problem 8). Show that $\nu^*(g) = \mu^*(g \circ \pi)$ for all functions $g \geqq 0$ on Y. (Consider first the case in which g has compact support, and then use Section 12.7, Problem 2(b).) Show that $N \subset Y$ is ν-negligible if and only if $\pi^{-1}(N)$ is μ-negligible.

13. Assume that there exists in $I = [0, 1]$ an increasing sequence (H_n) of nonmeasurable sets (with respect to Lebesgue measure λ) such that $\bigcup_n H_n = I$ and $\lambda_*(H_n) = 0$ for all n. Show that $\lambda^*\left(\inf_n (1 - \varphi_{H_n})\right) = 0$, but that $\lambda^*(1 - \varphi_{H_n}) = 1$ for all n.

14. (a) Let F be a convex function defined on a convex open set $A \subset \mathbf{R}^n$ (Section 8.5, Problem 8) and let f_k $(1 \leqq k \leqq n)$ be n integrable functions on X, such that the mapping $x \mapsto (f_k(x))_{1 \leqq k \leqq n}$ takes its values in A, and such that the composite function $x \mapsto F(f_1(x), \ldots, f_n(x))$ is integrable. Show that if μ is bounded (13.9.16) and $\mu(X) = 1$, then

$$\int F(f_1(x), \ldots, f_n(x))\, d\mu(x) \geqq F\left(\int f_1\, d\mu, \ldots, \int f_n\, d\mu\right).$$

(Use the Hahn–Banach theorem applied to the convex set of points

$$(t_1, \ldots, t_n, u) \in A \times \mathbf{R}$$

such that $u \geqq F(t_1, \ldots, t_n)$.)

Consider in particular the case $n = 1$, $F(t) = e^t$, and deduce the *inequality of the means*

$$u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_m^{\alpha_m} \leqq \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m,$$

where the u_k are >0, the α_k are $\geqq 0$ and $\sum_{k=1}^{m} \alpha_k = 1$.

(b) Suppose that μ is bounded and $\mu(X) = 1$. Show that if two functions $f \geqq 0$, $g \geqq 0$ are integrable and such that $fg \geqq 1$, then $\left(\int f\, d\mu\right)\left(\int g\, d\mu\right) \geqq 1$ (use (a)).

15. For each finite sequence $(A_j)_{1 \leqq j \leqq n}$ of μ-integrable sets, put

$$D(A_1, \ldots, A_n) = \left(\bigcup_{1 \leqq j \leqq n} A_j\right) - \left(\bigcap_{1 \leqq j \leqq n} A_j\right).$$

If not all the A_j are μ-negligible, put

$$\sigma_\mu(A_1, \ldots, A_n) = \mu(D(A_1, \ldots, A_n))/\mu(A_1 \cup \cdots \cup A_n).$$

Show that

$$\sigma_\mu(A_1, \ldots, A_n) \leqq \frac{1}{n-1} \sum_{i<j} \sigma_\mu(A_i, A_j)$$

if none of the A_j is μ-negligible. (Remark that a point of $D(A_1, \ldots, A_n)$ belongs to at least $n - 1$ of the sets $D(A_i, A_j)$ $(i < j)$.)

16. If λ is Lebesgue measure on the unit interval $I = [0, 1]$ and if $\mathfrak{F}(I)$ is the set of nonempty closed subsets of I (Section 3.16, Problem 3), show that the function $A \mapsto \lambda(A)$ is not continuous with respect to the Hausdorff distance on $\mathfrak{F}(I)$.

9. MEASURABLE FUNCTIONS

(13.9.1) *Let* A *be an integrable subset of* X. *Then there exists a partition of* A *consisting of a sequence* (K_n) *of compact sets and a negligible set* N.

We define the sets K_n inductively as follows, using (13.7.9), (13.7.7), and (13.7.6): $K_1 \subset A$ and $\mu(A - K_1) \leqq 1$; $K_n \subset A - \bigcup_{i=1}^{n-1} K_i$ and

$$\mu\left(\left(A - \bigcup_{i=1}^{n-1} K_i\right) - K_n\right) \leqq \frac{1}{n},$$

if $n > 1$. Then the intersection N of the sets $A - \bigcup_{i=1}^{n} K_i$ is negligible, by (13.8.7).

A subset of A is said to be *measurable* (with respect to μ) or *μ-measurable* if there exists a partition of A consisting of a *sequence of compact sets and a negligible set*. If K′ and K are compact sets such that $K' \subset K$, then $K - K'$ is integrable ((13.7.6) and (13.7.7)), and therefore, by virtue of (13.9.1), an equivalent statement is that A is the *union of a sequence of compact sets and a negligible set*. The same reasoning shows also that another equivalent statement is that A is the *union of a sequence of integrable sets*.

A set $A \subset X$ is said to be *universally measurable* if it is measurable with respect to *every* positive measure on X.

(13.9.2) *For a subset* A *of* X *to be measurable it is necessary and sufficient that* $A \cap K$ *should be integrable, for each compact set* K *in* X.

The condition is necessary, by virtue of (13.8.7(ii)). It is sufficient, because X is the union of an increasing sequence (K_n) of compact sets (3.18.3), and therefore A is the union of the sequence $(A \cap K_n)$ of integrable sets. In particular, we see that the space X itself is measurable.

(13.9.3) (i) *The complement of a measurable set is measurable.*

(ii) *Denumerable unions and denumerable intersections of measurable sets are measurable.*

(iii) *All open sets and all closed sets are universally measurable.*

Assertion (i) follows from (13.9.2) and (13.7.6); assertion (ii) from (13.9.2) and (13.8.7); assertion (iii) from (13.9.2) and (13.7.7) for closed sets, and then by (i) for open sets.

The sets obtained from the open (or closed) sets of X by iterated application of the operations (i) and (ii) above are therefore also *universally measurable.* The definition of a μ-measurable set A also shows that there exists a *universally measurable* set B contained in A such that $A - B$ is μ-negligible. By applying this result to $X - A$, we see that there also exists a *universally measurable* set C containing A such that $C - A$ is μ-negligible.

A mapping u of X into a topological space Y is said to be *measurable* (with respect to μ) or *μ-measurable* if there exists a *partition* of X into a *sequence of compact sets* K_n and a *μ-negligible* set N such that *each of the restrictions* $u \mid K_n$ *is continuous.* The mapping u is said to be *universally measurable* if it is measurable for *all* measures on X. A *continuous* function is universally measurable.

For a subset A of X to be measurable (resp. universally measurable) it is necessary and sufficient that its characteristic function φ_A should be measurable (resp. universally measurable). For if we have a partition of X consisting of a sequence of compact sets K_n and a negligible set N, such that $\varphi_A \mid K_n$ is continuous for all n, then K_n is the union of two disjoint compact subsets, namely $A \cap K_n$ and $K_n - (A \cap K_n)$ ((3.11.4) and (3.17.3)), and A is the union of the sets $A \cap K_n$ and $A \cap N$, and is therefore measurable. Conversely, if A is measurable, then so is $X - A$ (13.9.3), hence there exists a partition of X consisting of a sequence of compact sets K_n and a negligible set N, such that either $K_n \subset A$ or $K_n \subset X - A$ for all n. Consequently $\varphi_A \mid K_n$ is continuous for all n.

(13.9.4) *Let u be a mapping of* X *into a topological space* Y. *Then the following three properties are equivalent:*

(a) *u is μ-measurable.*

(b) *For each compact subset* K *of* X *and each* $\varepsilon > 0$, *there exists a compact subset* K′ *of* K *such that* $\mu(K - K') \leqq \varepsilon$ *and such that the restriction of u to* K′ *is continuous.*

(c) *For each compact subset* K *of* X, *there exists a partition of* K *consisting of a sequence* (L_n) *of compact sets and a μ-negligible set* M, *such that each of the restrictions* $u \mid L_n$ *is continuous.*

To show that (a) implies (c), we remark that the hypothesis (a) implies the existence of a partition of X consisting of a sequence (K_n) of compact sets and a negligible set N, such that $u \mid K_n$ is continuous for all n; hence (c) is satisfied by taking $L_n = K \cap K_n$ and $M = K \cap N$.

To show that (c) implies (b), we remark that it follows from (c) that $\mu(K) = \sum_{n=1}^{\infty} \mu(L_n)$ (13.8.7), and hence that there is an integer n such that $\mu(K - K') \leqq \varepsilon$, where $K' = \bigcup_{i<n} L_i$. Clearly K′ satisfies the conditions of (b).

Finally, we have to show that (b) implies (a). The space X is the union of an increasing sequence (H_n) of compact sets (3.18.3). For each n we use the hypothesis (b) together with (13.7.9) to define a sequence $(K_{mn})_{m\geq 1}$ of compact sets such that $K_{1n} \subset H_n - H_{n-1}$, $K_{mn} \subset (H_n - H_{n-1}) - \bigcup_{i<m} K_{in}$, and

$$\mu\left((H_n - H_{n-1}) - \bigcup_{i<m} K_{in}\right) \leqq \frac{1}{m}$$

and such that the restrictions $u \mid K_{mn}$ are continuous. The complement of the union of the sets K_{mn} for $n \geqq 1$ and $m \geqq 1$ is then negligible, and consequently u is measurable.

It follows that, for a mapping $f: X \to \overline{\mathbf{R}}$ to be measurable, it is necessary and sufficient that, for each compact subset K of X, the function $f\varphi_K$ (which *by convention* is taken to be zero at all points of $X - K$, even if f is *infinite* at such points (cf. (13.11)) should be *measurable.* For this condition is sufficient by virtue of (13.9.4), and it is necessary by (13.9.4) and the fact that $X - K$ is measurable (13.9.3).

The same reasoning shows that if f is a mapping of X into $\overline{\mathbf{R}}$ and K is a compact subset of X such that $f \mid K$ is *continuous*, then $f\varphi_K$ is *measurable.*

(13.9.5) *Let (Y_n) be a sequence of topological spaces, and for each n let u_n be a measurable mapping of X into Y_n. Then, given any compact subset K of X and any $\varepsilon > 0$, there exists a compact subset K′ of K such that $\mu(K - K') \leqq \varepsilon$ and such that $u_n \mid K'$ is continuous for all n.*

Using (13.9.4) we define inductively a decreasing sequence (K_n) of compact subsets of K such that $\mu(K - K_1) \leqq \frac{1}{2}\varepsilon$ and $\mu(K_n - K_{n+1}) \leqq \varepsilon/2^{n+1}$, and such that $u_n \mid K_n$ is continuous. Then by (13.8.7) it is clear that $K' = \bigcap_n K_n$ satisfies the required conditions.

(13.9.6) *Let $(Y_i)_{1\leqq i\leqq n}$ be a finite sequence of topological spaces, Z a topological space, and v a continuous mapping of $\prod_i Y_i$ into Z. Let $u_i : X \to Y_i$ $(1 \leqq i \leqq n)$ be measurable. Then the mapping $x \mapsto v(u_1(x), \ldots, u_n(x))$ is measurable.*

This follows immediately from (13.9.5) and (13.9.4).

(13.9.7) *If f, g are two measurable functions with values in $\overline{\mathbf{R}}$, then* $\sup(f, g)$ *and* $\inf(f, g)$ *are measurable. If u, v are measurable mappings of X into a (real or complex) vector space, then $u + v$ and au (where a is any scalar) are measurable.*

(13.9.8) *A function equivalent to a measurable function is measurable.*

This follows from (13.9.4) and the fact that if K is compact and $N \subset K$ is negligible, then there exists a compact set $K' \subset K - N$ with measure arbitrarily close to the measure of K (13.7.9).

A mapping f *defined almost everywhere* in X, with values in a topological space Y, is said to be *measurable* if every mapping of X into Y which is equivalent to f (13.6) is measurable. Clearly (13.9.8) it is enough for *one* such mapping to be measurable.

(13.9.8.1) In particular, it follows from (13.9.6) and (13.9.8) that if f and g (with values in $\overline{\mathbf{R}}$) are *defined and finite almost everywhere* in X and measurable, then $f + g$ and fg (which are defined almost everywhere) are measurable.

(13.9.9) *Let u be a measurable mapping of* X *into a topological space* Y. *If* M *is any open or closed subset of* Y, *then* $u^{-1}(M)$ *is measurable.*

Consider a partition of X consisting of a sequence of compact sets K_n and a negligible set N, such that the restrictions $u \mid K_n$ are continuous. If $M \subset Y$ is closed, then $u^{-1}(M) \cap K_n$ is compact (3.11.4), hence $u^{-1}(M)$ is measurable. If M is open, then $X - u^{-1}(M) = u^{-1}(Y - M)$ is measurable, and therefore so is $u^{-1}(M)$ (13.9.3).

(13.9.10) (Egoroff's theorem) *Let* Y *be a metric space and let* (f_n) *be a sequence of measurable mappings of* X *into* Y *such that, for almost all* $x \in X$, *the sequence* $(f_n(x))$ *converges to a limit* $f(x)$. *Then*

(1) f *is measurable;*

(2) *for each compact subset* K *of* X *and each* $\varepsilon > 0$, *there exists a compact subset* K′ *of* K *such that* $\mu(K - K') \leqq \varepsilon$ *and such that the restrictions* $f_n \mid K'$ *are continuous and converge* uniformly *in* K′ *to* $f \mid K'$.

Clearly it is sufficient to prove the second assertion ((13.9.4) and (7.2.1)). Let K_0 be a compact subset of K such that $\mu(K - K_0) \leqq \frac{1}{2}\varepsilon$ and such that the restrictions $f_n \mid K_0$ are continuous for all n (13.9.5). If d denotes the distance function on Y, let $B_{n,r}$ be the set of all $x \in K_0$ such that $d(f_p(x), f_q(x)) \geqq 1/r$ for *at least* one pair of integers (p, q) such that $p \geqq n$ and $q \geqq n$. If $A_{p,q,r}$ is the set of points $x \in K_0$ such that $d(f_p(x), f_q(x)) \geqq 1/r$, then we have

$$B_{n,r} = \bigcup_{p \geqq n,\, q \geqq n} A_{p,q,r}.$$

Now the sets $A_{p,q,r}$ are closed (3.11.4), hence $B_{n,r}$ is integrable (13.8.7). Moreover, by hypothesis, for each integer $r \geqq 1$ we have $\mu\left(\bigcap_n B_{n,r}\right) = 0$, and the sequence $(B_{n,r})_{n \geqq 1}$ is decreasing, so that by (13.8.7) we have $\lim_{n \to \infty} \mu(B_{n,r}) = 0$. Hence there exists an integer n_r such that $\mu(B_{n_r,r}) \leqq \varepsilon/2^{r+2}$. Let $B = \bigcup_r B_{n_r,r}$; the set B is integrable, and we have $\mu(B) \leqq \sum_{r=1}^{\infty} \varepsilon/2^{r+2} = \frac{1}{4}\varepsilon$ by (13.8.7). The set $C = K_0 - B$ is also integrable, and by definition the sequence $(f_n | C)$ converges uniformly to $f | C$. Hence we may take K′ to be any compact subset of C such that $\mu(C - K') \leqq \frac{1}{4}\varepsilon$ (13.7.9).

(13.9.11) *Let (f_n) be any sequence of measurable functions on* X *with values in* $\overline{\mathbf{R}}$. *Then the functions* $\inf_n f_n$, $\sup_n f_n$, $\limsup_{n \to \infty} f_n$ *and* $\liminf_{n \to \infty} f_n$ *are measurable.*

For $\sup_n f_n$ is the limit of the increasing sequence of functions $g_n = \sup_{1 \leqq i \leqq n} f_i$, which are measurable (13.9.7); and $\limsup_{n \to \infty} f_n$ is the limit of the decreasing sequence of functions $\sup_{p \geqq 0} f_{n+p}$, which are measurable, by what has just been proved.

(13.9.11.1) The propositions (13.9.6), (13.9.7), (13.9.9), and (13.9.11) remain valid when the word "measurable" is replaced throughout by "universally measurable." On the other hand, a function cannot be universally measurable unless it is *everywhere defined*, because any nonempty subset N of X has measure $\mu(N) \neq 0$ if for example μ is the Dirac measure at a point of N.

In particular (12.7.8), every function belonging to $\mathscr{I}$ (resp. $\mathscr{S}$) is universally measurable. If f is any lower semicontinuous function on X, then by writing

$$f = \inf_n (\sup(f, -n)) \qquad \text{and} \qquad \sup(f, -n) = -n + \sup(f + n, 0)$$

and using (13.9.6), it follows that

(13.9.11.2) *Every (upper or lower) semicontinuous function on* X *with values in* $\overline{\mathbf{R}}$ *is universally measurable.*

The same will be true of all functions obtained from semi-continuous functions by repeated applications of the operations in (13.9.6), (13.9.7), and (13.9.11).

A function $f: X \to \overline{\mathbf{R}}$ is a *step function* if it takes only a *finite* number of values a_k ($1 \leqq k \leqq n$). In that case we have $f = \sum_{k=1}^{n} a_k \varphi_{A_k}$, where $A_k = f^{-1}(a_k)$, with the *convention* that if a_k is $\pm\infty$, the value of $a_k \varphi_{A_k}$ is a_k in A_k and 0 in $X - A_k$ (13.11). It follows immediately from this and from (13.9.9) that f is measurable if and only if each of the A_k is measurable.

(13.9.12) *Let $f: X \to \overline{\mathbf{R}}$ be measurable. Then there exists a sequence (g_n) of universally measurable step functions with compact support, such that $|g_n(x)| \leqq |f(x)|$ for all $x \in X$ and such that the sequence $(g_n(x))$ converges almost everywhere to $f(x)$ (which implies that f is equivalent to a universally measurable function).*

Consider a partition of X consisting of a sequence (K_n) of compact sets and a negligible set N. For each integer $i \leqq n$, there exists a finite covering $(U_{ij}^{(n)})_{1 \leqq j \leqq q_{in}}$ of K_i by sets which are open *in* K_i, such that the oscillation of f in each $U_{ij}^{(n)}$ is $\leqq 1/n$ (with respect to a distance on $\overline{\mathbf{R}}$, cf. (3.16.5)). Now we have the following lemma:

(13.9.12.1) *Let $(F_h)_{1 \leqq h \leqq p}$ be a finite family of integrable sets. Then there exists a finite family $(G_k)_{1 \leqq k \leqq r}$ of pairwise disjoint integrable sets, such that each F_h is the union of some of the G_k.*

Consider the $2^p - 1$ sets of the form $\bigcap_{h=1}^{p} Z_h$, where each Z_h is either F_h or $X - F_h$, and $Z_h = F_h$ for at least one index h. Then the family of distinct non-empty sets of this form satisfies the conditions of the lemma, because these sets are integrable by (13.7.6) and each F_h is the union of the sets $\bigcap_{l=1}^{p} Z_l$ for which $Z_h = F_h$.

By applying this lemma to each of the finite families $(U_{ij}^{(n)})_{1 \leqq j \leqq q_{in}}$, we obtain a *partition* $(A_{ik}^{(n)})_{1 \leqq k \leqq r_{in}}$ of K_i into universally measurable, integrable sets. Let

$$a_{ik}^{(n)} = \begin{cases} 0 & \text{if } f \text{ changes sign in } A_{ik}^{(n)}, \\ \inf f(A_{ik}^{(n)}) & \text{if } f \geqq 0 \text{ on } A_{ik}^{(n)}, \\ \sup f(A_{ik}^{(n)}) & \text{if } f \leqq 0 \text{ on } A_{ik}^{(n)}, \end{cases}$$

and define g_n as follows: $g_n(x) = a_{ik}^{(n)}$ if $x \in A_{ik}^{(n)}$ ($1 \leqq i \leqq n$, $1 \leqq k \leqq r_{in}$), and $g_n(x) = 0$ otherwise. Then it is clear that the functions g_n have the required properties.

(13.9.13) *In order that a mapping* $f: X \to \overline{\mathbf{R}}$ *should be integrable, it is necessary and sufficient that* f *should be measurable and that the upper integral* $\mu^*(|f|)$ *be finite.*

Necessity. Since $|f|$ is integrable (13.7.4), clearly $\mu^*(|f|)$ is finite. Next we remark that, for each integer n, there exists by hypothesis a function $f_n \in \mathscr{I}$ such that $f \leqq f_n$ and $\mu(f_n - f) \leqq 1/n$ (13.7.1). Replacing f_n by $\inf_{1 \leqq i \leqq n} f_i$ (12.7.5), we may assume that the sequence (f_n) is decreasing. If g is the limit of this sequence (equal to its lower envelope), we have $f \leqq g$ and $\mu(g - f) = \lim_{n \to \infty} \mu(f_n - f) = 0$ (13.8.1). In other words, f is almost everywhere equal to the limit of the sequence (f_n). Now all functions belonging to $\mathscr{I}$ are measurable (13.9.11), and another application of Egoroff's theorem (13.9.10) then proves that f is measurable.

Sufficiency. We may assume that f is finite, by replacing f by an equivalent function (13.6.4). By (13.9.12) there exists a sequence (g_n) of measurable step functions with compact supports such that $|g_n| \leqq |f|$ and the sequence (g_n) converges to f almost everywhere. Now g_n is a linear combination of characteristic functions of relatively compact measurable sets; since such sets are integrable (13.9.2), it follows that g_n is integrable. Hence, applying (13.8.4) again, f is integrable.

(13.9.14) (i) *If* f *is an integrable function and* A *is a measurable set, then the function* $f\varphi_A$ *is integrable.*

(ii) *If* (A_n) *is an increasing sequence of integrable sets, whose union is the complement of a negligible set* N, *then*

$$\int f \, d\mu = \lim_{n \to \infty} \int f\varphi_{A_n} \, d\mu.$$

We may assume without loss of generality that f is everywhere finite (13.6.4). Then $f\varphi_A$ is measurable (13.9.6), and we have

$$\mu^*(|f\varphi_A|) \leqq \mu^*(|f|) \leqq +\infty,$$

from which (i) follows by (13.9.13). The assertion (ii) follows from the dominated convergence theorem (13.8.4).

In the situation of (13.9.14), we put $\int f\varphi_A \, d\mu = \int_A f \, d\mu$, or $\int_A f(x) \, d\mu(x)$; this number is called the *integral of* f *over* A. Also (if A is measurable) we write $\int_A^* f \, du$ in place of $\int^* f\varphi_A \, d\mu$.

(13.9.15) *If f is integrable and if* A, B *are two measurable sets such that* $A \cap B$ *is negligible, then*

$$\int_{A \cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu.$$

For $\varphi_{A \cup B}$ and $\varphi_A + \varphi_B$ are then equivalent functions (12.7.3).

Remark

(13.9.16) If I is a compact interval $[a, b]$ in $\mathbf{R}$, then for any regulated function f with compact support we have $\int_I f\,d\lambda = \int_a^b f(x)\,dx$, where λ is Lebesgue measure (13.8.8). If now f is an arbitrary regulated function defined on $\mathbf{R}$, then f is measurable because it is the limit of the integrable functions $f\varphi_{I_n}$, where $I_n = [-n, n]$ (13.9.11). For f to be integrable, it is necessary and sufficient that $\lim_{n\to\infty} \int_{-n}^{+n} |f(x)|\,dx < +\infty$, and in that case we have $\int f\,d\lambda = \lim_{n\to\infty} \int_{-n}^{+n} f(x)\,dx$ ((13.8.2) and (13.8.4)); this integral is also denoted by $\int_{-n}^{+n} f(x)\,dx$. It can happen that the limit $\lim_{n\to\infty} \int_{-n}^{+n} f(x)\,dx$ exists although $\lim_{n\to\infty} \int_{-\infty}^{+\infty} |f(x)|\,dx = +\infty$ (this is the case, for example, if $f(x) = (\sin x)/x$ for $x \neq 0$). The interpretation of most of these "improper integrals" belongs to the theory of distributions (Section 17.5, Problem 4, and Chapter XXIII).

For the reasons given above, if $I \subset \overline{\mathbf{R}}$ is any interval with endpoints a, b (with $a \leqq b$), we write $\int_a^b f(x)\,dx$ in place of $\int_I f\,d\lambda$, for any function f such that $f\varphi_I$ is integrable.

The (positive) measure μ is said to be *bounded* if the constant function 1 is integrable, or in other words (13.9.13), if $\mu^*(X) = \mu^*(1)$ is *finite*: the number $\mu(X) = \mu(1) = \int_X d\mu$ is then called the *total mass* of μ (cf. (13.20)).

(13.9.17) *If μ is bounded, then every bounded measurable function $f : X \to \overline{\mathbf{R}}$ is integrable* (in particular, every *bounded semicontinuous* function is integrable) *and*

$$\left| \int f\,d\mu \right| \leqq \|f\| \cdot \mu(X). \tag{13.9.17.1}$$

This follows directly from (13.9.13).

(13.9.18) *Let μ be a bounded positive measure on* X. *Then for each $\varepsilon > 0$ there exists a compact subset* K *of* X *such that* $\mu(X - K) \leqq \varepsilon$.

This is a particular case of (13.7.9).

(13.9.19) *Suppose that μ is bounded. Let (f_n) be a uniformly bounded sequence of measurable functions* (i.e., $\sup_n \|f_n\| < +\infty$) *such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *exists for almost all $x \in X$. Then f is integrable, and we have*

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu. \tag{13.9.19.1}$$

This is a particular case of (13.8.4).

Example

(13.9.20) Let μ be a positive measure on X and let Y be a *closed* subspace of X. It follows from the Tietze–Urysohn theorem (4.5.1) that every function $f \in \mathscr{K}_{\mathbf{R}}(Y)$ is the restriction to Y of a function $g \in \mathscr{C}^{\infty}_{\mathbf{R}}(X)$ such that $\|g\| = \|f\|$. The function $f^Y = g\varphi_Y$, which is equal to f in Y and zero outside Y, is therefore μ-integrable (13.9.13), and $|\mu(f^Y)| \leqq \|f\| \cdot \mu(\mathrm{Supp}(f))$. Write $\mu_Y(f) = \mu(f^Y)$ (which clearly depends only on f); then the discussion above shows that $f \mapsto \mu_Y(f)$ is a *positive measure on* Y, which is said to be the measure *induced by* μ on Y. From (12.7.8), the remarks above and (13.5.7), it follows immediately that if f belongs to $\mathscr{I}(Y)$ and if as above we denote by f^Y the function which is equal to f in Y and is zero in $X - Y$, then we have $\mu_Y^*(f) = \mu^*(f^Y)$. Hence we conclude (13.5.5) that this formula is valid for *any* mapping $f: Y \to \overline{\mathbf{R}}$ (with f^Y defined as before). Hence, finally, a function $f: Y \to \overline{\mathbf{R}}$ is μ_Y-integrable if and only if f^Y is μ-integrable, and that in this case we have

$$\int f\,d\mu_Y = \int f^Y\,d\mu. \tag{13.9.20.1}$$

Likewise, a mapping u of Y into a topological space Z is μ_Y-measurable if and only if the mapping $u' : X \to Z$ which extends u to X by giving $u'(x)$ an (arbitrary) constant value for $x \in X - Y$, is μ-measurable.

Conversely, consider a closed subset Y of X and a positive measure ν on Y, and let μ be the canonical extension of ν to X (13.1.7). Clearly we have $\nu = \mu_Y$. Also, if $f \in \mathscr{I}(Y)$, then the function g which is equal to f on Y and to $+\infty$ on $X - Y$ belongs to $\mathscr{I}(X)$, and we have $\nu^*(f) = \mu^*(g)$. It follows

directly (13.5.5) that, for *any* mapping $g : X \to \overline{\mathbf{R}}$, we have $\mu^*(g) = \nu^*(g \mid Y)$, and hence that such a mapping g is μ-integrable if and only if $g \mid Y$ is ν-integrable, in which case we have

$$\int (g \mid Y)\, d\nu = \int g\, d\mu. \tag{13.9.20.2}$$

Likewise, a mapping u of X into a topological space Z is μ-measurable if and only if $u \mid Y$ is ν-measurable.

(13.9.21) If $\pi : X \to X'$ is a *homeomorphism*, then a mapping u' of X' into a topological space Y is $\pi(\mu)$-measurable if and only if $u' \circ \pi$ is μ-measurable.

PROBLEMS

1. Let (f_{mn}) be a double sequence of μ-measurable mappings of X into a metric space Y. Suppose that for each m the sequence $(f_{mn})_{n \geqslant 1}$ converges almost everywhere to a function g_m, and that the sequence (g_m) converges almost everywhere to a function h. Show that there exist two strictly increasing sequences of integers (m_k), (n_k) such that the sequence $(f_{m_k n_k})_{k \geqslant 1}$ converges almost everywhere to h. (First prove the result when X is compact by observing that, for each $\varepsilon > 0$, there exists a compact subset $K \subset X$ such that $\mu(X - K) \leqq \varepsilon$ and such that the sequence (g_m) and each of the sequences $(f_{mn})_{n \geqslant 1}$ are uniformly convergent on K. Then pass to the general case by using the diagonal trick.)

2. (a) Let f, g be two real-valued functions $\geqq 0$, and suppose that g is μ-measurable. Show that $\mu^*(fg) = \inf \mu^*(ug)$, where u runs through the set of all μ-measurable functions $\geqq f$.
(b) Let f, g, h be three functions $\geqq 0$ on X, finite or not, and suppose that g and h are μ-measurable. Show that $\mu^*(f(g+h)) = \mu^*(fg) + \mu^*(fh)$ (using the convention of (13.11) for the product of 0 and $+\infty$).

3. Let $f : X \to \mathbf{R}$ be μ-measurable. For each $n \in \mathbf{Z}$, let A_n be the set of all $x \in X$ such that $2^{n-1} < |f(x)| \leqq 2^n$. Show that f is integrable if and only if $\sum_{n \in \mathbf{Z}} 2^n \mu(A_n) < +\infty$.

4. Let (f_n) be a sequence of integrable functions on X, converging simply to a function f.
(a) If f is integrable and $\int f\, d\mu = \lim_{n \to \infty} \int f_n\, d\mu$, show that for each $\varepsilon > 0$ there exists an integrable set A, an integrable function $g \geqq 0$ and an integer n_0 such that, for all $n \geqq n_0$, we have $\left| \int_{X-A} f_n\, d\mu \right| \leqq \varepsilon$ and $|f_n(x)| \leqq |g(x)|$ for all $x \in A$. (Consider an integrable set B such that $\int_{X-B} |f|\, d\mu \leqq \frac{1}{2}\varepsilon$ and such that f is bounded on B, and apply Egoroff's theorem.)

(b) Suppose that, for each $\varepsilon > 0$, there exists a measurable set A, an integrable
tion $g \geqq 0$, and an integer n_0 such that, for all $n \geqq n_0$, we have $\int_{X-A} |f_n|\, d\mu \leqq$
$|f_n(x)| \leqq |g(x)|$ for all $x \in A$. Show that f is integrable and that $\lim_{n\to\infty} \int |f - f_n|\, d$
Consider the converse.
(c) Show by examples that the conditions of (a) are not sufficient and that the
ditions of (b) are not necessary for f to be integrable and $\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$.

5. If A is measurable and B is *any* subset of X, show that

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap (X - A)).$$

(If $\mu^*(B) < +\infty$, consider an integrable set $B_1 \supset B$ such that $\mu^*(B) = \mu(B_1)$.)
versely, show that if A satisfies this condition (for all $B \subset X$), then A is measu
(cf. Section 13.8, Problem 3).

6. Suppose that X is compact. A bounded real-valued function f on X is said
continuous almost everywhere (with respect to μ) on X if the set of points of d
tinuity of f is negligible.
(a) Give an example of a function f which is continuous almost everywhere and
that there exists no continuous function g which is equal to f almost everywhere.
(b) Suppose that the support of μ is equal to X. Show that a bounded real-v
function f defined on X is equal almost everywhere to an almost everywhere c
uous function on X if and only if there exists a subset A of X such that $X -$
negligible and $f|A$ is continuous. (To show that the condition is sufficient, ol
that A is dense in X and that the lower semicontinuous extension of $f|A$ to
continuous at every point of A.) Deduce that f is measurable. Show that there
a sequence (f_n) of continuous functions on X which converges at *every* point of
whose limit is almost everywhere equal to f (cf. Section 13.11, Problem 3).
(c) Show that a real-valued function f on $\mathbf{R}$ which is *continuous on the righ*
such that $f(x+) = f(x)$ for all $x \in \mathbf{R}$) is continuous except at the points of an at
denumerable set, and is therefore continuous almost everywhere with resp
Lebesgue measure. (Apply Section 3.9, Problem 3 to the set A_n of points $x \in \mathbf{R}$ at
the oscillation of f is $> 1/n$.)

7. A partition $\varpi = (A_\alpha)$ of a set E is said to be *finer* than a partition $\varpi' = (A'_\beta)$ o
for each index α, there exists β such that $A_\alpha \subset A'_\beta$. The partitions of E form an or
set with respect to this relation.
(a) Let X be a compact metric space. For each finite partition $\varpi = (A_k)$ of X
sisting of integrable sets and each bounded real-valued function f on X, put

$$s_\varpi(f) = \sum_k \inf_{x \in A_k} f(x) \cdot \mu(A_k),$$

$$S_\varpi(f) = \sum_k \sup_{x \in A_k} f(x) \cdot \mu(A_k)$$

("Riemann sums" relative to f and the partition ϖ). Show that

$$s_\varpi(f) \leqq \mu_*(f) \leqq \mu^*(f) \leqq S_\varpi(f),$$

and that if ϖ is finer than ϖ', then $s_{\varpi'}(f) \leqq s_\varpi(f)$ and $S_\varpi(f) \leqq S_{\varpi'}(f)$.

(b) A sequence (ϖ_n) of finite partitions of X is said to be *fundamental* if ϖ_{n+1} is finer than ϖ_n for all n and if the maximum of the diameters of the sets of ϖ_n tends to 0 as n tends to $+\infty$. Show that if f is a bounded integrable function on X, then there exists *at least one* fundamental sequence (ϖ_n) of finite partitions of X, consisting of integrable sets, and such that the sequences $(s_{\varpi_n}(f))$ and $(S_{\varpi_n}(f))$ both tend to $\int f\,d\mu$.

(c) Let f be bounded and continuous almost everywhere on X (Problem 6). Show that, for *all* fundamental sequences (ϖ_n) of finite partitions of X consisting of integrable sets, the sequences $(s_{\varpi_n}(f))$ and $(S_{\varpi_n}(f))$ both tend to $\int f\,d\mu$. (Observe that if A_n is the closed set of points $x \in X$ at which the oscillation of f (Section 3.14) is $\geqq 1/n$, then $\mu(A_n)$ tends to 0 as n tends to $+\infty$; each A_n has an open neighborhood V_n such that the points of V_n have a distance $<1/n$ from A_n and such that $\mu(V_n)$ tends to 0. For a partition ϖ_k whose sets all have diameter $<1/n$, consider separately the sets of the partition which meet V_n and those which do not.) If f is bounded and lower semi-continuous on X, show that $s_{\varpi_n}(f)$ tends to $\int f\,d\mu$ for *every* fundamental sequence (ϖ_n) of finite partitions of X consisting of integrable sets.

(d) A subset A of X is said to be *quadrable* (with respect to μ) if its characteristic function φ_A is continuous almost everywhere, or equivalently if the frontier of A is μ-negligible. Show that every point x_0 in X has a fundamental system of quadrable open neighborhoods. (For each neighborhood V of x_0, let $f: X \to [0, 1]$ be a continuous mapping such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in X - V$. For each $\alpha \in]0, 1[$, consider the set of points $x \in X$ such that $f(x) > \alpha$.) Deduce that there exists a fundamental sequence of finite partitions of X consisting of sets which are either open or negligible. Give an example of a nonquadrable closed set (Section 13.8, Problem 4(a); cf. Section 13.21, Problem 2).

(e) Let (ϖ_n) be a fundamental sequence of finite partitions of X consisting of sets which are either open or negligible. If f is a bounded function on X, let g be the largest lower semicontinuous function $\leqq f$; show that $s_{\varpi_n}(f) = s_{\varpi_n}(g)$. Deduce that the sequences $(s_{\varpi_n}(f))$ and $(S_{\varpi_n}(f))$ tend to the same limit if and only if f is continuous almost everywhere.

(f) Use (e) to give an example of a negligible function f and a fundamental sequence (ϖ_n) of finite partitions of X consisting of integrable sets, such that the sequences $(s_{\varpi_n}(f))$ and $(S_{\varpi_n}(f))$ do not tend to the same limit.

8. Let f be a measurable mapping of X into a complete metric space E, and let K be a compact subset of X. Show that $f|K$ can be approximated *uniformly* by measurable step-functions if and only if $f(K)$ is relatively compact in E.

9. Suppose that X is compact. Let (f_n) be a sequence of μ-measurable finite real-valued functions on X. Show that the following properties are equivalent:

(1) There exists a subsequence of (f_n) which tends to 0 almost everywhere in X.

(2) There exists a sequence (t_n) of finite real numbers, such that $\limsup_{n\to\infty} |t_n| > 0$ and such that the series with general term $t_n f_n(x)$ converges almost everywhere in X.

(3) There exists a sequence (t_n) of finite real numbers such that $\sum_n |t_n| = +\infty$ and such that the series with general term $t_n f_n(x)$ converges absolutely almost everywhere in X.

(To show that (1) implies (2) and (3), use Egoroff's theorem. To show that (3 plies (1). show that (3) implies that there exists an increasing sequence of inte subsets A_k of X and a subsequence (f_{n_k}) of the sequence (f_n), such that $\mu(A_k)$ te $\mu(X)$ and $\int_{A_k} |f_{n_k}|\, d\mu$ tends to 0, as k tends to $+\infty$.)

10. Let X, Y be two locally compact spaces and $\pi : X \to Y$ a *proper* continuous map Let μ be a positive measure on X, and let $\nu = \pi(\mu)$ (Section 13.4, Problem 8).

(a) A mapping g of Y into a topological space is ν-measurable if and only if is μ-measurable.

(b) A mapping $g : Y \to \bar{\mathbf{R}}$ is ν-integrable if and only if $g \circ \pi$ is μ-integrable, and we have $\int g\, d\nu = \int (g \circ \pi)\, d\mu$. (Use Problem 12 of Section 13.8.)

(c) Show that the support of ν is the closure of π (Supp(μ)).

11. Let X be a locally compact space, $u : X \to X$ a continuous mapping, and μ a po measure on X. Suppose that for each μ-negligible set N, the inverse image u is μ-negligible (cf. Problem 17). We shall write $u^{-n}(A)$ for $(u^n)^{-1}(A)$, for each in $n \geqq 1$ and each subset A of X, and $u^0 = 1_X$.

If A is any μ-measurable subset of X, let $A_{ent} = \bigcup_{n=0}^{\infty} u^{-n}(A)$ be the set of p which "enter A at least once," and let $A_{ret} = A \cap u^{-1}(A_{ent})$ be the set of points which "return at least once to A". Also put $A_{ret\ inf} = A \cap \bigcap_{n=0}^{\infty} u^{-n}(A_{ent})$, the s points of A which "return to A infinitely often." The set A is said to be *wand* (relative to u) if the sets $u^{-n}(A)$ ($n \geqq 0$) are pairwise disjoint.

The mapping u is said to be *incompressible* if, for each μ-measurable subset A such that $A \subset u^{-1}(A)$, the set $u^{-1}(A) - A$ is negligible; or equivalently if, for μ-measurable subset B of X such that $u^{-1}(B) \subset B$, the set $B - u^{-1}(B)$ is negli If u is not incompressible it is said to be *compressible*.

(a) Show that the following four properties are equivalent:

(α) u is incompressible.

(β) The wandering sets (relative to u) are negligible.

(γ) For each measurable set A, the set $A - A_{ret}$ is negligible.

(δ) For each measurable set A, the set $A - A_{ret\ inf}$ is negligible.

(Observe that $u^{-1}(A_{ent}) \subset A_{ent}$ and that $A - A_{ret}$ is wandering.)

(b) Show that u is compressible if and only if there exists a measurable real-v function f on X such that $f(u(x)) \geqq f(x)$ for all $x \in X$ and such that the set of p $x \in X$ at which $f(u(x)) > f(x)$ has measure > 0. (To show that the condition is suffi argue by contradiction to show that it implies that there exists a rational num for which the set of points $x \in X$ such that $f(x) < r < f(u(x))$ is not negligible.) De that if u^n is incompressible for some integer $n > 1$, then u is incompressible.

(c) Suppose that X is compact and that the measure μ is *invariant* with respec (cf. Section 13.4, Problem 8). Show that u is incompressible ("Poincaré's recur theorem").

(d) Show that if u is incompressible, then there exists a negligible set N in X that, for every $x \notin N$ and every neighborhood V of x in X, there are infinitely n integers n such that $u^n(x) \in V$. (Consider the sets $U - U_{ret\ inf}$, as U runs throu denumerable basis of open sets of X.)

12. (a) Let $(t_i)_{0 \leqslant i \leqslant n}$ be a sequence of real numbers, and let m be an integer $\leqslant n$. Let L_m be the set of indices i with the following property: there exists an integer p such that $0 \leqslant p \leqslant m$ and $t_i + t_{i+1} + \cdots + t_{i+p-1} \geqslant 0$. Show that $\sum_{i \in L_m} t_i \geqslant 0$. (Observe that if $i \in L_m$, and if p is the *smallest* of the integers with the above property, then $i+1, \ldots, i+p-1$ belong to L_m.)

(b) Let X be a locally compact space and μ a positive measure on X. Let $u : X \to X$ be a proper continuous mapping such that $u(\mu) = \mu$. If f is any μ-integrable function on X, put $f_0 = f$ and $f_k = f \circ u^k$ ($k \geqslant 1$). For each integer m, let A_m be the measurable set of points $x \in X$ such that one of the sums $f_0(x) + f_1(x) + \cdots + f_p(x)$, where $p \leqslant m$, is $\geqslant 0$. Show that $\int_{A_m} f(x)\, d\mu(x) \geqslant 0$. (For each integer $n > 0$ and each $x \in X$, consider the sequence $(f_i(x))_{0 \leqslant i \leqslant n+m}$, and apply (a) to this sequence, denoting the corresponding set of indices by $L_m(x)$. For each $k \leqslant n+m$, let B_k be the set of $x \in X$ such that $k \in L_m(x)$, and deduce from (a) that $\sum_{k=0}^{n+m} \int_{B_k} f_k(x)\, d\mu(x) \geqslant 0$. Now use the fact that $B_k = u^{-k}(A_m)$ for $0 \leqslant k \leqslant n$ to deduce that

$$(n+1) \int_{A_m} f(x)\, d\mu(x) + m \int |f(x)|\, d\mu(x) \geqslant 0$$

for all n.) If A is the union of the A_m, conclude that $\int_A f(x)\, d\mu(x) \geqslant 0$ ("maximal ergodic theorem").

(c) Let a be a real number and C an integrable set such that

$$a < \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_k(x)$$

for all $x \in C$. Show that $a\mu(C) \leqslant \int |f(x)|\, d\mu(x)$. (Apply (b) to the function $f - a\varphi_C$.)

(d) Let a, b be two real numbers such that $a < b$. If E is the set of points $x \in X$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_k(x) < a < b < \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_k(x),$$

show that E is μ-negligible. (Deduce first from (c) that E is integrable, and then apply (b) to each of the functions $(f-b)\varphi_E$ and $(a-f)\varphi_E$, using the fact that $u(E) \subset E$.) Hence show that, *for almost all* $x \in X$, the sequence

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} f(u^k(x)) \right)$$

converges to a limit $f^*(x)$, that f^* is integrable and that $f^*(u(x)) = f^*(x)$ almost everywhere (G. D. Birkhoff's ergodic theorem). (Take a, b to be all pairs of rational numbers such that $a < b$.)

(e) If X is compact, show that $\int f^*\, d\mu = \int f\, d\mu$. (Reduce to the case $f \geqslant 0$, and consider first the case where f is bounded; then pass to the general case by observing that $\int |f^*|\, d\mu \leqslant \int |f|\, d\mu$.)

(f) Suppose that X is compact and $\mu(X) = 1$. Consider a sequence (g_n) of μ-integrable functions such that $\lim_{n\to\infty} g_n(x) = f(x)$ almost everywhere. Suppose moreover that there exists a μ-integrable function $G \geqq 0$ such that $|g_n| \leqq G$ almost everywhere, for each integer n. Show that under these conditions the sequence

$$\left(\frac{1}{n}\sum_{k=0}^{n-1} g_k(u^k(x))\right)$$

converges almost everywhere to $f^*(x)$. (Reduce to the case $f = 0$. For each $\varepsilon \in]0, 1[$, there exists $\delta > 0$ such that, for each μ-integrable subset B of X satisfying $\mu(B) \leqq \delta$, we have $\int_B G\, d\mu \leqq \varepsilon^2$ (13.15.5). Next, there exists an integer m such that the set A_ε of points $x \in X$ satisfying

$$\sup_{n \geqq m} |g_n(x)| \leqq \varepsilon^2$$

has measure $\mu(A_\varepsilon) \geqq 1 - \delta$. Let G_ε be the function which is equal to ε^2 on A_ε and to G on $X - A_\varepsilon$; then for $n \geqq m$ and all $x \in X$, we may write

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} g_k(u^k(x))\right| \leqq \left|\frac{1}{n}\sum_{k=0}^{m-1} g_k(u^k(x))\right| + \frac{1}{n}\sum_{k=0}^{n-1} G_\varepsilon(u^k(x)).$$

Deduce from (c) that the set of points $x \in X$ such that

$$\limsup_{n\to\infty} \left|\frac{1}{n}\sum g_k(u^k(x))\right| \geqq \varepsilon$$

has measure $\leqq 2\varepsilon$.)

13. Let X be a locally compact space, μ a positive measure on X, and $u : X \to X$ a continuous mapping such that, for each μ-negligible set N, the set $u^{-1}(N)$ is μ-negligible. A measurable real-valued function f on X is said to be *μ-invariant with respect to u* if $f \circ u$ and f are equal almost everywhere (relative to μ). A μ-measurable subset A of X is said to be *μ-invariant with respect to u* if its characteristic function is μ-invariant with respect to u.

(a) For a measurable function f to be μ-invariant with respect to u, it is necessary and sufficient that, for each rational number r, the set of points $x \in X$ such that $f(x) \geqq r$ should be μ-invariant with respect to u. (To show that the condition is sufficient, consider the set of points $x \in X$ such that $f(u(x)) \neq f(x)$.)

(b) From now on, suppose that u is *proper* and the measure μ-invariant with respect to u (i.e., $u(\mu) = \mu$). The mapping u is said to be *ergodic* with respect to μ (or μ is ergodic with respect to u) if the only μ-measurable functions which are μ-invariant with respect to u are constant almost everywhere. Then, for every μ-integrable function f on X, the function

$$f^*(x) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} f(u^k(x))\right)$$

is *constant almost everywhere*. If X is compact, the value of the constant is $\left(\int f(x)\, d\mu(x)\right)/\mu(X)$. Conversely, if X is compact and if f^* is constant almost everywhere, for every μ-integrable function f, then u is ergodic.

(c) Take X to be the circle $\mathbf{U}: |z| = 1$ in $\mathbf{C}$, and μ to be the image of Lebesgue measure under the mapping $t \mapsto e^{2\pi i t}$ of [0, 1] onto U. If θ is irrational, show that the mapping $u: z \mapsto e^{2\pi i\theta} z$ leaves μ invariant and is *ergodic* with respect to μ. (Use Bohl's theorem, Section 13.4, Problem 7.)

(d) Suppose that X is compact and u ergodic with respect to μ. If A and B are μ-measurable subsets of X, show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(u^{-k}(\mathrm{A}) \cap \mathrm{B}) = \mu(\mathrm{A})\mu(\mathrm{B})/\mu(\mathrm{X}).$$

Conversely, if this formula is satisfied for all pairs of measurable subsets A, B in X, then u is ergodic with respect to μ.

(e) Suppose that X is compact and u ergodic with respect to μ. If a measurable function $f \geqq 0$ is such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(u^k(x))$$

exists almost everywhere, then f is integrable. (Consider f as the limit of an increasing sequence of bounded functions.)

(f) Suppose that X is compact, that u is ergodic with respect to μ, and that $\mathrm{Supp}(\mu) = \mathrm{X}$. Show that, for almost all $x \in \mathrm{X}$, the set of points $u^n(x)$ $(n \geqq 0)$ is dense in X.

14. Let X be a compact space, $u: \mathrm{X} \to \mathrm{X}$ a continuous mapping, and μ a positive measure on X, invariant with respect to u and of total mass 1. Let A be a μ-measurable subset of X. For each $x \in \mathrm{A}$, let $n(x)$ denote the smallest integer $n \geqq 1$ such that $u^n(x) \in \mathrm{A}$. If we put $\mathrm{A}_n = u^{-n}(\mathrm{A})$ and $\mathrm{B}_n = u^{-n}(\mathrm{X} - \mathrm{A})$, then the set of points $x \in \mathrm{A}$ for which $n(x) = m$ is $\mathrm{A}_0 \cap \mathrm{B}_1 \cap \mathrm{B}_2 \cap \cdots \cap \mathrm{B}_{m-1} \cap \mathrm{A}_m$, and hence the function $x \mapsto n(x)$ is measurable.

(a) Put $\alpha_0 = 1$ and $\alpha_m = \mu(\mathrm{B}_0 \cap \mathrm{B}_1 \cap \cdots \cap \mathrm{B}_{m-1})$ $(m \geqq 1)$. Show that

$$\mu(\mathrm{A}_0 \cap \mathrm{B}_1 \cap \mathrm{B}_2 \cap \cdots \cap \mathrm{B}_{m-1} \cap \mathrm{B}_m \cap \mathrm{A}_{m+1}) = \alpha_m - 2\alpha_{m+1} + \alpha_{m+2}$$

for all $m \geqq 0$.

(b) Show that the series whose general term is $\alpha_m - \alpha_{m+1}$ is convergent and that its terms form a decreasing sequence (interpret $\alpha_m - \alpha_{m+1}$ as the measure of a set). Deduce that $\lim_{m\to\infty} m(\alpha_m - \alpha_{m+1}) = 0$.

(c) Let $\mathrm{B}_\infty = \bigcap_n \mathrm{B}_n$. Show that

$$\int_{\mathrm{A}} n(x)\, d\mu(x) = 1 - \mu(\mathrm{B}_\infty)$$

("Kac's theorem": use (a) and (b)). Consider the case where u is ergodic with respect to μ, and $\mu(\mathrm{A}) > 0$.

(d) Suppose that u is *bijective* and *ergodic* with respect to μ, and that $\mu(\mathrm{A}) > 0$. Let E_m be the set of points $x \in \mathrm{A}$ such that $n(x) = m$. Show that the complement in A of $\bigcup_{m=1}^{\infty} \mathrm{E}_m$ is negligible (cf. Problem 11(a)). Show that the sets $u^p(\mathrm{E}_m)$ for $m \geqq 1$ and $0 \leqq p < m$ are mutually disjoint, and that the complement of their union in X is negligible ("Kakutani's skyscraper").

15. Let f be a lower (or upper) semicontinuous real-valued function on $\mathbf{R}^n$. Let X be a locally compact space and $u_1, \ldots, u_n$ universally measurable finite real-valued functions. Show that $f(u_1, \ldots, u_n)$ is universally measurable (cf. Problem 17).

16. With the notation of Section 12.7, Problem 3, show that there exist n universally measurable mappings $t \mapsto z_k(t)$ of $\mathbf{C}^n$ into $\mathbf{C}$ $(1 \leqq k \leqq n)$ such that

$$P_t(X) = X^n + t_1 X^{n-1} + \cdots + t_n = \prod_{k=0}^{n} (X - z_k(t)).$$

(By induction on the degree n, using Problem 15.)

17. Let f be the continuous function on $I = [0, 1]$ which was defined in Section 4.2, Problem 2(d), such that the restriction of f to the Cantor set K is a bijection of K onto I. Let g be the function defined by $g(x) = x + f(x)$; then g is a *homeomorphism* of I onto $2I = [0, 2]$, such that $g(K)$ is a compact set of measure 1 (with respect to Lebesgue measure), although K is negligible. If $h(x) = g^{-1}(2x)$, then h is a homeomorphism of I onto I such that $h^{-1}(K)$ is not negligible with respect to Lebesgue measure. Moreover, if A is a nonmeasurable set (with respect to Lebesgue measure) contained in $g(K)$, then the set $B = h(\frac{1}{2}A)$ is negligible and therefore measurable. The function $\varphi_B \circ h$ is not measurable, although h is continuous and φ_B is measurable (cf. (16.23)).

18. (a) Let f be a finite continuous real-valued function on $\mathbf{R}$. Show that there exists a real-valued function g on $\mathbf{R}$ which is universally measurable and such that $f(g(x)) = x$ for all $x \in f(\mathbf{R})$ (cf. Section 12.7, Problem 1).

(b) Assume that there exists a partition of $I = [0, 1]$ into a family $(H_n)_{n \in \mathbf{Z}}$ of nonmeasurable sets (with respect to Lebesgue measure) such that $\mathrm{Card}(H_n) = \mathrm{Card}(\mathbf{R})$ for all $n \in \mathbf{Z}$. Let K be the Cantor set (Section 4.2, Problem 2) and $F = \bigcup_{n \in \mathbf{Z}} (K + n)$, so that F is a negligible closed set. Show that there exists a bijection $f: \mathbf{R} \to \mathbf{R}$ such that $f(F \cap]n, n+1[) = H_n$ for all n, and such that f is linear in each of the component intervals of $\complement F$. Then f is measurable with respect to Lebesgue measure, but f^{-1} is not.

19. Let X be a locally compact space and μ a positive measure on X. Let D be a denumerable dense subset of $\mathbf{R}$. Show that a mapping $f: X \to \bar{\mathbf{R}}$ is μ-measurable if and only if, for each $r \in D$, the set of points $x \in X$ such that $f(x) \geqq r$ is μ-measurable.

20. Suppose μ bounded and $\mu(X) = 1$. If $f \geqq 0$ is an integrable function, show that $(1 + f^2)^{1/2}$ is integrable and that, if $A = \int f \, d\mu$, then

$$(1 + A^2)^{1/2} \leqq \int (1 + f^2)^{1/2} \, d\mu \leqq 1 + A.$$

(Use Problem 14(a) of Section 13.8.)

21. Let X, Y, Z be three locally compact spaces. if $f: X \to Y$ and $g: Y \to Z$ are universally measurable, show that $g \circ f$ is universally measurable. (To show that $g \circ f$ is μ-measurable where μ is any positive measure on X, reduce to the case where X is compact and consider the measure $f(\mu)$ on Y (Section 13.4, Problem 8).) In particular, if $B \subset Y$ is universally measurable, then $f^{-1}(B)$ is universally measurable.

22. Let X be a locally compact space, Y a compact space, μ a positive measure on X, and $(f_n)_{n\geq 0}$ a sequence of μ-measurable mappings from X to Y. Show that there exists a mapping $(p, x)\mapsto \sigma_p(x)$ of $\mathbf{N}\times \mathrm{X}$ into N with the following properties: (1) for each integer $p\geq 0$ and each integer $n\geq 0$, the set $\sigma_p^{-1}(n)$ is μ-measurable; (2) for each $x\in \mathrm{X}$, the sequence $(f_{\sigma_p(x)}(x))_{p\geq 0}$ converges in Y. (One method is as follows. Taking a distance function on Y, define universally measurable subsets A_s of Y satisfying conditions (i) and (ii) of Section 4.2, Problem 3(a) and such that (in the same notation) $\mathrm{A}_{s'}\cap \mathrm{A}_{s''}=\varnothing$. Then define subsets B_s of X inductively as follows: if B_s has been defined in such a way that, for each $x\in \mathrm{B}_s$, infinitely many terms of the sequence $(f_n(x))$ belong to A_s, we define $\mathrm{B}_{s'}$ as the set of all points $x\in \mathrm{B}_s$ such that infinitely many terms of the sequence $(f_n(x))$ belong to $\mathrm{A}_{s'}$, and we define $\mathrm{B}_{s''}$ to be the complement of $\mathrm{B}_{s'}$ in B_s. Now let $x\in \mathrm{X}$ and let $(\varepsilon_p)_{p\geq 0}$ be the sequence of 0's and 1's such that, if $s_p=(\varepsilon_i)_{0\leq i\leq p}$, then $x\in \mathrm{B}_{s_p}$ for all p. Define $\sigma_p(x)$ by induction as follows: $\sigma_0(x)$ is the smallest integer n such that $f_n(x)\in \mathrm{A}_{s_0}$, and $\sigma_p(x)$ is the smallest integer $n>\sigma_{p-1}(x)$ such that $f_n(x)\in \mathrm{A}_{s_p}$. Use Problem 21 to verify that this definition of $\sigma_p(x)$ satisfies the conditions of the problem.)

23. Let μ be a bounded positive measure on X and let $f\geq 0$ be a μ-integrable function. Show that there exists a lower semicontinuous function g such that $g\geq 1/f$ (with the convention that $1/0=+\infty$) and such that gf is integrable (with the convention that the product of 0 and $+\infty$ is 0). (Reduce to the case where f is bounded; consider the sets A_n of points $x\in \mathrm{X}$ such that $f(x)\leq 1/n$, and apply (13.7.9) to these sets to construct g.)

24. Let X, Y be locally compact spaces, μ a positive bounded measure on X, and π a μ-measurable mapping from X to Y. For each $f\in \mathscr{K}_{\mathbf{R}}(\mathrm{Y})$, show that $f\circ\pi$ is μ-integrable and that the mapping $f\mapsto \int (f\circ\pi)\,d\mu$ is a *positive* measure ν on Y (called the *image* of μ under π, and denoted by $\pi(\mu)$). Generalize the results of Section 13.8, Problem 12, and Section 13.9, Problems 10 to 14.

25. Let X be a compact space and let (ϖ_n) be a sequence of finite partitions of X consisting of integral sets, such that ϖ_m is finer then ϖ_n whenever $m>n$ (Problem 7). An *elementary martingale* relative to the sequence (ϖ_n) is by definition a sequence (f_n) of integrable functions ≥ 0 such that:

(i) f_n is constant on each set of the partition ϖ_n;

(ii) If $m>n$, for each set $\mathrm{A}\in\varpi_n$ we have

$$\int_{\mathrm{A}} f_n\,d\mu=\int_{\mathrm{A}} f_m\,d\mu$$

(in other words, on each set $\mathrm{A}\in\varpi_n$ the value of f_n is equal to the average of f_m over A).

(a) Let a, b be real numbers such that $0\leq a<b$, and let E_{ab} be the integrable set of points $x\in \mathrm{X}$ such that

$$\liminf_{n\to\infty} f_n(x)<a<b<\limsup_{n\to\infty} f_n(x).$$

Show that E_{ab} is negligible. (Suppose if possible that $\mu(\mathrm{E}_{ab})>0$. For each integer $p>0$, let F_p be the union of the sets $\mathrm{A}\in\varpi_n$ $(n\geq p)$ such that $\mathrm{A}\cap \mathrm{E}_{ab}\neq\varnothing$; if $\mathrm{F}=\bigcap_p \mathrm{F}_p$, then $\mu(\mathrm{F})\geq \mu(\mathrm{E}_{ab})>0$. Show that for each integer p and each $\mathrm{A}\in\varpi_n$ $(n\geq p)$ such that $\mathrm{A}\cap \mathrm{E}_{ab}\neq\varnothing$, we have $b\cdot\mu(\mathrm{A}\cap \mathrm{F}_q)\leq a\cdot\mu(\mathrm{A})$ for all $q\geq n$, by using the

definition of a martingale; now let $q \to +\infty$, then $p \to +\infty$, and so obtain a contradiction.)

(b) Deduce from (a) that the sequence (f_n) converges almost everywhere to an integrable function f. (Give a, b rational values, and use Fatou's lemma.)

(c) Give an example where $\int f\,d\mu$ is not equal to the (constant) value of the integrals $\int f_n\,d\mu$. (Take $X = [0, 1]$ and take μ to be Lebesgue measure; take ϖ_n to consist of the intervals $[0, 2^{-n}[$, $[2^{-n}, 2^{-n+1}[$, $\ldots$, $[\frac{1}{2}, 1]$.)

(d) Let $F(x) = \sup_n f_n(x)$. Show that there exists a constant $C > 0$ such that, for all $\alpha > 0$, if B^α is the set of points x such that $F(x) > \alpha$, we have $\mu(B_\alpha) \leqq C/\alpha$. (Remark that B_α is the union of the sets $B_{\alpha,n} = \{x \in X : f_n(x) > \alpha\}$.)

26. With the hypotheses of Problem 25, let σ be another finite partition of X into μ-integrable sets. For each $x \in X$ and each integer n, let $B \in \sigma$ and $A \in \varpi_n$ be the sets of the partitions σ and ϖ_n which contain x; put $f_n(x) = 0$ if $\mu(A) = 0$, and $f_n(x) = \mu(B \cap A)/\mu(A)$ if $\mu(A) > 0$. Show that the sequence $(f_n(x))$ converges almost everywhere to a limit $f(x) \leqq 1$, and that the sequence (f_n) converges in mean to f. (For each set $B \in \sigma$, observe that the functions $\varphi_B f_n$ form an elementary martingale for the measure $\varphi_B \cdot \mu$, and use Problem 25.)

27. Let X be a compact space with measure $\mu(X) = 1$, and let $\alpha = (A_i)_{1 \leqslant i \leqslant n}$ be a finite partition of X into integrable subsets. For each $x \in X$, if $A_i \in \alpha$ is the set containing x, the number

$$j(\alpha; x) = -\log \mu(A_i)$$

is called the *information* at the point x corresponding to the partition α. The number

$$H(\alpha) = \int j(\alpha; x)\,d\mu(x) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i) \geqq 0$$

(with the convention that $t \log t = 0$ when $t = 0$) is called the *entropy* of the partition α (relative to the measure μ). It is the "average" of the information corresponding to α.

If $\beta = (B_k)_{1 \leqslant k \leqslant m}$ is another finite partition of X into integrable subsets, the number

$$H(\alpha/\beta) = -\sum_{i,k} \mu(A_i \cap B_k) \log(\mu(A_i \cap B_k)/\mu(B_k)) \geqq 0$$

(in which the terms for which $\mu(B_k) = 0$ are replaced by 0) is called the *entropy of* α *relative to* β. If ω is the partition consisting of the single set X, then $H(\alpha/\omega) = H(\alpha)$.

(a) If $(\alpha_j)_{1 \leqslant j \leqslant n}$ is a finite sequence of finite partitions of X into integrable sets, we denote by $\bigvee_{j=1}^{n} \alpha_j$ or $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$ the coarsest of the partitions which are finer than all the α_j, that is to say the partition consisting of all nonempty intersections

$$A_1 \cap A_2 \cap \cdots \cap A_n,$$

where $A_j \in \alpha_j$ for each j.

If α, β, γ are three finite partitions of X into integrable sets, show that

$$H((\alpha \vee \beta)/\gamma) = H(\alpha/\gamma) + H(\beta/(\alpha \vee \gamma))$$

and in particular

$$H(\alpha \vee \beta) = H(\alpha) + H(\beta/\alpha).$$

Deduce that if α is coarser than β, then $H(\alpha/\gamma) \leqq H(\beta/\gamma)$, and in particular $H(\alpha) \leqq H(\beta)$.

(b) With the same notation, show that if β is coarser than γ then $H(\alpha/\beta) \geqq H(\alpha/\gamma)$ (use the fact that the function $t \mapsto t \log t$ is convex in [0, 1]). In particular, we have $H(\alpha) \geqq H(\alpha/\gamma)$. For arbitrary α, β, γ, we have

$$H((\alpha \vee \beta)/\gamma) \leqq H(\alpha/\gamma) + H(\beta/\gamma)$$

and in particular

$$H(\alpha \vee \beta) \leqq H(\alpha) + H(\beta).$$

28. Let X be a compact space with measure $\mu(X) = 1$, and let $u : X \to X$ be a μ-measurable mapping such that $u(\mu) = \mu$ (Problem 24). Let α be a finite partition of X into integrable subsets; if $\alpha = (A_k)_{1 \leqslant k \leqslant m}$, we denote by $u^{-n}(\alpha)$ the partition formed by the sets $u^{-n}(A_k), 1 \leqq k \leqq m$.

(a) With the notation of Problem 27, show that the limit

$$h(u, \alpha) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} u^{-j}(\alpha)\right)$$

exists and is finite. (If $a_n = H\left(\bigvee_{j=0}^{n-1} u^{-j}(\alpha)\right)$, show by the use of Problem 27(b) that $a_{m+n} \leqq a_m + a_n$, and then copy the proof of (15.2.7(i)). The number $h(u, \alpha)$ is called the *entropy of u relative to* α. Show that $h(u, \alpha) \leqq H(\alpha)$ (use Problem 27).

(b) If α, β are two finite partitions of X into integrable subsets, show that

$$h(u, \alpha \vee \beta) \leqq h(u, \alpha) + h(u, \beta)$$

$$h(u, \alpha) \leqq h(u, \beta) + H(\alpha/\beta).$$

If also α is coarser than β, then $h(u, \alpha) \leqq h(u, \beta)$. (Use Problem 27.)

(c) Show that

$$h(u, \alpha) = \lim_{n \to \infty} H\left(\alpha \Big/ \left(\bigvee_{j=1}^{n} u^{-j}(\alpha)\right)\right).$$

(Prove by induction on n that

$$H\left(\bigvee_{j=0}^{n-1} u^{-j}(\alpha)\right) = H(\alpha) + \sum_{j=0}^{n-1} H\left(\alpha \Big/ \left(\bigvee_{k=1}^{j} u^{-k}(\alpha)\right)\right),$$

by using Problem 27.)

(d) Show that, for each integer $m \geqq 1$,

$$h(u, \alpha) = h\left(u, \bigvee_{j=0}^{m} u^{-j}(\alpha)\right).$$

(Observe that the right-hand side is equal to

$$\lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{j=0}^{m+n-1} u^{-j}(\alpha)\right)$$

and use Problem 27.) If in addition u is bijective and u^{-1} is μ-measurable, then for

each integer $m \geqq 1$ we have

$$h(u, \alpha) = h\left(u, \bigvee_{j=-m}^{m} u^j(\alpha)\right)$$

(same method).

(e) Put $h(u) = \sup_{\alpha} h(u, \alpha)$, where the supremum is taken over all finite partitions of X into μ-integrable subsets. The number $h(u)$ is called the *entropy* of u (relative to the measure μ). Show that, for each integer $m \geqq 1$, we have $h(u^m) = mh(u)$. (Remark that

$$h(u^m, \alpha) \leqq h\left(u^m, \bigvee_{j=0}^{m-1} u^{-j}(\alpha)\right) = mh(u, \alpha).)$$

If moreover u is bijective and u^{-1} is μ-measurable, then $h(u^{-1}) = h(u)$.

(f) Show that $h(1_X) = 0$, where 1_X is the identity mapping of X. Deduce that if $u^p = 1_X$ for some integer $p > 0$, then $h(u) = 0$.

10. INTEGRALS OF VECTOR-VALUED FUNCTIONS

Consider first a mapping $\mathbf{f}$ of X into a *finite*-dimensional real vector space E. Let $(\mathbf{e}_i)_{1 \leqq i \leqq m}$ be a basis of E, and put $\mathbf{f}(x) = \sum_{i=1}^{m} f_i(x)\mathbf{e}_i$ for each $x \in X$. Then $\mathbf{f}$ is said to be *integrable* (with respect to μ) or *μ-integrable* if each of the real-valued functions f_i is integrable, and we define

$$\int \mathbf{f}(x)\, d\mu(x) = \sum_{i=1}^{m} \left(\int f_i(x)\, d\mu(x)\right)\mathbf{e}_i . \tag{13.10.1}$$

It is immediately checked that this definition is independent of the basis chosen. If $\|\mathbf{z}\|$ is a norm defining the topology of E, then we have the following criterion:

(13.10.2) *A mapping $\mathbf{f} : X \to E$ is integrable if and only if $\mathbf{f}$ is measurable and $\int^* \|\mathbf{f}(x)\|\, d\mu(x) < +\infty$. The function $x \mapsto \|\mathbf{f}(x)\|$ is then integrable.*

For $\mathbf{f}$ is measurable if and only if the $f_i (1 \leqq i \leqq n)$ are measurable, by (13.9.6); also there exist two constants a, b such that

$$a \cdot \sum_i |f_i(x)| \leqq \|\mathbf{f}(x)\| \leqq b \cdot \sum_i |f_i(x)|$$

by (5.9.1). The result now follows from (13.9.13) and (13.9.6).

In particular, a *complex* function f on X is integrable if and only if $\mathscr{R}f$

and $\mathscr{I}f$ are integrable, and we have

$$\int f\,d\mu = \int (\mathscr{R}f)\,d\mu + i\int (\mathscr{I}f)\,d\mu,$$

from which it follows immediately that

$$\int \bar{f}\,d\mu = \overline{\int f\,d\mu}.$$

The set $\mathscr{L}^1_{\mathbf{C}}(X, \mu)$ of μ-integrable functions on X with values in $\mathbf{C}$ is a vector space over $\mathbf{C}$, and $f \mapsto \int f\,d\mu$ is a linear form on this vector space. Furthermore, if f is any complex-valued integrable function, then $|f|$ is integrable and

$$\left|\int f\,d\mu\right| \leqq \int |f|\,d\mu. \tag{13.10.3}$$

For since $|f| = ((\mathscr{R}f)^2 + (\mathscr{I}f)^2)^{1/2}$, it follows that $|f|$ is measurable (13.9.6), and we have $\mu^*(|f|) \leqq \mu^*(|\mathscr{R}f|) + \mu^*(|\mathscr{I}f|)$. Hence the first assertion, by (13.9.13). Also there exists a complex number ζ of absolute value 1 such that $\zeta\mu(f) = |\mu(f)|$, and hence $|\mu(f)| = \mathscr{R}(\mu(\zeta f)) = \mu(\mathscr{R}(\zeta f)) \leqq \mu(|f|)$, which establishes (13.10.3).

If E′ is the dual of the finite-dimensional vector space E, then to say that the f_i are integrable is equivalent to saying that, for *each* $\mathbf{z}' \in \mathrm{E}'$, the mapping $x \mapsto \langle \mathbf{f}(x), \mathbf{z}' \rangle$ is integrable (because it is a linear combination of the f_i). We generalize this as follows: if I is any set, then a mapping $x \mapsto f_x$ of X into the vector space K^{I} (where $\mathrm{K} = \mathbf{R}$ or $\mathbf{C}$) is *scalarly μ-integrable* if for *each* $\alpha \in \mathrm{I}$ the function $x \mapsto f_x(\alpha)$ is *integrable*.

(13.10.4) *Let* E *be a Fréchet space,* E′ *its dual* (12.15). *Let* $x \mapsto \mathbf{f}_x$ *be a scalarly integrable mapping of* X *into* E′ *such that, for each convergent sequence* $(\mathbf{a}_n)$ *in* E, *there exists a function* $g \geqq 0$ *defined on* X *such that* $\mu^*(g) < +\infty$ *and* $|\mathbf{f}_x(\mathbf{a}_n)| \leqq g(x)$ *for all* n, *almost everywhere in* X. *Then there exists a continuous linear form* $\mathbf{z}' \in \mathrm{E}'$ *on* E *such that* $\langle \mathbf{z}, \mathbf{z}' \rangle = \int \mathbf{f}_x(\mathbf{z})\,d\mu(x)$ *for all* $\mathbf{z} \in \mathrm{E}$.

By (3.13.4) it is enough to show that if $(\mathbf{a}_n)$ is a sequence in E with limit $\mathbf{z}$ then $\int \mathbf{f}_x(\mathbf{z})\,d\mu(x) = \lim_{n\to\infty} \int \mathbf{f}_x(\mathbf{a}_n)\,d\mu(x)$; and this follows from (13.8.4).

The linear form $\mathbf{z}'$ so defined is called the *integral* (or *weak integral*) of the function $x \mapsto \mathbf{f}_x$ with respect to μ, and is denoted by $\int \mathbf{f}_x\,d\mu(x)$. Hence, for

all $\mathbf{x} \in E$, we have

$$\left\langle \mathbf{z}, \int \mathbf{f}_x \, d\mu(x) \right\rangle = \int \mathbf{f}_x(\mathbf{z}) \, d\mu(x). \tag{13.10.5}$$

If in particular E is a *Hilbert space*, then since there is a semilinear isometry $\mathbf{z} \mapsto j(\mathbf{z})$ of E onto its dual E′ (12.15), we can define the notion of a *scalarly integrable* mapping $x \mapsto \mathbf{f}(x)$ of X into E: this means that, for each $\mathbf{z} \in E$, the complex-valued function $x \mapsto (\mathbf{f}(x) \,|\, \mathbf{z})$ is integrable. If the function $x \mapsto \| \mathbf{f}(x) \|$ is integrable, then by (13.10.4) there exists a unique element of E, denoted by $\int \mathbf{f}(x) \, d\mu(x)$ and called the *integral* (or *weak integral*) of $\mathbf{f}$, with the property that

$$\left(\int \mathbf{f}(x) \, d\mu(x) \,|\, \mathbf{z} \right) = \int (\mathbf{f}(x) \,|\, \mathbf{z}) \, d\mu(x) \tag{13.10.6}$$

for all $\mathbf{z} \in E$.

PROBLEMS

1. If a mapping $x \mapsto \mathbf{f}_x$ of X into $\mathbf{R}^I$ is scalarly μ-integrable, then the element

$$\alpha \mapsto \int \mathbf{f}_x(\alpha) \, d\mu(x)$$

of $\mathbf{R}^I$ is called the *integral* of the mapping and is denoted by $\int \mathbf{f}_x \, d\mu(x)$. If, for all $x \in X$, $\mathbf{f}_x$ belongs to a weakly closed convex set A in $\mathbf{R}^I$ and if X is integrable and $\mu(X) = 1$, then the integral $\int \mathbf{f}_x \, d\mu(x)$ also belongs to A. (Use Problem 13 of Section 12.15.)

2. Let E be a separable real Fréchet space and E′ its dual (12.15). It follows from the Hahn–Banach theorem (Section 12.15, Problem 4) that the linear mapping c_E which maps each $\mathbf{z} \in E$ to the linear form $\mathbf{z}' \mapsto \langle \mathbf{z}, \mathbf{z}' \rangle$ on E′, is an *injective* mapping of E into $\mathbf{R}^{E'}$ and is continuous with respect to the product topology on $\mathbf{R}^{E'}$. If K is any compact subset of E, the restriction $c_E \,|\, K$ is a homeomorphism of K onto the subspace $c_E(K)$ of $\mathbf{R}^{E'}$ (12.3.6).

(a) Suppose that K is a *compact convex* subset of E. A mapping $\mathbf{f}$ of X into K is said to be *scalarly μ-integrable* if, for each $\mathbf{z}' \in E'$, the real-valued function $x \mapsto \langle \mathbf{f}(x), \mathbf{z}' \rangle$ is μ-integrable. Using the remarks above and Problem 1, show that there then exists a unique vector $\mathbf{z} \in K$ such that $\langle \mathbf{z}, \mathbf{z}' \rangle = \int \langle \mathbf{f}(x), \mathbf{z}' \rangle \, d\mu(x)$ for all $\mathbf{z}' \in E'$. This vector $\mathbf{z}$ is denoted by $\int \mathbf{f} \, d\mu$ or $\int \mathbf{f}(x) \, d\mu(x)$, and is called the *integral* of $\mathbf{f}$.

In particular, if $X = K$ and $\mathbf{f}$ is the identity mapping 1_K, for any positive measure μ on K with *total mass equal to* 1, the element $\mathbf{b}_\mu = \int \mathbf{z} \, d\mu$ of K is called the *barycenter* of μ.

(b) Let H be a compact subset of E. Show that the closed convex hull K of H is

compact (by proving that it is precompact) and coincides with the set of barycenters $\mathbf{b}_\mu$ of positive measures with total mass 1 and *support contained* in H. (Remark that as μ runs through the set $P \subset M_+(H)$ of positive measures with total mass 1, the mapping $\mu \mapsto \mathbf{b}_\mu$ is continuous with respect to the vague topology on P and the topology induced on K by the product topology on $\mathbf{R}^{E'}$, and use Section 13.4, Problem 12.)

3. Let E be a separable real Fréchet space and A a closed convex subset of E. Show that every convex function f (Section 13.9, Problem 14) which is lower semicontinuous on A is the upper envelope (12.7) of a family of functions which are the restrictions to A of continuous *affine* linear functions on E (i.e., functions of the form $\mathbf{z} \mapsto a + u(\mathbf{z})$, where u is a continuous linear form on E, and $a \in \mathbf{R}$). (Apply the Hahn–Banach theorem to the closed convex set of points $(\mathbf{z}, \zeta) \in E \times \mathbf{R}$ such that $\mathbf{z} \in A$ and $\zeta \geqq f(\mathbf{z})$.)

4. Let E be a separable real Fréchet space, K a compact convex subset of E, μ a positive measure of total mass 1 on K, and $\mathbf{b}_\mu$ the barycenter of μ (Problem 2). Show that if f is a convex real-valued function $\geqq 0$ which is lower semicontinuous on K, then $f(\mathbf{b}_\mu) \leqq \int^* f\, d\mu$ (use Problem 3).

5. (a) Let E be a separable Fréchet space, K a compact convex subset of E, and $\mathbf{z}$ a point of K. Show that every positive measure μ on K, with total mass 1, which has $\mathbf{z}$ as barycenter, lies in the vague closure of the set of positive measures of total mass 1 and *finite* support which have $\mathbf{z}$ as barycenter. (Let U be a neighborhood of μ with respect to the vague topology, consisting of measures ν on K such that $|\mu(f_i) - \nu(f_i)| \leqq \delta$ for some functions $f_i \in \mathscr{C}_{\mathbf{C}}(K)$. Show that there exists a finite number of points $\mathbf{a}_j \in K$, and for each j a compact convex neighborhood W_j of $\mathbf{a}_j$ in K, such that the W_j cover K and $|f_i(\mathbf{y}) - f_i(\mathbf{a}_j)| \leqq \frac{1}{2}\delta$ for each i and each $\mathbf{y} \in W_j$. Show (by using a continuous partition of unity) that we can write μ in the form $\sum_j \alpha_j \mu_j$, where each μ_j is a positive measure of total mass 1 and support contained in W_j, and each $\alpha_j \geqq 0$ and $\sum_j \alpha_j = 1$. If $\mathbf{z}_j$ is the barycenter of μ_j, show that the barycenter of $\nu = \sum_j \alpha_j \varepsilon_{\mathbf{z}_j}$ is $\mathbf{z}$, and that $\nu \in U$.)

(b) Let K′ be a compact subset of K, such that K is the closed convex hull of K′. For a point $\mathbf{z} \in K$ to be an extremal point of K (Section 12.15, Problem 5) it is necessary and sufficient that $\varepsilon_{\mathbf{z}}$ should be the only positive measure on K′ with total mass 1 and $\mathbf{z}$ as barycenter. (Use (a) to prove that the condition is necessary.)

6. Let E be a separable real Fréchet space and K a compact convex subset of E. Show that there exists a *strictly convex* continuous real-valued function on K. (The set $\mathscr{A}$ of restrictions to K of continuous affine linear functions on E is a subspace of the separable metrizable space $\mathscr{C}_{\mathbf{R}}(K)$, hence contains a dense sequence (h_n). Show that, for suitable scalars $\alpha_n > 0$, the function $\sum_n \alpha_n h_n^2$ satisfies the required conditions, by using the Hahn–Banach theorem.)

7. Let E be a separable real Fréchet space, K a compact convex subset of E; let u be a convex continuous real-valued function on K, let $G \subset E \times \mathbf{R}$ be the graph of u, and let S be the closed convex hull of G in $E \times \mathbf{R}$.

(a) Let v be the lower envelope of the continuous affine linear functions w on E such that $w(\mathbf{z}) \geqq u(\mathbf{z})$ for all $\mathbf{z} \in K$. Show that S is the set of points $(\mathbf{z}, \zeta) \in E \times \mathbf{R}$ such that

$\mathbf{z} \in K$ and $u(\mathbf{z}) \leqq \zeta \leqq v(\mathbf{z})$. (By the Hahn–Banach theorem, the points $(\mathbf{z}, \zeta)$ belonging to S are those with the following property: if h is any continuous affine linear function on $E \times \mathbf{R}$ such that $h(\mathbf{y}, u(\mathbf{y})) \geqq 0$ for all $\mathbf{y} \in K$, then $h(\mathbf{z}, \zeta) \geqq 0$. (Use Problem 5.)

(b) Show that if u is strictly convex in K, then $u(\mathbf{z}) < v(\mathbf{z})$ for all nonextremal points $\mathbf{z}$ in K.

(c) Show that, for each point $\mathbf{z} \in K$, there exists a positive measure μ on K having $\mathbf{z}$ as barycenter and such that $v(\mathbf{z}) = \int u(\mathbf{y})\, d\mu(\mathbf{y})$. (Apply Problem 2(b) to the closed convex hull S of the compact set G in $E \times \mathbf{R}$.) Deduce from Problem 4 that $u(\mathbf{z}) = v(\mathbf{z})$ almost everywhere in K, with respect to μ.

8. Let E be a separable real Fréchet space and K a compact convex subset of E.

(a) Show that the set M of extremal points of K is the intersection of a sequence of open sets of K, and therefore universally measurable. (In the space $K \times K \times [0, 1]$, consider the set of triples $(\mathbf{z}', \mathbf{z}'', \lambda)$ such that $\mathbf{z}' \neq \mathbf{z}''$ and $0 < \lambda < 1$, and its image $K - M$ under the mapping $(\mathbf{z}', \mathbf{z}'', \lambda) \mapsto \lambda\mathbf{z}' + (1 - \lambda)\mathbf{z}''$.)

(b) Prove that every point of K is the barycenter of a positive measure μ on K, with total mass 1, such that $\mu(K - M) = 0$ ("Choquet's theorem"). (Use Problems 6 and 7.)

9. Extend the results of Problems 2 to 8 to the case of a convex and vaguely compact subset K of $M_+(X)$, where X is a compact space (cf. Section 12.15, Problem 13).

11. THE SPACES L^1 AND L^2

Throughout the remainder of this chapter we shall make the following convention: the function $(x, y) \mapsto xy$ will be extended to $\overline{\mathbf{R}} \times \overline{\mathbf{R}}$ by putting $xy = 0$ whenever one of the factors is 0 and the other is $\pm\infty$ (of course, this extension to $\overline{\mathbf{R}}$ of the multiplication on $\mathbf{R}$ is not continuous). With this convention, we have $x(yz) = (xy)z$ for all x, y, $z \in \overline{\mathbf{R}}$, and the relations $x \leqq y$ and $z \geqq 0$ imply $xz \leqq yz$.

If f is any mapping of X into $\overline{\mathbf{R}}$, we put

$$\text{(13.11.1)} \qquad N_1(f) = \int^* |f|\, d\mu, \qquad N_2(f) = \left(\int^* |f|^2\, d\mu\right)^{1/2} = N_1(|f|^2)^{1/2}.$$

(13.11.2) *For all mappings $f \geqq 0$ and $g \geqq 0$ of X into $\overline{\mathbf{R}}$, we have*

$$\text{(13.11.2.1)} \qquad N_p(f + g) \leqq N_p(f) + N_p(g) \qquad (p = 1, 2).$$

This has already been proved for $p = 1$ (13.5.6). For $p = 2$ we shall first prove (with the above convention for products in $\overline{\mathbf{R}}$) the *generalized Cauchy–Schwarz inequality*

$$\text{(13.11.2.2)} \qquad N_1(fg) \leqq N_2(f)N_2(g).$$

This inequality is true if one of the factors on the right-hand side is zero. If for example $N_2(g) = 0$, then g^2 is negligible and therefore so is g (13.6.3)

hence also fg (with the above convention) (13.6.3), and so both sides of (13.11.2.2) are zero. Next, the inequality is true when one of the factors on the right-hand side is $+\infty$ (for the same reason). Hence we may assume that both factors on the right-hand side of (13.11.2.2) are *finite* and nonzero, which implies (13.6.4) that f and g are *finite almost everywhere*. Now if a, b are any two real numbers $\geqq 0$, we have

$$2ab \leqq ta^2 + t^{-1}b^2$$

for all scalars $t \geqq 0$. Bearing in mind (13.6.5) and (13.5.6), it follows that

$$2\int^* fg\, d\mu \leqq \int^* (tf^2 + t^{-1}g^2)\, d\mu \leqq t\int^* f^2\, d\mu + t^{-1}\int^* g^2\, d\mu.$$

Putting $t = N_2(g)/N_2(f)$, we have the inequality (13.11.2.2).

To prove (13.11.2.1) when $p = 2$, we may again restrict ourselves to the case where $N_2(f)$ and $N_2(g)$ are finite, and f and g are $\geqq 0$. Since $(f+g)^2 \leqq 2(f^2+g^2)$, it follows that $N_2(f+g)$ is finite. We may assume that $N_2(f+g) \neq 0$. Moreover, we have

$$(f+g)^2 = f(f+g) + g(f+g)$$

with the convention at the beginning of this section; hence ((13.5.6) and (13.11.2.2))

$$\begin{aligned}(N_2(f+g))^2 = N_1((f+g)^2) &\leqq N_1(f(f+g)) + N_1(g(f+g)) \\ &\leqq N_2(f)N_2(f+g) + N_2(g)N_2(f+g)\end{aligned}$$

from which (13.11.2.1) follows, by dividing through by $N_2(f+g) \neq 0$.

(13.11.3) *If (f_n) is any sequence of functions on X with values in $\overline{\mathbf{R}}$, then*

$$\text{(13.11.3.1)} \qquad N_p\left(\sum_{n=1}^{\infty} f_n\right) \leqq \sum_{n=1}^{\infty} N_p(f_n) \qquad (p = 1, 2).$$

This also has already been proved for $p = 1$ (13.5.8). The proof is similar for $p = 2$: the sequence of functions $g_n = \left(\sum_{k=1}^{n} f_k\right)^2$ is increasing and has $\left(\sum_{n=1}^{n} f_n\right)^2$ as its upper envelope. Since $(N_1(g_n))^{1/2} \leqq \sum_{k=1}^{n} N_2(f_k)$, by (13.11.2.1) and induction on n, we can apply (13.5.7) to the sequence (g_n) to obtain the result.

We have already (13.7.3) defined the set $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$ of (finite) real-valued μ-integrable functions, and we have seen that it is a vector space. Likewise,

the set of all real-valued *μ-measurable* functions f such that f^2 is integrable is a *vector space* $\mathscr{L}^2_{\mathbf{R}}(X, \mu)$ (also denoted by $\mathscr{L}^2_{\mathbf{R}}(\mu)$ or $\mathscr{L}^2_{\mathbf{R}}$). For if f and g are measurable and f^2 and g^2 are integrable, then $f+g$ is measurable by virtue of (13.11.2.1), and it follows from (13.9.13) that $(f+g)^2$ is integrable.

There exist functions f which are *non-measurable*, such that f^2 is integrable (Problem 2).

By abuse of language, the space $\mathscr{L}^2_{\mathbf{R}}(X, \mu)$ is called the *space of square-integrable functions*. Since $N_p(af) = |a|N_p(f)$ for $p = 1, 2$ and any scalar $a \neq 0$, it follows from (13.11.2.1) that N_p is a *seminorm* on the space $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$. The set $\mathscr{N}$ of functions f such that $N_p(f) = 0$ is in both cases the vector subspace of *negligible* (finite) real-valued functions (13.6.3). Hence the space $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ is not in general Hausdorff with respect to the topology defined by the seminorm N_p. *The quotient space* $L^p_{\mathbf{R}}(X, \mu) = \mathscr{L}^p_{\mathbf{R}}(X, \mu)/\mathscr{N}$ (also denoted by $L^p_{\mathbf{R}}(\mu)$ or $L^p_{\mathbf{R}}$) is the space of *equivalence classes* $\tilde{f}$ of integrable functions when $p = 1$, square-integrable functions when $p = 2$ (13.6). A function which is defined and finite almost everywhere in X is said to be *square-integrable* if its class belongs to $L^2_{\mathbf{R}}$.

The number $N_p(f)$ is the same for all functions f belonging to a class $\tilde{f} \in L^p_{\mathbf{R}}$, and is denoted by $N_p(\tilde{f})$. From (12.14.8) it follows that $\tilde{f} \mapsto N_p(\tilde{f})$ is a *norm* on $L^p_{\mathbf{R}}$ $(p = 1, 2)$.

(13.11.4) (Fischer–Riesz Theorem) *The normed space* $L^p_{\mathbf{R}}(X, \mu)$ *is complete* $(p = 1, 2)$ (in other words, it is a Banach space). *More precisely:*

(i) *If* (g_n) *is a sequence of functions on* X *whose classes belong to* $L^p_{\mathbf{R}}$, *and if* $\sum_{n=1}^{\infty} N_p(g_n) < +\infty$, *then the series whose general term is* $g_n(x)$ *converges absolutely in* **R** *almost everywhere. If* $g(x) = \sum_{n=1}^{\infty} g_n(x)$, *the class* $\tilde{g}$ *of the function* g (defined almost everywhere) *belongs to* $L^p_{\mathbf{R}}$, *and we have* $\tilde{g} = \sum_{n=1}^{\infty} \tilde{g}_n$ *in the normed space* $L^p_{\mathbf{R}}$.

(ii) *If* (f_n) *is a sequence of functions such that the sequence* $(\tilde{f}_n)$ *of their equivalence classes is a Cauchy sequence in* $L^p_{\mathbf{R}}$, *then there exists a subsequence* (f_{n_k}) *such that* $(f_{n_k}(x))$ *converges to a limit* $f(x)$ *in* **R** *almost everywhere. For each such subsequence* (f_{n_k}), *the class* $\tilde{f}$ *of* f *belongs to* $L^p_{\mathbf{R}}$ *and is the limit of the sequence* $(\tilde{f}_n)$ *in the normed space* $L^p_{\mathbf{R}}$.

(iii) *Let* (f_n) *be a sequence of functions belonging to* $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ *such that the sequence* $(f_n(x))$ *converges almost everywhere to a limit* $f(x)$, *and suppose that there exists a function* $h \in \mathscr{L}^p_{\mathbf{R}}(X, \mu)$ *such that* $|f_n(x)| \leqq h(x)$ *almost everywhere, for all* n. *Then the class* $\tilde{f}$ *of* f *belongs to* $L^p_{\mathbf{R}}$ *and is the limit of the sequence* $(\tilde{f}_n)$ *in the normed space* $L^p_{\mathbf{R}}$.

To show that $L^p_{\mathbf{R}}$ is complete, it is clearly enough to prove (i) and (ii). We have already proved (i) in the case $p = 1$ (13.8.5). When $p = 2$, it follows from the hypotheses and (13.11.3.1) that the function $\sum_{n=1}^{\infty} |g_n|$ is finite almost everywhere (13.6.4), and hence the series $\sum_{n=1}^{\infty} |g_n(x)|$ converges in $\mathbf{R}$ for almost all $x \in X$. Hence the same is true for $\sum_{n=1}^{\infty} g_n(x)$, and since $|g(x)| \leqq \sum_{n=1}^{\infty} |g_n(x)|$ almost everywhere, we have $N_2(g) < +\infty$ by (13.11.3.1). Moreover, g is measurable (13.9.11), hence its class belongs to $L^2_{\mathbf{R}}$. Since

$$\left| g(x) - \sum_{k=1}^{n} g_k(x) \right| \leqq \sum_{k=n+1}^{\infty} |g_k(x)|$$

for all n and almost all $x \in X$, it follows from (13.11.3.1) that

$$N_2\left(g - \sum_{k=1}^{n} g_k \right) \leqq \sum_{k=n+1}^{\infty} N_2(g_k)$$

which is arbitrarily small for large n. Hence $\tilde{g} = \sum_{n=1}^{\infty} \tilde{g}_n$.

(ii) To prove the first assertion, it is enough to show that, for a suitably chosen subsequence (f_{n_k}), the series $f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$ converges almost everywhere, and that if $f(x)$ is its sum (defined almost everywhere) then $\tilde{f} = \lim_{n \to \infty} \tilde{f}_n$. We define the sequence (n_k) inductively to satisfy

$$N_p(f_{n_{k+1}} - f_{n_k}) \leqq 2^{-k}$$

for all $k \geqq 1$ (and $n_1 = 1$); then it follows from (i) that the series

$$f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

converges almost everywhere and that, if $f(x)$ is its sum, then $\tilde{f} = \lim_{k \to \infty} \tilde{f}_{n_k}$ in $L^p_{\mathbf{R}}$. Since $(\tilde{f}_n)$ is a Cauchy sequence, it follows that $\tilde{f} = \lim_{n \to \infty} \tilde{f}_n$ in $L^p_{\mathbf{R}}$, by (3.14.2). To prove the second assertion, the preceding argument may be applied by replacing the sequence (f_n) by any subsequence (f_{n_k}) which has a limit almost everywhere.

(iii) For each n let $g_n = \sup_{h \geqq 0,\, k \geqq 0} |f_{n+h} - f_{n+k}|$. Then g_n is $\geqq 0$ and measurable (13.9.11), and belongs to $\mathscr{L}^p_{\mathbf{R}}$ (13.9.13) because $|g_n| \leqq 2h$. The sequence (g_n) is decreasing and by hypothesis tends to 0 almost everywhere, hence

$\lim_{n\to\infty} N_p(g_n) = 0$ (13.8.1). This shows that the sequence $(\tilde{f}_n)$ is a Cauchy sequence in $L^p_{\mathbf{R}}$, and converges to $\tilde{f}$ by virtue of (ii) and the hypotheses.

A sequence (f_n) in $\mathscr{L}^p_{\mathbf{R}}$ which converges to $f \in \mathscr{L}^p_{\mathbf{R}}$ with respect to the topology defined by the seminorm N_p is said to *converge in the mean* to f when $p = 1$, and to *converge in square mean* to f when $p = 2$. It can happen that f_n tends to f in this sense but that the sequence $(f_n(x))$ does not converge in $\mathbf{R}$ for *any* $x \in X$ (Problem 1).

(13.11.5) *Let $\mathscr{E} \subset \mathscr{L}^p_{\mathbf{R}}$ be a dense set of functions (with respect to the topology defined by the seminorm N_p). Then, for every function $f \in \mathscr{L}^p_{\mathbf{R}}$, there exists a sequence of functions $g_n \in \mathscr{E}$ such that the sequence $(g_n(x))$ converges almost everywhere to $f(x)$ and such that g_n tends to f in $\mathscr{L}^p_{\mathbf{R}}$.*

This follows immediately from (13.11.4) and the fact that in the metric space $L^p_{\mathbf{R}}$ there exists a sequence $(\tilde{g}_n)$ of classes of functions belonging to $\mathscr{E}$, with $\tilde{f}$ as limit (3.13.13).

This result applies in particular when $\mathscr{E} = \mathscr{K}_{\mathbf{R}}(X)$:

(13.11.6) (i) *The subspace $\mathscr{K}_{\mathbf{R}}(X)$ is dense in $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ ($p = 1, 2$).*

(ii) *For each compact subset K of X, the topology induced on $\mathscr{K}_{\mathbf{R}}(X; K)$ by the topology of $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ is coarser than the topology defined by the norm $\|f\|$.*

(iii) *The spaces $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ are separable ($p = 1, 2$).*

Assertion (i) has already been proved for $p = 1$ (13.7.2). To prove it for $p = 2$, it is enough to show that every function $f \geqq 0$ in $\mathscr{L}^2_{\mathbf{R}}$ lies in the closure of $\mathscr{K}_{\mathbf{R}}(X)$. Now by (13.7.2), given any $\varepsilon > 0$, there exists a function $u \in \mathscr{K}_{\mathbf{R}}(X)$ such that $N_1(f^2 - u) \leqq \varepsilon$, and since $|f^2 - u^+| \leqq |f^2 - u|$ we may assume that $u \geqq 0$, hence that $u = v^2$ with $v \in \mathscr{K}_{\mathbf{R}}(X)$ and $v \geqq 0$. But then $|f - v|^2 \leqq |f^2 - v^2|$, and consequently

$$(N_2(f - v))^2 = N_1(|f - v|^2) \leqq N_1(|f^2 - v^2|) \leqq \varepsilon.$$

To prove (ii) it is enough to observe that for any $g \in \mathscr{K}_{\mathbf{R}}(X; K)$ we have $N_p(g) \leqq \|g\| \cdot (\mu(K))^{1/p}$.

Finally, let (K_n) be an increasing sequence of compact sets which cover X and are such that $\mathscr{K}_{\mathbf{R}}(X)$ is the union of the $\mathscr{K}_{\mathbf{R}}(X; K_n)$ (3.18.3). By (7.4.4) and (3.10.9), for each n there exists a dense sequence $(g_{mn})_{m \geqq 1}$ in the Banach space $\mathscr{K}_{\mathbf{R}}(X; K_n)$. From (i) and (ii) above, the double sequence $(g_{mn})_{m \geqq 1, n \geqq 1}$ is dense in $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$.

We remark that, given $f \in \mathscr{L}^1_{\mathbf{R}}(X, \mu)$, it is not always possible to find a function f' *equivalent* to f and a sequence of functions $f_n \in \mathscr{K}_{\mathbf{R}}(X)$ such that the sequence $(f_n(x))$ converges to $f'(x)$ for *all* $x \in X$ (Problem 3).

We have already defined (13.10) the vector space $\mathscr{L}^1_{\mathbf{C}}(X, \mu)$ of *complex*-valued integrable functions. Likewise, we denote by $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$ (or $\mathscr{L}^2_{\mathbf{C}}(\mu)$, or $\mathscr{L}^2_{\mathbf{C}}$) the set of *complex*-valued μ-measurable functions such that $|f|^2$ is μ-integrable, and we define $N_1(f)$ and $N_2(f)$ by the formula (13.11.1) for *any* complex-valued function f. Since a complex-valued function is measurable if and only if $\mathscr{R}f$ and $\mathscr{I}f$ are measurable, and since

$$\sup(|\mathscr{R}f|, |\mathscr{I}f|) \leq |f| \leq |\mathscr{R}f| + |\mathscr{I}f|,$$

it follows that

$$\mathscr{L}^2_{\mathbf{C}}(X,\mu) = \mathscr{L}^2_{\mathbf{R}}(X,\mu) \oplus i\mathscr{L}^2_{\mathbf{R}}(X,\mu)$$

All the properties of $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ proved above extend immediately to $\mathscr{L}^p_{\mathbf{C}}(X, \mu)$ $(p = 1, 2)$, and the Banach space $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$ (also denoted by $\mathscr{L}^2_{\mathbf{C}}(\mu)$ or $\mathscr{L}^2_{\mathbf{C}}$) is defined exactly as in the real case.

Furthermore, we have

(13.11.7) *If f, g belong to $\mathscr{L}^2_{\mathbf{R}}(X, \mu)$* (resp. $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$) *then their product fg belongs to $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$* (resp. $\mathscr{L}^1_{\mathbf{C}}(X, \mu)$). *The space $L^2_{\mathbf{R}}(X, \mu)$* (resp. $L^2_{\mathbf{C}}(X, \mu)$) *is a separable Hilbert space with respect to the Hermitian form*

$$(\tilde{f}, \tilde{g}) \mapsto (\tilde{f} \mid \tilde{g}) = \int f(x)\,\overline{g(x)}\,d\mu(x), \tag{13.11.7.1}$$

and the corresponding norm is $N_2(\tilde{f})$.

The product fg is measurable (13.9.8.1), hence the first assertion follows from (13.9.13) and (13.11.2.2). The second follows from the first together with the Fischer–Riesz theorem (13.11.4) and (13.11.6).

PROBLEMS

1. Let λ be Lebesgue measure on $I = [0, 1[$. For each integer $n = 2^h + k$ $(0 \leqq k < 2^h)$, let f_n be the function which is equal to 1 on the interval $[k \cdot 2^{-h}, (k+1) \cdot 2^{-h}[$, and zero elsewhere. Show that the sequence (f_n) converges to 0 in mean and in square mean, but that the sequence $(f_n(x))$ does not converge for any $x \in I$.

2. Let A be a subset of $I = [0, 1]$ which is not Lebesgue-measurable, and let f be the function which is equal to 1 on A and to -1 on $I - A$. Then f^2 is integrable but f is not measurable.

3. Consider the sequence (A_n) defined in Problem 4(b) of Section 13.8, and let H be the union of the sets A_{2n+1} $(n > 0)$. Show that, if J is any nonempty open interval contained in I, then the sets $J \cap H$ and $J \cap (I - H)$ both have strictly positive measure. Now let f be a function equal to φ_H almost everywhere. Show that there does not exist any sequence (f_n) of continuous functions on I which converges at *every* point of I to the function f. (Remark that f must be discontinuous at every point of I, and use Problem 3 of Section 12.16.)

4. Let μ be a bounded measure of total mass 1 on a locally compact space X. Let $(A_k)_{1 \leqslant k \leqslant n}$ be a finite sequence of integrable subsets of X, such that $\mu(A_k) = c > 0$ for all k. If $nc > 1$ and $\varepsilon > c(1 - c)/(n - 1)$, show that there exist two distinct indices i, j such that $\mu(A_i \cap A_j) \geqq c^2 - \varepsilon$. (Argue by contradiction, using the Cauchy-Schwarz inequality.)

5. Let (f_n) be a sequence of functions belonging to $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ $(p = 1$ or $2)$ which converges almost everywhere to a function f.
(a) Show that, if $N_p(f_n) \leqq a$ for all n, then $f \in \mathscr{L}^p_{\mathbf{R}}(X, \mu)$ and $N_p(f) \leqq a$. (Use Fatou's lemma.) Give an example (with $p = 1$) where $N_p(f_n)$ does not tend to $N_p(f)$.
(b) Now suppose in addition that $N_p(f_n) \to N_p(f)$ as $n \to +\infty$. Show that $N_p(f - f_n) \to 0$. (Show that for each $\varepsilon > 0$ there exists a compact subset K of X and an integer n_0 such that $\int_{X-K} |f_n|^p \, d\mu \leqq \varepsilon$ for all $n \geqq n_0$.)

6. In the space $\mathscr{L}^2_{\mathbf{R}}(I, \lambda)$, where $I =]0, 1]$ and λ is Lebesgue measure, consider the functions t^α (α real), which belong to this space provided that $\alpha > -\frac{1}{2}$.

Show that a sequence of distinct exponents $\alpha_n > -\frac{1}{2}$ is such that the functions t^{α_n} form a *total* sequence in $\mathscr{L}^2_{\mathbf{R}}(I, \lambda)$ if and only if the sequence (α_n) satisfies one of the following three conditions:

(1) there exists a subsequence of the sequence (α_n) tending to a finite limit $> -\frac{1}{2}$;

(2) $\lim_{n \to \infty} \alpha_n = -\frac{1}{2}$ and $\sum_n \left| \alpha_n + \frac{1}{2} \right| = +\infty$,

(3) $\lim_{n \to \infty} \alpha_n = +\infty$ and $\sum_n \alpha_n^{-1} = +\infty$.

(Using Weierstrass' theorem (7.4.1), calculate for each integer $m > 0$ the minimum of $N_2(t^m - f(t))$ as f runs through the set of all linear combinations of the first n functions t^{α_k}. For this purpose, use Problem 3(b) of Section 6.6, and the formula (Cauchy's determinant)

$$\det\left(\frac{1}{a_i + b_j}\right) = \frac{\prod_{i<j} (a_i - a_j)(b_i - b_j)}{\prod_{i,j} (a_i + b_j)}$$

in which (a_i), (b_i) are any two sequences of n distinct numbers > 0.) Deduce that the

sequence of functions consisting of 1 and t^{α_n} ($n \geqq 1$), where $\lim_{n\to\infty} \alpha_n = +\infty$, is a total sequence in the space $\mathscr{C}_{\mathbf{R}}(\mathrm{I})$, if and only if $\sum_n \alpha_n^{-1} = +\infty$ ("Muntz's theorem").

In $\mathscr{L}^2_{\mathbf{R}}(\mathrm{X}, \mu)$ or $\mathscr{L}^2_{\mathbf{C}}(\mathrm{X}, \mu)$ a sequence (f_n) of functions is said to be *orthonormal* if the sequence of classes $\tilde{f}_n$ of these functions is an orthonormal system in the corresponding Hilbert space.

(a) Let (f_n) be an infinite orthonormal sequence, uniformly bounded on X. If μ is bounded, show that there exists a set A with measure >0 such that $\sum_n |f_n(x)|^2 = +\infty$ for all $x \in \mathrm{A}$. (Argue by contradiction, observing that the sequence (f_n) cannot tend to 0 almost everywhere.)

(b) Suppose that μ is bounded. Let (f_n) be a uniformly bounded infinite orthonormal sequence. If a sequence (a_n) of scalars is such that the series with general term $a_n f_n(x)$ converges almost everywhere, show that $\lim_{n\to\infty} a_n = 0$ (use Egoroff's theorem applied to the sequence $(|a_n f_n|^2)$). Show that there exists a sequence of scalars (b_n) such that $\lim_{n\to\infty} b_n = 0$ and $\sum_n |b_n|^2 |f_n(x)|^2 = +\infty$ on a set of measure >0. (Choose the sequence (b_n) so that $\sum_n |b_n|^2 = +\infty$, and argue by contradiction, by applying Egoroff's theorem to the partial sums of the series $\sum_n |b_n|^2 |f_n|^2$.)

(c) Let (f_n) be a sequence in $\mathscr{L}^2$. Show that the following three conditions are equivalent:

α) If a sequence (a_n) is such that the series with general term $a_n f_n(x)$ converges on a set of measure >0, then $\lim_{n\to\infty} a_n = 0$.

β) If a sequence (a_n) is such that the series with general term $a_n f_n(x)$ converges absolutely on a set of measure >0, then $\lim_{n\to\infty} a_n = 0$.

γ) For every measurable set A of measure >0, we have

$$\liminf_{n\to\infty} \int_{\mathrm{A}} |f_n| \, d\mu > 0.$$

(To prove that γ) implies α) and β), use Egoroff's theorem. To prove the converse, argue by contradiction, by considering a subsequence (f_{n_k}) such that $\int_{\mathrm{A}} |f_{n_k}| \, d\mu > 1/k^3$, and taking $a_{n_k} = k$ and $a_n = 0$ if $n \neq n_k$.)

(d) Suppose that μ is bounded and (f_n) is a sequence in $\mathscr{L}^2$. Prove that the following three conditions are equivalent:

α) For each sequence (a_n) such that the series with general term $a_n f_n(x)$ converges almost everywhere, we have $\lim_{n\to\infty} a_n = 0$.

β) For each sequence (a_n) such that the series with general term $a_n f_n(x)$ converges absolutely almost everywhere, we have $\lim_{n\to\infty} a_n = 0$.

γ) There exists $\delta > 0$ such that, for every measurable set A whose complement has measure $\leqq \delta$, we have $\liminf_{n\to\infty} \int_{\mathrm{A}} |f_n| \, d\mu > 0$. (Same method as (c).) Deduce that, if the sequence (f_n) is orthonormal and uniformly bounded, it satisfies these three conditions, and therefore the series with general term $a_n f_n(x)$ converges *absolutely* almost everywhere if and only if $\sum_n |a_n| < +\infty$ (use (b)).

8. (a) Let $(f_k)_{1 \leqslant k \leqslant n}$ be a finite orthonormal sequence in $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$. Show that exists a constant c *independent of* n such that, for every finite sequence $(a_k)_{1 \leqslant k}$ complex numbers, we have

$$\int \sup_{1 \leqslant i \leqslant j \leqslant n} \left| \sum_{h=i}^{j} a_h f_h \right|^2 d\mu \leqq c(\log n)^2 \sum_{k=1}^{n} |a_k|^2.$$

(Suppose first that $n = 2^r$. For each $j \leqq n$, let $s_j = \sum_{h=1}^{j} a_h f_h$. Split up this sum in most $r+1$ partial sums corresponding to the dyadic expansion of the number that $s_j = \sum_{q=0}^{r} p_q$ where, for each q, either $p_q = 0$ or else p_q is of the form

$$\sum_{h.2^q \leqslant k < (h+1)2^q} a_k f_k,$$

with $0 \leqq h < 2^{r-q}$. Using Cauchy's inequality $|s_j|^2 \leqq r \cdot \sum_{q=0}^{r} |p_q|^2$, show that

$$\int |s_j|^2 \, d\mu \leqq r(r+1) \sum_{k=1}^{n} |a_k|^2.$$

Finally, if $2^r \leqq n < 2^{r+1}$, take the a_k to be 0 for $k > n$.)

(b) Let $(w(n))_{n \geqslant 1}$ be an increasing sequence of integers >0, and tending to Let $(f_n)_{n \geqslant 1}$ be an orthonormal sequence in $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$. If the sequence $(a$ is such that $\sum_n w(n)|a_n|^2 < +\infty$, and if (n_k) is a sequence of integers such $k \leqq w(n_k) < k+1$, then the sequence of partial sums $s_{n_k} = \sum_{j=1}^{n_k} a_j f_j$ converges al everywhere in X. (If $f \in \mathscr{L}^2_{\mathbf{C}}(X, \mu)$ has the a_n as coefficients (6.5.2) with respect t system (f_n), evaluate $N_2(f - s_{n_k}) = r_{n_k}$ say, and remark that the series with ge term $r^2_{n_k}$ converges, using Abel's partial summation technique by writing r $(k+1-k)r^2_{n_k}$ (Section 13.21, Problem 6).)

(c) Suppose that the sequence (a_n) is such that $\sum_n |a_n|^2 (\log n)^2 < +\infty$. Show tha series $\sum_n a_n f_n$ converges almost everywhere ("Rademacher–Menchoff theore (First deduce from (b) that the sequence (s_{2k}) converges almost everywhere. use (a) to show that, if $\delta_k = \sup_{2^k \leqslant n < 2^{k+1}} |s_n - s_{2k}|$, the series with general term $\int \delta$ converges.)

9. Suppose that the measure μ is bounded, so that $L^2(X, \mu) \subset L^1(X, \mu)$. Suppose that $\mu(X) = 1$. Let U be a continuous endomorphism of L^1 with norm $\leqq 1$ (a " traction") whose restriction to L^2 is also a contraction (i.e., $N_2(U \cdot f) \leqq N_2(f)$ in Let P be the endomorphism of L^2 which is the strong limit of $\frac{1}{n} \sum_{k=0}^{n-1} U^k$ (Section 1 Problem 12), so that $P = PU = UP = P^2$. Show that P can be extended by tinuity to an endomorphism of L^1 satisfying the same relations and that

$$P \cdot f = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k \cdot f$$

for all $f \in L^1$ (use the fact that L^2 is dense in L^1 and that $N_1(f) \leqq N_2(f)$.)

10. Suppose that the measure μ is bounded and that $\mu(X) = 1$. Let u be a μ-measurable mapping of X into itself which leaves *invariant* the measure μ (Section 13.9, Problem 24). Then for every function $f \in \mathscr{L}^1(X, \mu)$ the class $U \cdot f = f \circ u$ depends only on the class of f, hence defines an endomorphism of L^1, also denoted by $\tilde{f} \mapsto U \cdot \tilde{f}$, such that $N_1(U \cdot \tilde{f}) = N_1(\tilde{f})$. The restriction of U to L^2 is a *unitary* operator on L^2.
(a) If P is the orthogonal projection of L^2 on the subspace of vectors which are invariant under U, defined in Problem 9, show that $N_1(\tilde{f})^2 \leqq (\tilde{f} \mid P \cdot \tilde{f})$ for all $\tilde{f} \in L^2$.
(b) Show that for each $\tilde{f} \in L^1$ and each $\varepsilon > 0$ there exists an integer n such that, for all integers $m > 0$,

$$N_1\left(P \cdot \tilde{f} - \frac{1}{n} \sum_{k=m}^{m+n-1} U^k \cdot \tilde{f}\right) \leqq \varepsilon.$$

(c) Deduce that, for each measurable subset A of X and each $\varepsilon > 0$, there exists an integer $n > 0$ with the property that for each integer $m > 0$, there exists an integer k such that $m \leqq k \leqq m + n - 1$ and $\mu(A \cap u^{-k}(A)) \geqq (\mu(A))^2 - \varepsilon$ ("Khintchine's statistical recurrence theorem").
(d) For each $f \in \mathscr{L}^2_{\mathbf{R}}$ and each integer n, put

$$f_n = \frac{1}{n} \sum_{k=0}^{n-1} U^k \cdot f.$$

Show that the limit

$$F_n(x) = \lim_{m \to \infty} \frac{1}{m} \sum_{h=0}^{m-1} ((U^h \cdot f_n)(x) - (P \cdot f)(x))^2,$$

(where $P \cdot f$ denotes a function in the class $P \cdot \tilde{f}$) exists almost everywhere and that $\lim_{n \to \infty} \int F_n \, d\mu = 0$. (Use Birkhoff's ergodic theorem.)
(e) Let $f \in \mathscr{L}^1(X, \mu)$ be such that $f(x) \geqq 0$ almost everywhere. Show that, for almost all $x \in X$, either $f(x) = 0$ or $(P \cdot f)(x) > 0$. (Consider the set N of points $x \in X$ at which $(P \cdot f)(x)$ is either undefined or equal to zero.)

1. Let X be a metrizable compact space and let $u : X \to X$ be a homeomorphism. The set I of measures $\geqq 0$ on X of total mass 1 and *invariant* under u is then a nonempty vaguely compact subset of M(X) (Section 13.4, Problem 8). A point $x \in X$ is said to be *quasi-regular* (relative to u) if the sequence of measures

$$\rho_{n,x} = \frac{1}{n} \sum_{k=0}^{n-1} u^k(\varepsilon_x)$$

converges vaguely as $n \to +\infty$ (necessarily to a measure $\mu_x \in I$). Let Q denote the set of quasi-regular points of X. A point $x \in Q$ is said to be *ergodic* (relative to u) if the measure μ_x is ergodic (Section 13.9, Problem 13). Let E be the set of ergodic points $x \in Q$. A point $x \in Q$ is said to be *dense* if x belongs to the support of μ_x. Let D be the set of dense points $x \in Q$. The points belonging to $R = E \cap D$ are said to be *regular* (relative to μ).
(a) Show that the complement of R (and hence also the complement of Q, E and D) *is negligible with respect to any invariant measure* $\nu \in I$. (To show that Q has measure 1 for any measure $\nu \in I$, apply Birkhoff's ergodic theorem to the functions belonging to a dense sequence in $\mathscr{C}(X)$. To show that D has measure 1, consider a denumerable basis (U_n) for the topology of X, and for each pair (m, n) such that $\bar{U}_m \subset U_n$, a continuous

mapping $g_{mn} : X \to [0, 1]$ such that $g_{mn}(x) = 1$ for all $x \in U_m$ and $g_{mn}(x) = 0$ for all $x \in X - U_n$; apply Problem 10(e) to each of the functions g_{mn}. Finally, to show that E has measure 1, show (with the notation of Problem 10) that, for any measure $\nu \in I$,

$$\int ((P \cdot f)(y) - (P \cdot f)(x))^2 \, d\mu_x(y) = 0$$

for almost all $x \in X$ and all $f \in \mathscr{C}(X)$, by applying Problem 10(d); now let f run through a dense sequence of functions in $\mathscr{C}(X)$.)

(b) With respect to the vague topology on M(X) show that, for every measure $\nu \in I$,

$$\nu = \int_R \mu_x \, d\nu(x)$$

in the sense of (13.10), and that the external points of I are the measures which are *ergodic* with respect to u (cf. Section 13.10, Problem 9).

(c) For every *ergodic* measure $\nu \in I$, let Q_ν (resp. R_ν) be the set of points $x \in Q$ (resp. $x \in R$) such that $\mu_x = \nu$. The measure ν is concentrated on R_ν (13.18). The sets Q_ν and R_ν are called the *quasi-ergodic set* and the *ergodic set* corresponding to ν. The Q_ν (resp. R_ν) form a partition of Q (resp. R), and we have $u(Q_\nu) = Q_\nu$ and $u(R_\nu) = R_\nu$. Show that, for each closed set F such that $u(F) = F$, we have either $F \cap R_\nu = R_\nu$ or $F \cap R_\nu = \varnothing$.

(d) For every nonempty closed set F such that $u(F) = F$, show that $R \cap F \neq \varnothing$ (consider the points which are regular with respect to $u \mid F$).

(e) Let $\mathscr{M}$ be the monoid generated by 1_X and u. Show that if Z is any minimal closed orbit with respect to $\mathscr{M}$ (Section 12.10, Problem 6) we have $u(Z) = Z$ and hence $R \cap Z \neq \varnothing$. Deduce that, for each $x \in X$, the closed orbit $O(x)$ of x with respect to $\mathscr{M}$ intersects R, and that $O(x) \cap R$ is a union of ergodic sets R_ν.

(f) For each $x \in X$, let μ be a measure which is a cluster point of the sequence $(\rho_{n,x})$ in M(X). Show that $\mu(O(x)) = 1$.

(g) Show that for a measure $\mu \in I$ to be ergodic it is necessary and sufficient that there should exist an ergodic set R_ν such that $\mu(R_\nu) = 1$, and that then we have $\mu = \nu$. (Use (b) to prove that the condition is sufficient.)

(h) If $x \in X$ is such that $O(x)$ contains only one ergodic set, show that x is an ergodic point (use (f) and (g)).

(i) Suppose that I consists of a *single* measure ν_0. Then $Q = X$, and $R = D$ is the support of ν_0 (use (h)) and is the only minimal closed orbit with respect to M. Moreover, for each function $f \in \mathscr{C}(X)$, the sequence of numbers $\frac{1}{n} \sum_{k=0}^{n-1} f(u^k(x))$ converges to $\int f \, d\nu_0$ *uniformly* in $x \in X$. (Observe that, for each n, the mapping $x \mapsto \rho_{n,x}$ of X into M(X) is continuous with respect to the vague topology, and use (7.5.6).)

12. For every real number p such that $0 < p < +\infty$, and every mapping $f : X \to \bar{\mathbf{R}}$, put $N_p(f) = \left(\int^* |f|^p \, d\mu \right)^{1/p}$.

(a) If $p > 1$, put $q = p/(p-1)$. If f, g are any two mappings of X into $\bar{\mathbf{R}}$, show that $N_1(fg) \leqq N_p(f) N_q(g)$ ("Hölder's inequality"). (Show that the set of points $(t_1, t_2) \in \mathbf{R}^2$ such that $t_1 \geqq 0$ and $t_2 \geqq 0$ and $t_1^{1/p} t_2^{1/q} \geqq 1$ is convex, and argue as in Section 13.8, Problem 14(c), making use of (13.5.6). Notice that the proof of (13.11.2.2) is a particular case.)

(b) If $f \geq 0, g \geq 0$ and $p > 1$, then $N_p(f + g) \leq N_p(f) + N_p(g)$ ("Minkowski's inequality"). (Same method as (a): consider the set of points (t_1, t_2) such that $t_1 \geq 0$ and $t_2 \geq 0$ and $t_1^{1/p} + t_2^{1/p} \geq 1$.)

(c) Extend the results of Section 13.11 to the case where p is any number such that $1 < p < +\infty$.

(d) Suppose that $p > 1$. Let $f \in \mathscr{L}^p_{\mathbf{R}}, g \in \mathscr{L}^q_{\mathbf{R}}$ be two functions $\geqq 0$. Show that we have $N_1(fg) = N_p(f)N_q(g)$ if and only if there exist two constants $\alpha > 0, \beta > 0$ such that $\alpha(f(x))^p = \beta(g(x))^q$ almost everywhere.

(e) Suppose that the measure μ is bounded and that $\mu(X) = 1$. For each $r > 0$ and each measurable function $f \geqq 0$ such that f^r is integrable, show that the mapping $p \mapsto N_p(f)$ is an increasing function on the interval $]0, r]$ (use Hölder's inequality). As $p \to 0$, its limit is $\exp\left(\int^* \log |f| \, d\mu\right)$, or 0 if $\int^* \log |f| \, d\mu = -\infty$. If the limit is $\neq 0$, it follows that $f(x) \neq 0$ almost everywhere. If $\exp\left(\int^* \log |f| \, d\mu\right) = \int |f| \, d\mu$, the function $|f|$ is constant almost everywhere.

(f) Let $f \geqq 0$ be a μ-measurable function, and for each $\alpha > 0$ let A_α be the set of all $x \in X$ such that $f(x) \geqq \alpha$. If $f \in \mathscr{L}^p_{\mathbf{C}}$, then $\alpha^p \mu(A_\alpha) \leqq \int f^p \, d\mu$. Conversely, if the measure μ is bounded and if there exist constants $C > 0, \varepsilon > 0$ such that $\mu(A_\alpha) \leqq C \cdot \alpha^{-p-\varepsilon}$ for all $\alpha > 0$, then $f \in \mathscr{L}^p_{\mathbf{C}}$ (cf. Section 13.9, Problem 3).

13. Let f be a real valued function $\geqq 0$ defined on $\mathbf{R}^*_+ =]0, +\infty[$, which is Lebesgue-measurable and such that $|f|^p$ is Lebesgue-integrable $(1 < p < +\infty)$.

(a) The function f is integrable on every compact subset of $[0, +\infty[$, and in particular $F(x) = \int_0^x f(t) \, dt$ is defined for all $x > 0$ (use Hölder's inequality). As x tends to 0 or to $+\infty$, the quotient $F(x)/x^{(p-1)/p}$ tends to 0 (same method).

(b) Show that the function $F(x)/x$ is pth-power-integrable on $]0, +\infty[$ and that

$$\int_0^{+\infty} \left(\frac{F(x)}{x}\right)^p dx \leqq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} (f(x))^p \, dx$$

("Hardy's inequality"). (Consider first the case where $f \in \mathscr{K}(\mathbf{R}^*_+)$. For each compact interval $[a, b] \subset \mathbf{R}^*_+$, majorize the integral $\int_a^b (F(t)/t)^p \, dt$, by integrating by parts and using Hölder's inequality.)

14. Let p be a real number > 0 and let f be a complex μ-measurable function such that (1) $|f|^p$ is μ-integrable and (2) e^{tf} is μ-integrable for $0 < t < t_0$. Show that $N_p(t^{-1}(e^{tf} - 1) - f) \to 0$ as $t \to 0$. (Let $f = u + iv$ where u and v are real; observe that if we put $w_t = e^{tu} - 1 - tu$, we have $w_s/s \leqq w_t/t$ for $0 \leqq s \leqq t$. If $0 < p < 1$, use the elementary inequality $(a + b)^p \leqq a^p + b^p$ when a, b are $\geqq 0$. Finally, apply (13.8.1) and (13.8.4).)

15. Let $(\lambda_n)_{n \geqslant 1}$ be a strictly increasing sequence of integers > 0, and let h be a rational integer. For each integer $N \geqq 1$, put

$$f_N(x) = \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h \lambda_n x}.$$

Show that the series

$$\sum_{m=1}^{\infty} \int_0^1 |f_{m^2}(x)|^2 \, dx$$

converges. Note that if $m^2 \leqq N < (m+1)^2$, we have

$$\left| f_N(x) - \frac{m^2}{N} f_{m^2}(x) \right| \leqq 2/\sqrt{N},$$

and deduce that the sequence $(f_N(x))_{N \geqslant 1}$ tends to 0 for almost all $x \in [0, 1]$ respect to Lebesgue measure). Hence show that, for *almost all* $x \in [0, 1]$, the sequ $(\lambda_n x - [\lambda_n x])$ is *equirepartitioned* with respect to Lebesgue measure (Section Problem 7).

16. Let U be a continuous endomorphism of the space $L^1_{\mathbf{R}}(X, \mu)$ with norm $\leqq 1$ (in o words, such that $N_1(U \cdot \tilde{f}) \leqq N_1(\tilde{f})$ and such that the relation $\tilde{f} \geqq 0$ implies $U \cdot \tilde{f}$ (13.6). If f is any function in the class $\tilde{f}$, we denote by $U \cdot f$ any function in the $U \cdot \tilde{f}$.

(a) Let (f_n) be a sequence of functions in $\mathscr{L}^1_{\mathbf{R}}$ which converges almost everyw to a function f and is such that there exists a function $h \in \mathscr{L}^1_{\mathbf{R}}$ satisfying $|f_n| \leqq$ all n. Show that under these conditions the sequence $(U \cdot f_n)$ converges almost ev where to $U \cdot f$. (Reduce to the case $f = 0$, and consider the sequence of funct $g_n = \sup_p |f_{n+p}|$.)

(b) For each finite sequence $(t_k)_{1 \leqslant k \leqslant n}$ of real numbers, put

$$s_n(t_1, \ldots, t_n) = \sup_{1 \leqslant k \leqslant n} (t_1 + \cdots + t_k)^+.$$

If $s_n(t_1, \ldots, t_n) > 0$, then

$$s_n(t_1, \ldots, t_n) = \sup_{1 \leqslant k \leqslant n} (t_1 + \cdots + t_k).$$

Show that if $s_n(t_1, \ldots, t_n) > 0$, then for all $t_{n+1} \in \mathbf{R}$ we have

$$t_1 + s_n(t_2, \ldots, t_{n+1}) \geqq s_n(t_1, \ldots, t_n). \tag{1}$$

(c) For each function $f \in \mathscr{L}^1_{\mathbf{R}}$, show that

$$s_n(U \cdot f, \ldots, U^{n+1} \cdot f) \leqq U \cdot s_n(f, \ldots, U^n \cdot f)$$

almost everywhere. (Use (1) and the fact that for any n functions $f_1, \ldots, f_n \in$ we have

$$\sup_{1 \leqslant k \leqslant n} U \cdot f_k \leqq U \cdot \sup_{1 \leqslant k \leqslant n} f_k$$

almost everywhere.)

(d) Let $E(f)$ be the measurable set of points $x \in X$ such that

$$f(x) + (U \cdot f)(x) + \cdots + (U^{n-1} \cdot f)(x) > 0$$

for some integer $n > 0$. Show that

$$\int_{E(f)} f \, d\mu \geqq 0$$

(E. Hopf's maximal ergodic theorem). (For each integer $n \geqq 1$, let $E_n(f)$ denote set of $x \in X$ such that

$$f(x) + (U \cdot f)(x) + \cdots + (U^k \cdot f)(x) > 0$$

for some integer k between 0 and $n-1$; show that $\int_{E_n(f)} f\,d\mu \geqq 0$ by using (c) above.) Hence deduce the maximal ergodic theorem of Section 13.9, Problem 12.

17. Suppose that U satisfies the conditions of Problem 16 and also that for each $t > 0$ the relation $|f(x)| \leqq t$ almost everywhere in X implies that $|(U \cdot f)(x)| \leqq t$ almost everywhere. (This will be the case whenever μ is bounded and $U \cdot c = c$ for each constant c.)
(a) Show that, for each function $f \in \mathscr{L}^1_{\mathbf{R}}$ and each $t > 0$,

$$(U \cdot f - t)^+ \leqq U \cdot (f-t)^+ \tag{1}$$

almost everywhere. (Consider the function which is equal to f when $|f(x)| \leqq t$, to t when $f(x) > t$, and to $-t$ when $f(x) < -t$.)
(b) Show that, for each function $f \in \mathscr{L}^1_{\mathbf{R}} \cap \mathscr{L}^\infty_{\mathbf{R}}$, the function $U \cdot f$ belongs to $\mathscr{L}^p_{\mathbf{R}}$ for $1 \leqq p \leqq +\infty$ and that

$$N_p(U \cdot f) \leqq N_p(f).$$

(Deduce from (13.21.9) that for each $f \in \mathscr{L}^1_{\mathbf{R}} \cap \mathscr{L}^\infty_{\mathbf{R}}$, the function $(x,t) \mapsto t^{p-2}(f(x)-t)^+$ is integrable relative to the measure on $X \times]0, +\infty[$ which is the product of μ and Lebesgue measure, and then integrate the inequality

$$t^{p-2}(U \cdot f - t)^+ \leqq t^{p-2}(f-t)^+.)$$

Deduce that U can be extended to an endomorphism of norm $\leqq 1$ of each of the spaces $\mathscr{L}^p_{\mathbf{R}}$. Denote this extension also by U.
(c) Let $f \in \mathscr{L}^p_{\mathbf{R}}$ for some $p \in [1, +\infty[$. For each $t > 0$ and each integer $n \geqq 0$, let $E_{n,t}(f)$ denote the set of points $x \in X$ such that

$$f(x) + (U \cdot f)(x) + \cdots + (U^{n-1} \cdot f)(x) > nt,$$

and let $E_t(f)$ denote the union of the $E_{n,t}(f)$. Show that each of the sets $E_{n,t}(f)$ is integrable and that

$$\int_{E_{n,t}(f)} (f-t)\,d\mu \geqq 0.$$

(Observe that $E_{n,t}(f)$ is contained in the set of points $x \in X$ satisfying one or other of the n relations $(U^k \cdot f)(x) > t$, where $0 \leqq k \leqq n-1$; then continue as in Problem 16, by remarking that

$$(U \cdot f + U^2 \cdot f + \cdots + U^{k-1} \cdot f - (k+1)t)^+ \leqq U \cdot (f + U \cdot f + \cdots + U^k \cdot f - (k+1)t)^+.)$$

(d) If $f \in \mathscr{L}^1_{\mathbf{R}}$, deduce from (c) that

$$\mu(E_t(f)) \leqq \frac{1}{t} N_1(f)$$

for all $t > 0$.

18. Suppose that U satisfies the conditions of Problem 16. Let $f, g \in \mathscr{L}^1_{\mathbf{R}}$ be such that $f \geqq 0$ and $g \geqq 0$.
(a) Let ε be a real number > 0, and let A_n be the set of $x \in X$ which satisfy the inequality

$$(U^n \cdot f)(x) > \varepsilon(g(x) + (U \cdot g)(x) + \cdots + (U^n \cdot g)(x)).$$

Show that the series $\varphi_{A_1} g + \varphi_{A_2} g + \cdots + \varphi_{A_n} g + \cdots$ converges almost everywhere. (Observe that

$$\varphi_{A_n} g + \left(U^n \cdot f - \sum_{k=0}^{n} U^k \cdot g \right)^+ \leqq U \cdot \left(U^{n-1} \cdot f - \sum_{k=0}^{n-1} U^k \cdot g \right)^+$$

almost everywhere; integrate both sides, and use (13.8.5).)

(b) Let B be the measurable set of points $x \in X$ such that $(U^n \cdot g)(x) > 0$ for at least one integer $n \geqq 0$. Show that in B the sequence of functions

$$U^n \cdot f / (g + U \cdot g + \cdots + U^n \cdot g)$$

converges to 0 almost everywhere. (Deduce from (a) above that the intersection of $\bigcap_n A_n$ with the set $\{x \in X : g(x) > 0\}$ is negligible. Now replace g and f successively by $U^m \cdot g$ and $U^m \cdot f$, where $m \geqq 1$.)

(c) Let B_0 be the measurable set of points $x \in X$ such that $g(x) > 0$. For each $x \in B_0$, let

$$R_n(f, g)(x) = \frac{f(x) + (U \cdot f)(x) + \cdots + (U^{n-1} \cdot f)(x)}{g(x) + (U \cdot g)(x) + \cdots + (U^{n-1} \cdot g)(x)}$$

and let $R^*(f, g) = \sup_n |R_n(f, g)|$. Show that $R^*(f, g)(x) < +\infty$ for almost all $x \in B_0$. (For each $t > 0$, let A_t be the set of points $x \in B_0$ such that $R^*(f, g)(x) > t$. Show that $t \int_{A_t} g \, d\mu \leqq N_1(f)$ by applying Problem 16(d) to $|f| - t \cdot g$.)

(d) Let $\Phi \in \mathscr{L}^1_{\mathbf{R}}$ be such that $\Phi(x) > 0$ for all $x \in X$ (cf. (13.15.7)). Show that the set X_∞ of points $x \in X$ such that

$$\Phi(x) + (U \cdot \Phi)(x) + \cdots + (U^n \cdot \Phi)(x) + \cdots = +\infty$$

does not depend on the choice of Φ, but only on U.

For each $x \in X$, let

$$G_n(x) = \Phi(x) + (U \cdot \Phi)(x) + \cdots + (U^{n-1} \cdot \Phi)(x)$$

and let $G(x) = \lim_{n \to \infty} G_n(x)$ (so that $G(x) = +\infty$ for $x \in X_\infty$). Show that for each integrable set $A \subset X_\infty$ and each $t > 0$ we have $t \cdot (U \cdot \varphi_A) \leqq G$, and hence, that the function $U \cdot \varphi_A$ is zero almost everywhere in $X_0 = \complement X_\infty$. (Observe that, if A_n is the set of $x \in A$ such that $G_n(x) > t$, then $t\varphi_{A_n} \leqq G_n$.)

12. THE SPACE L^∞

If f is a mapping of X into $\overline{\mathbf{R}}$, the *maximum in measure* (resp. *minimum in measure*) of f on X (with respect to the measure μ) is defined to be the greatest lower bound (resp. least upper bound) of the real numbers a such that $f(x) \leqq a$ (resp. $f(x) \geqq a$) *almost everywhere* with respect to μ, and is denoted by $M_\infty(f)$, or ess sup f, or $\operatorname{ess\,sup}_{x \in X} f(x)$ (resp. $m_\infty(f)$, or ess inf f, or $\operatorname{ess\,inf}_{x \in X} f(x)$). Clearly we have $m_\infty(f) = -M_\infty(-f)$.

Suppose that $M_\infty(f) < +\infty$. Then for each $a > M_\infty(f)$ the set of points $x \in X$ such that $f(x) > a$ is negligible. Now the set of points $x \in X$ such that $f(x) > M_\infty(f)$ is the union of the sets of $x \in X$ such that $f(x) > r_n$, where (r_n) is a decreasing sequence with $M_\infty(f)$ as limit. Hence (13.6.2) we have $m_\infty(f) \leqq f(x) \leqq M_\infty(f)$ *almost everywhere* in X. It follows that $m_\infty(f) \leqq M_\infty(f)$ if $\mu \neq 0$. (If $\mu = 0$, then $m_\infty(f) = +\infty$ and $M_\infty(f) = -\infty$.) The relation $m_\infty(f) = M_\infty(f)$ is equivalent to saying that f is *equivalent to a constant* if $\mu \neq 0$. Also (still assuming that $\mu \neq 0$) we have

$$\inf_{x \in X} f(x) \leqq \operatorname*{ess\,inf}_{x \in X} f(x) \leqq \operatorname*{ess\,sup}_{x \in X} f(x) \leqq \sup_{x \in X} f(x).$$

If two functions f, g are equivalent, then

$$m_\infty(f) = m_\infty(g) \qquad \text{and} \qquad M_\infty(f) = M_\infty(g).$$

Hence we can define $m_\infty(f)$ and $M_\infty(f)$ for a function f which is defined only almost everywhere in X: we choose any function $g \in f$ such that g is defined everywhere in X, and put $m_\infty(f) = m_\infty(g)$ and $M_\infty(f) = M_\infty(g)$.

If f and g are such that $f + g$ is defined almost everywhere, then

$$M_\infty(f + g) \leqq M_\infty(f) + M_\infty(g) \tag{13.12.1}$$

wherever the right-hand side is defined. Likewise, if f and g are both $\geqq 0$, then

$$M_\infty(fg) \leqq M_\infty(f) M_\infty(g)$$

with the product convention of (13.11).

A function f, defined almost everywhere on X, with values in $\mathbf{R}$ or $\mathbf{C}$, is said to be *bounded in measure* or *essentially bounded* (with respect to μ) if $M_\infty(|f|) < +\infty$; it follows that f is then *finite almost everywhere*. A bounded function is bounded in measure.

(13.12.2) (Mean value theorem) *Let $f : X \to \overline{\mathbf{R}}$ be measurable and bounded in measure. For every integrable function $g \geqq 0$, the function fg (which is defined and finite almost everywhere) is integrable, and*

$$m_\infty(f) \int g \, d\mu \leqq \int fg \, d\mu \leqq M_\infty(f) \int g \, d\mu. \tag{13.12.2.1}$$

Furthermore, two of the three terms in (13.12.2.1) *are equal only if, in the (measurable) set* S *of points $x \in X$ such that $g(x) \neq 0$, we have either $f(x) = M_\infty(f)$ almost everywhere, or $f(x) = m_\infty(f)$ almost everywhere.*

The function fg is measurable (13.9.8.1) and we have

$$m_\infty(f)g(x) \leqq f(x)g(x) \leqq M_\infty(f)g(x)$$

almost everywhere. Hence fg is integrable (13.9.13), and the inequalities (13.12.2.1) follow. On the other hand, the function $M_\infty(f)g - fg$ is defined almost everywhere and equal to $(M_\infty(f) - f)g$, hence is almost everywhere $\geqq 0$ in X. Hence the relation $\int (M_\infty(f) - f)g\, d\mu = 0$ implies that $(M_\infty(f) - f)g$ is negligible, which completes the proof.

In particular, if A is any *integrable* set, we have

$$m_\infty(f)\mu(A) \leqq \int_A f\, d\mu \leqq M_\infty(f)\mu(A). \tag{13.12.2.2}$$

For any function f on X with values in $\overline{\mathbf{R}}$ or $\mathbf{C}$, we put $N_\infty(f) = M_\infty(|f|)$ if $\mu \neq 0$, and $N_\infty(f) = 0$ if $\mu = 0$. By virtue of (13.12.1), the set $\mathscr{L}^\infty_{\mathbf{R}}(X, \mu)$ (resp. $\mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$) of real-valued (resp. complex-valued) *measurable* functions on X such that $N_\infty(f) < +\infty$ is a real (resp. complex) *vector space*, and N_∞ is a *seminorm* on this space. The set of functions f such that $N_\infty(f) = 0$ is once again the vector subspace $\mathscr{N}$ of negligible functions. The quotient of $\mathscr{L}^\infty_{\mathbf{R}}(X, \mu)$ (resp. $\mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$) by this subspace is therefore the space of *equivalence classes* $\tilde{f}$ of measurable functions bounded in measure. It is denoted by $L^\infty_{\mathbf{R}}(X, \mu)$, or simply by $L^\infty_{\mathbf{R}}(\mu)$ or $L^\infty_{\mathbf{R}}$ (resp. $L^\infty_{\mathbf{C}}(X, \mu)$, or $L^\infty_{\mathbf{C}}(\mu)$ or $L^\infty_{\mathbf{C}}$). The number $N_\infty(f)$ is the same for all functions f belonging to the same class $\tilde{f} \in L^\infty_{\mathbf{R}}$ (resp. $\tilde{f} \in L^\infty_{\mathbf{C}}$); it is denoted also by $N_\infty(\tilde{f})$, and the function $\tilde{f} \mapsto N_\infty(\tilde{f})$ is a *norm* on $L^\infty_{\mathbf{R}}$ (resp. $L^\infty_{\mathbf{C}}$). Clearly we have $L^\infty_{\mathbf{C}} = L^\infty_{\mathbf{R}} \oplus iL^\infty_{\mathbf{R}}$.

(13.12.3) *For a sequence (f_n) in $\mathscr{L}^\infty_{\mathbf{R}}$ (resp. $\mathscr{L}^\infty_{\mathbf{C}}$) to converge to a function f, it is necessary and sufficient that $f_n(x)$ should tend uniformly to $f(x)$ in the complement of a negligible set.*

The condition is clearly sufficient. Conversely, if $\lim_{n\to\infty} N_\infty(f - f_n) = 0$, then for each integer m there exists a negligible set H_m and an integer n_0 such that, for all $n \geqq n_0$, we have $|f(x) - f_n(x)| \leqq 1/m$ for all $x \in \complement H_m$. The union H of the sets H_m is negligible, and $f_n(x)$ tends to $f(x)$ uniformly in $\complement H$.

(13.12.4) *The normed space $L^\infty_{\mathbf{R}}(X, \mu)$ (resp. $L^\infty_{\mathbf{C}}(X, \mu)$) is complete* (i.e., a Banach space).

Let (f_n) be a sequence such that $(\tilde{f}_n)$ is a Cauchy sequence in $L^\infty_{\mathbf{R}}$ (resp. $L^\infty_{\mathbf{C}}$). For each integer $n \geqq 1$, there exists an integer k_n such that

$$N_\infty(f_r - f_s) \leqq 1/n$$

for $r \geqq k_n$ and $s \geqq k_n$. For each pair (r, s) such that $r \geqq k_n$ and $s \geqq k_n$, let A_{rsn} be the negligible set of points $x \in X$ such that $|f_r(x) - f_s(x)| > 1/n$, and let A be the union of the sets A_{rsn} ($n \geqq 1$, $r \geqq k_n$, $s \geqq k_n$), so that A is a negligible set. It is clear that in $X - A$ the sequence $(f_n(x))$ converges uniformly to a limit $f(x)$. The function f, which is defined almost everywhere, is measurable (13.9.10) and bounded in $X - A$. Hence $\tilde{f} \in L^\infty_{\mathbf{R}}$ (resp. $\tilde{f} \in L^\infty_{\mathbf{C}}$). Clearly $N_\infty(f - f_r) \leqq 1/n$ for all $r \geqq k_n$, and therefore $\tilde{f}$ is the limit of the sequence $(\tilde{f}_n)$.

(13.12.5) *If* $f \in \mathscr{L}^p_{\mathbf{C}}(X, \mu)$ (*where* $p = 1$ *or* 2) *and* $g \in \mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$, *then* $fg \in \mathscr{L}^p_{\mathbf{C}}(X, \mu)$, *and* $N_p(fg) \leqq N_p(f)N_\infty(g)$.

For fg is measurable (13.9.8.1), and $|f(x)g(x)| \leqq |f(x)|N_\infty(g)$ almost everywhere; hence the result.

Remark

(13.12.6) Evidently we have $\mathscr{C}^\infty_{\mathbf{R}}(X) \subset \mathscr{L}^\infty_{\mathbf{R}}(X, \mu)$; but in general the canonical image of $\mathscr{C}^\infty_{\mathbf{R}}(X)$ in $L^\infty_{\mathbf{R}}(X, \mu)$ is *not dense* in the latter space, and $L^\infty_{\mathbf{R}}(X, \mu)$ is *not separable* in general (Problem 1).

PROBLEMS

1. If λ is Lebesgue measure, show that the space $L^\infty_{\mathbf{R}}(\mathbf{R}, \lambda)$ is not separable. (If (A_n) is an infinite sequence of nonnegligible measurable sets, no two of which intersect, consider the functions which are constant on each A_n and take only the values ± 1.)

2. (a) For every μ-integrable subset A of X and every $\delta > 0$, let $V(A, \delta)$ denote the set of μ-measurable real-valued functions f such that the set M of points $x \in A$ for which $|f(x)| > \delta$ has measure $\mu(M) \leqq \delta$. Let $\mathscr{S}(X, \mu)$ denote the vector space of (finite) real-valued μ-measurable functions on X. Show that the sets $V(A, \delta)$ form a fundamental system of neighborhoods of 0 for a topology on $\mathscr{S}(X, \mu)$ compatible with its vector space structure (Section 12.14, Problem 1). This topology is called the topology of *convergence in measure* (with respect to μ). A sequence (f_n) which tends to a limit f in this topology is said to *converge in measure* to f.
(b) Show that the intersection of all the neighborhoods $V(A, \delta)$ is the subspace $\mathscr{N}$ of negligible functions, and that the quotient space $S(X, \mu) = \mathscr{S}(X, \mu)/\mathscr{N}$ is metrizable.
(c) If (f_n) is any sequence in $\mathscr{S}(X, \mu)$ (12.9) such that $(\tilde{f}_n)$ is a Cauchy sequence in $S(X, \mu)$, show that there exists a subsequence (f_{n_k}) of (f_n) such that the sequence $(f_{n_k}(x))$ converges for almost all $x \in X$. Deduce that the metrizable vector space $S(X, \mu)$ is *complete*.
(d) Every sequence (f_n) of measurable real-valued functions which converges almost everywhere to a function f, converges in measure to f.
(e) For every finite $p \geqq 1$, show that the space $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ is dense in $\mathscr{S}(X, \mu)$, and that the topology induced on $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ by the topology of convergence in measure is coarser than that defined by the seminorm N_p.

(f) Suppose that X is compact and that the measure μ is diffuse (13.18). Show that, for every neighborhood V of 0 in $\mathscr{S}(X, \mu)$ and every function $f \in \mathscr{S}(X, \mu)$, there exists an integer n such that all the functions af (where a is any real number) belong to $V + \cdots + V$ (n summands). Deduce that every continuous linear form on $\mathscr{S}(X, \mu)$ is *identically zero*, and hence that every vector subspace of *finite* codimension in $\mathscr{S}(X, \mu)$ is *dense* in $\mathscr{S}(X, \mu)$.

3. Suppose that X is compact and the measure μ diffuse (13.18).
(a) Let $(\tilde{f}_n)_{n \geqslant 1}$ be a Hilbert basis of $L^2_{\mathbf{R}}(X, \mu)$. Show that, for each real number $\delta > 0$, there exists a compact subset Y of X such that $\mu(X - Y) \leqq \delta$ and the sequence $(\tilde{f}_n)_{n \geqslant 2}$ is total in $L^2_{\mathbf{R}}(Y, \mu_Y)$. (By using Problems 2(e) and 2(f), show that there exists a sequence of linear combinations of the f_n ($n \geqq 2$) which converges in measure to f_1; then use Problem 2(c) and Egoroff's theorem.)
(b) Show that there exists a bounded measurable function h such that the sequence $(h\tilde{f}_n)_{n \geqslant 2}$ is total in $L^2_{\mathbf{R}}(X, \mu)$. (Choose $h > 0$ such that $f_1/h \notin \mathscr{L}^2_{\mathbf{R}}(X, \mu)$, and then show that no nonnegligible function can be orthogonal to hf_n for all $n \geqq 2$.)

4. Let p be a finite real number $\geqq 1$. A subset H of $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ is said to be *equi-integrable* if, for each $\varepsilon > 0$, there exists a compact subset K of X such that $\int_{X-K} |f|^p \, d\mu \leqq \varepsilon$ for all $f \in H$, and a real number $\delta > 0$ such that $\int_A |f|^p \, d\mu \leqq \varepsilon$ for all $f \in H$ and all integrable sets A of measure $\mu(A) \leqq \delta$.
(a) On an equi-integrable set H, show that the topology of convergence in measure is the same as that defined by the seminorm N_p. Is the conclusion true if H is merely bounded in $\mathscr{L}^p$?
(b) A sequence (f_n) in $\mathscr{L}^p_{\mathbf{R}}(X, \mu)$ is convergent if and only if it is equi-integrable and convergent in measure.
(c) Suppose that the measure μ is bounded and that $\mu(X) = 1$. Let (f_n) be a sequence of functions belonging to $\mathscr{L}^1_{\mathbf{C}}$, and suppose that $\lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \int |\mathscr{R} f_n| \, d\mu = 1$, $\lim_{n \to \infty} N_1(1 - |f_n|) = 0$. Show that $\lim_{n \to \infty} \int |\mathscr{I} f_n| \, d\mu = 0$. (Use (b).)
(d) With μ as in (c), let (f_n) be a sequence of functions belonging to $\mathscr{L}^1_{\mathbf{C}}$ such that

$$\lim_{n \to \infty} \int f_n \, d\mu = \lim_{n \to \infty} \int |f_n| \, d\mu = \lim_{n \to \infty} \int |f_n|^{1/2} \, d\mu = 1.$$

Show that $\lim_{n \to \infty} N_1(1 - f_n) = 0$. (Reduce to the situation of (c) above by using the Cauchy–Schwarz inequality; write

$$|1 - f_n| \leqq |1 - |f_n|| + ||f_n| - f_n|$$

and

$$|1 - |f_n|| = |1 - |f_n|^{1/2}| \cdot |1 + |f_n|^{1/2}|.)$$

5. Let F be a real-valued function with period 1 on **R** which is integrable on the unit interval I = [0, 1] (with respect to Lebesgue measure).
(a) If f is any bounded measurable function on [0, 1], then

$$\lim_{n \to \infty} \int_0^1 f(t) F(nt) \, dt = \left(\int_0^1 F(t) \, dt \right) \left(\int_0^1 f(t) \, dt \right).$$

If also F is bounded on [0, 1], then this equality is valid for every integrable tion f on [0, 1]. (Start with the case where F is bounded and f is continuous on], then (for $f \in \mathscr{L}^1$) approximate f by continuous functions. When F is unbounded, ce to the case $F \geqq 0$ and approximate F by an increasing sequence of bounded tions.)

Deduce from (a) that, for every function f which is integrable on an interval] in $\mathbf{R}$, we have

$$\lim_{n\to\infty} \int_a^b f(t) \sin nt \, dt = \lim_{n\to\infty} \int_a^b f(t) \cos nt \, dt = 0.$$

Let φ be the canonical mapping of $\mathbf{R}$ onto $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, and let μ be the measure which is the image under $\varphi \mid \mathrm{I}$ of Lebesgue measure on I; then μ is invariant er translations in the compact group T (cf. (14.4)). Let k be an integer >1, and let $\to \mathbf{T}$ be the continuous mapping such that $u(\varphi(t)) = u(kt)$ for $t \in \mathrm{I}$. The mapping not injective, but μ is *invariant* with respect to u. Deduce from (a) that u is *ergodic* respect to μ (use Problem 13(b) of Section 13.9). Deduce that $\limsup_{n\to\infty} \mathrm{F}(k^n t)$ most everywhere equal to the constant $\operatorname{ess\,sup}_{t \in \mathrm{I}} \mathrm{F}(t)$, and that the same is true of $\mathrm{F}(nt)$.

$f \in \mathscr{L}^\infty_{\mathbf{C}}(\mu)$. Show that $\left|\int f \, d\mu\right| = \int |f| \, d\mu$ if and only if there exists $c \in \mathbf{C}$ with $= 1$ such that $f(x) = c|f(x)|$ almost everywhere.

pose that μ is bounded and that $\mu(\mathbf{X}) = 1$. Let $f \in \mathscr{L}^1_{\mathbf{C}}$. If we have $\int |1 + \zeta f| \, d\mu \geqq 1$ every complex number ζ, show that $\int f \, d\mu = 0$. (For a fixed $\zeta \in \mathbf{C}$, consider $\lim_{t>0} \dfrac{|1 + t\zeta f| - 1}{t}$.)

If $1 \leqq p < r < +\infty$, then $L^r_{\mathbf{C}} \cap L^\infty_{\mathbf{C}} \subset L^p_{\mathbf{C}} \cap L^\infty_{\mathbf{C}}$ and $L^r_{\mathbf{R}} \cap L^\infty_{\mathbf{R}} \subset L^p_{\mathbf{R}} \cap L^\infty_{\mathbf{R}}$. On space $L^r_{\mathbf{C}} \cap L^\infty_{\mathbf{C}}$, the function $N_{r,\infty} = N_r + N_\infty$ is a norm with respect to which space is complete. Give an example in which the norm induced on $L^r_{\mathbf{C}} \cap L^\infty_{\mathbf{C}}$ by $_\infty$ is not equivalent to $N_{r,\infty}$. (In $L^r_{\mathbf{C}} \cap L^\infty_{\mathbf{C}}$ we always have $N_{r,\infty}(f) \leqq 2N_{p,\infty}(f)$.) e space $L^r_{\mathbf{C}} \cap L^\infty_{\mathbf{C}}$ is a Banach algebra relative to the product fg defined in Section 5; it posesses an identity element only if μ is bounded (in which case it is identical h $L^\infty_{\mathbf{C}}$).

The set $\mathrm{I}(\mathrm{X}, \mu)$ (or $\mathrm{I}(\mu)$) of idempotents in the algebra $L^r_{\mathbf{C}} \cap L^\infty_{\mathbf{C}}$ is independent , and consists of the classes $\tilde{\varphi}_{\mathrm{A}}$, where A runs through the set of integrable subsets X. If A, B are integrable subsets of X, the relation $\tilde{\varphi}_{\mathrm{A}} = \tilde{\varphi}_{\mathrm{B}}$ signifies that the set A, B) $= \mathrm{A} \cup \mathrm{B} - \mathrm{A} \cap \mathrm{B}$ (Section 13.8, Problem 15) is negligible; the relation D(A, B)) $= 0$ is an equivalence relation on the set of integrable subsets of X, and quotient set may be identified with $\mathrm{I}(\mathrm{X}, \mu)$. We shall write $\tilde{\mathrm{A}}$ in place of $\tilde{\varphi}_{\mathrm{A}}$. the norms $N_{r,\infty}$ induce the same topology on $\mathrm{I}(\mathrm{X}, \mu)$, and this topology is therefore ined by the distance $d(\tilde{\mathrm{A}}, \tilde{\mathrm{B}}) = N_1(\varphi_{\mathrm{A}} - \varphi_{\mathrm{B}}) = \mu(\mathrm{D}(\mathrm{A}, \mathrm{B}))$, with respect to which , $\mu)$ is a complete space. The mappings $(\tilde{\mathrm{A}}, \tilde{\mathrm{B}}) \mapsto \sup(\tilde{\mathrm{A}}, \tilde{\mathrm{B}}) = (\mathrm{A} \cup \mathrm{B})^\sim$ and $\tilde{\mathrm{B}}) \mapsto \inf(\tilde{\mathrm{A}}, \tilde{\mathrm{B}}) = (\mathrm{A} \cap \mathrm{B})^\sim$ are continuous for this topology.

Let Y be another locally compact space, ν a positive measure on Y. If there sts an isometry U of $\mathrm{I}(\mathrm{X}, \mu)$ onto $\mathrm{I}(\mathrm{Y}, \nu)$ such that $U(\tilde{\varphi}) = \tilde{\psi}$, the measures μ and ν said to be *isometric*. For each number $p \in [1, +\infty[$, there then exists a unique

linear isometry U_p of $L^p_{\mathbb{C}}(X, \mu)$ onto $L^p_{\mathbb{C}}(Y, \nu)$ which extends U. (Show that U unique extension to a linear bijection of the space $E(X, \mu)$ of classes of μ-int step functions on X, onto the space $E(Y, \nu)$ of classes of ν-integrable step fu on Y. Begin by proving that if $\inf(\tilde{A}, \tilde{B}) = \tilde{\phi}$, then also $\inf(U(\tilde{A}), U(\tilde{B})) = U(\sup(\tilde{A}, \tilde{B})) = \sup(U(\tilde{A}), U(\tilde{B}))$; then deduce that $U(\tilde{A}) \leqq U(\tilde{B})$ whenever finally, show that for arbitrary $\tilde{A}, \tilde{B}$ we have $U(\inf(\tilde{A}, \tilde{B})) = \inf(U(\tilde{A}), U(\tilde{B}))$ $U(\sup(\tilde{A}, \tilde{B})) = \sup(U(\tilde{A}), U(\tilde{B}))$. Observe then that $N_p(U(\tilde{f})) = N_p(\tilde{f})$ for $\tilde{f} \in E$ Show that the image of $L^p_{\mathbb{C}}(X, \mu) \cap L^\infty_{\mathbb{C}}(X, \mu)$ under U_p is $L^p_{\mathbb{C}}(Y, \nu) \cap L^\infty_{\mathbb{C}}(Y,$ that the restriction of U_p to $L^p_{\mathbb{C}}(X, \mu) \cap L^\infty_{\mathbb{C}}(X, \mu)$ is an isomorphism of algebras.

(d) Conversely, for a p such that $1 \leqq p < +\infty$, let V be a linear isome $L^p_{\mathbb{C}}(X, \mu)$ onto $L^p_{\mathbb{C}}(Y, \nu)$ such that the restriction of V to $L^p_{\mathbb{C}}(X, \mu) \cap L^\infty_{\mathbb{C}}(X, \mu$ algebra isomorphism onto $L^p_{\mathbb{C}}(Y, \nu) \cap L^\infty_{\mathbb{C}}(Y, \nu)$. Then the restriction of V to is an isometry onto $I(Y, \nu)$ which maps $\tilde{\phi}$ to $\tilde{\phi}$.

9. Let X be a compact space such that $\mu(X) = 1$. Let $\mathfrak{T}$ be a set of integrable of X, such that the relations $A \in \mathfrak{T}$ and $B \in \mathfrak{T}$ imply that $X - A \in \mathfrak{T}$ and $A \cap$ Suppose also that the set $\{\tilde{A} : A \in \mathfrak{T}\}$ is *dense* in the metric space $I(X, \mu)$ (Prob

(a) Let $(C_j)_{1 \leqslant j \leqslant n}$ be a finite partition of X into μ-integrable subsets. Sho for each $\varepsilon > 0$ there exists a partition $(A_j)_{1 \leqslant j \leqslant n}$ consisting of sets in $\mathfrak{T}$ su $\mu(D(C_j, A_j)) \leqq \varepsilon$ for $1 \leqq j \leqq n$. (Observe that if $\mu(D(C_j, B_j)) \leqq \delta$ for $1 \leqq j \leqq$ then $\mu(B_j \cap B_k) \leqq 2\delta$ for $1 \leqq j < k \leqq n - 1$; if N is the union of the $B_j \cap B_k$, c the sets $A_j = B_j - N$ $(1 \leqq j \leqq n - 1)$ and $A_n = X - \bigcup_{j=1}^{n-1} A_j$.

(b) Deduce from (a) that for each finite partition $\gamma = (C_j)_{1 \leqslant j \leqslant n}$ of X into int subsets, and each $\varepsilon > 0$, there exists a finite partition $\alpha = (A_j)_{1 \leqslant j \leqslant n}$ consis sets in $\mathfrak{T}$ and such that (in the notation of Section 13.9, Problem 27)

$$H(\alpha/\gamma) + H(\gamma/\alpha) \leqq \varepsilon.$$

(Reduce to the case where none of the C_j is negligible, and use the fact that the f $t \mapsto t \log t$ is zero and continuous at $t = 1$.)

10. Let X be a compact space such that $\mu(X) = 1$, and let $u : X \to X$ be a μ-mea mapping such that $u(\mu) = \mu$.

(a) Let α be a finite partition of X into integrable subsets, and let $\mathfrak{T}$ be the all finite unions of subsets of X belonging to one or other of the partitions $\bigvee_{j=0}^{n-1}$ Suppose that the set of classes $\{\tilde{A} : A \in \mathfrak{T}\}$ is dense in $I(X, \mu)$ (Problem 8). Sho we then have $h(u) = h(u, \alpha)$. (It is enough to show that $h(u, \beta) \leqq h(u, \alpha)$ fo finite partition β of X into integrable subsets; observe that we have

$$h(u, \beta) \leqq h\left(u, \bigvee_{j=0}^{n} u^{-j}(\alpha)\right) + H\left(\beta \Big/ \bigvee_{j=0}^{n} u^{-j}(\alpha)\right),$$

and use Problem 28(d) of Section 13.9, and Problem 9 above.)

(b) Under the hypotheses of (a), show that if in addition u is bijective and μ-measurable, then $h(u) = 0$. (Observe that the set of classes $\{(u^{-1}(A))^\sim : A$ again dense in $I(X, \mu)$, and use Problem 28(c) of Section 13.9.)

(c) Suppose that u is bijective and u^{-1} is μ-measurable. Let $\mathfrak{T}'$ be the set c unions of subsets of X belonging to one or other of the partitions $\bigvee_{j=-n}^{n} u^j(\alpha)$, and s

that the set of classes $\tilde{A}$, as A runs through $\mathfrak{T}'$, is dense in $I(X, \mu)$. Show that $h(u) = h(u, \alpha)$ (Kolmogoroff–Sinai theorem). (Same method as in (a).)

(d) Take X to be the unit circle $U : |z| = 1$, let μ be the image of Lebesgue measure under the mapping $t \mapsto e^{2\pi i t}$ of $[0, 1]$ onto U, and let u be the mapping $z \mapsto e^{2\pi i\theta} z$. Show that $h(u) = 0$. (Distinguish two cases according as θ is rational or irrational. In the latter case use (b), by taking α to be a partition into two half-open semicircles.)

11. Let X, Y be two compact spaces and let μ (resp. ν) be a positive measure on X (resp. Y) such that $\mu(X) = \nu(Y) = 1$. A μ-measurable (resp. ν-measurable) mapping $u : X \mapsto X$ (resp. $v : Y \mapsto Y$) such that $u(\mu) = \mu$ (resp. $v(\nu) = \nu$) defines an endomorphism $U : \tilde{f} \mapsto (f \circ u)^{\sim}$ of $L^1_C(X, \mu)$ (resp. an endomorphism $V : \tilde{g} \mapsto (g \circ v)^{\sim}$ of $L^1_C(Y, \nu)$) (Section 13.11, Problem 10). The mappings u, v are said to be *conjugate* if there exists an isometry T of $I(X, \mu)$ onto $I(Y, \nu)$ (Problem 8) such that $V \circ T = T \circ U$. Show that if this is the case then $h(u) = h(v)$.

12. Suppose that the measure μ is bounded, and let $p \in [1, +\infty[$. Let (U_n) be a sequence of continuous linear mappings of $L^p_{\mathbf{R}}(X)$ into the space $S(X, \mu)$ (Problem 2). For any function of $f \in \mathscr{L}^p_{\mathbf{R}}$ we denote by $U_n \cdot f$ any function belonging to the class $U_n \cdot \tilde{f}$. and put

$$(U^*_N \cdot f)(x) = \sup_{n \leqslant N} |(U_n \cdot f)(x)|,$$

$$(U^* \cdot f)(x) = \sup_n |(U_n \cdot f)(x)|.$$

For each $\alpha > 0$ let $E_{\alpha,N}(f)$ denote the set of all $x \in X$ such that $(U^*_N \cdot f)(x) > \alpha$, and let $E_\alpha(f)$ denote the set of all $x \in X$ such that $(U^* \cdot f)(x) > \alpha$.

(a) Show that the set of $\tilde{f} \in L^p_{\mathbf{R}}$ such that $\mu(E_{\alpha,N}(f)) \leqq \varepsilon$ is closed, for each $\varepsilon > 0$.

(b) Assume that for *all* $f \in \mathscr{L}^p_{\mathbf{R}}$ the function $U^* \cdot f$ is finite almost everywhere. Show that under these conditions the number

$$C(\alpha) = \sup_{N_p(f) \leqslant 1} \mu(E_\alpha(f))$$

tends to 0 as $\alpha \to +\infty$ (Banach's principle). (Use Baire's theorem in the complete space $L^p_{\mathbf{R}}$.)

(c) Under the hypotheses of (b), show that the set H of classes $\tilde{f}$ such that the sequence $((U_n \cdot f)(x))$ converges everywhere in X is closed in $L^p_{\mathbf{R}}$. (Write

$$R(f)(x) = \lim_{r \to \infty} \left(\sup_{m,n \geqslant r} |(U_m \cdot f)(x) - (U_n \cdot f)(x)| \right),$$

remark that $R(f)(x) = R(f - g)(x)$ almost everywhere for $\tilde{g} \in H$, and use (b).)

13. MEASURES WITH BASE μ

(13.13.1) *Let g be a mapping of X into* **C** *or into* $\overline{\mathbf{R}}$. *Then the following conditions are equivalent:*

(a) *For each $x \in X$, there exists a neighborhood V of x in X such that $g\varphi_V$ is integrable.*

(b) *The function g is measurable, and for each compact set $K \subset X$ we have*

$$\int^* |g|\varphi_K \, d\mu < +\infty.$$

(c) *For each function $h \in \mathscr{K}_C(X)$, the function gh is integrable.*

Since we may write $g = g_1 + ig_2$, where g_1 and g_2 are $\overline{\mathbf{R}}$-valued, and then $g_1 = g_1^+ - g_1^-$, $g_2 = g_2^+ - g_2^-$, we reduce straightaway ((13.9.6) and (13.10)) to the case where g is a mapping $\geqq 0$ of X into $\overline{\mathbf{R}}$. To show that (a) implies (b), cover K by a finite number of open sets V_j such that $g\varphi_{V_j}$ is integrable for each j. Then $\sup_j(g\varphi_{V_j})$ is integrable (13.7.4) and hence so is

$$g\varphi_K = \varphi_K \cdot \sup_j(g\varphi_{V_j})$$

(13.9.14). Hence (b) follows from (13.9.13). To show that (b) implies (c), observe first that g is almost everywhere finite (X being a denumerable union of compact sets), hence gh is measurable (13.9.8.1). Moreover, if $L = \mathrm{Supp}(h)$, we have $|gh| \leqq |g\varphi_L| \cdot \|h\|$, hence it follows from (b) and (13.9.13) that gh is integrable. Finally, to show that (c) implies (a), consider a compact neighborhood V of $x \in X$ and a continuous mapping $h : X \to [0, 1]$ which is equal to 1 on V and has compact support ((3.18.2) and (4.5.2)). By hypothesis gh is integrable, hence so is $g\varphi_V = (gh)\varphi_V$ (13.9.14).

When the equivalent conditions of (13.13.1) are satisfied, the function g is said to be *locally integrable* (with respect to μ) or *locally μ-integrable*. Clearly every integrable function is locally integrable. Every measurable function whose restriction to every compact subset of X is *bounded* almost everywhere (in particular, every function belonging to $\mathscr{L}^\infty_{\mathbf{R}}$ or $\mathscr{L}^\infty_{\mathbf{C}}$) is locally integrable. Every function belonging to $\mathscr{L}^2_{\mathbf{R}}$ or $\mathscr{L}^2_{\mathbf{C}}$ is locally integrable, by (13.11.7). Every function equivalent to a locally integrable function is locally integrable. We have remarked in the course of the proof of (13.13.1) that every locally integrable function is *finite almost everywhere*. The function on $\mathbf{R}$ which is equal to $1/|x|$ when $x \neq 0$ and is 0 when $x = 0$ is lower semi-continuous but *not* locally integrable with respect to Lebesgue measure.

For a complex-valued function g to be locally integrable, it is necessary and sufficient that $\mathscr{R}g$ and $\mathscr{I}g$ should be locally integrable. For a real-valued function g to be locally integrable, it is necessary and sufficient that g^+ and g^- should be locally integrable.

Let g be a locally μ-integrable function. Since $f \mapsto \int fg \, d\mu$ is defined on the whole of $\mathscr{K}_C(X)$, it is a *linear form* on this complex vector space. Moreover

this linear form is a (complex) *measure* on X, because if K is any compact subset of X, we have

$$\left| \int fg \, d\mu \right| \leqq \|f\| \cdot \int |g\varphi_K| \, d\mu$$

for all $f \in \mathscr{K}_C(X; K)$, by virtue of (13.10.3). The measure so defined is called *the measure with density g relative to* μ, and is denoted by $g \cdot \mu$; when g is continuous, this agrees with the definition (13.1.5). Measures of the form $g \cdot \mu$ are also called *measures with base* μ. It follows immediately from this definition that if g takes values in $\overline{\mathbf{R}}$ (resp. $\geqq 0$ almost everywhere), then the measure $g \cdot \mu$ is *real* (resp. *positive*). Furthermore, $g \cdot \mu$ does not change if we replace g by an equivalent function (with respect to μ), and therefore we may restrict ourselves to the case where g is everywhere *finite* and *universally measurable* (13.9.12).

If g_1 and g_2 are two locally integrable functions, then $g_1 + g_2$ and ag_1 are locally integrable (a being any complex scalar), and we have

(13.13.2) $$(g_1 + g_2) \cdot \mu = g_1 \cdot \mu + g_2 \cdot \mu, \qquad (ag_1) \cdot \mu = a(g_1 \cdot \mu).$$

For every complex-valued locally integrable function g, we have

(13.13.3) $$\overline{g \cdot \mu} = \bar{g} \cdot \mu, \qquad \mathscr{R}(g \cdot \mu) = (\mathscr{R}g) \cdot \mu, \qquad \mathscr{I}(g \cdot \mu) = (\mathscr{I}g) \cdot \mu.$$

(13.13.4) The set $\mathscr{L}^1_{\mathrm{loc},\mathbf{R}}(X, \mu)$ (resp. $\mathscr{L}^1_{\mathrm{loc},C}(X, \mu)$) of real-valued (resp. complex-valued) locally μ-integrable functions is a real (resp. complex) vector space, often denoted by $\mathscr{L}_{\mathrm{loc}}(X, \mu)$ or $\mathscr{L}_{\mathrm{loc}}(\mu)$ or $\mathscr{L}_{\mathrm{loc}}(X)$ or $\mathscr{L}_{\mathrm{loc}}$. For every compact subset K of X, the mapping $p_K : g \mapsto \int |g\varphi_K| \, d\mu$ is a *semi-norm* on this space. We shall always suppose $\mathscr{L}_{\mathrm{loc}}$ to be endowed with the topology defined by these seminorms. If (K_n) is an increasing sequence of compact subsets of X, such that $\bigcup_n K_n = X$ and $K_n \subset \mathring{K}_{n+1}$ (3.18.3), then it is immediately seen that the topology of $\mathscr{L}_{\mathrm{loc}}$ is defined by the seminorms p_{K_n}. In particular, if X is compact, then $\mathscr{L}^1_{\mathrm{loc},\mathbf{R}}(X, \mu)$ (resp. $\mathscr{L}_{\mathrm{loc},C}(X, \mu)$) is identical with $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$ (resp. $\mathscr{L}^1_{C}(X, \mu)$). The set of locally integrable real-valued functions g such that $p_K(g) = 0$ for all compact subsets K of X is the space $\mathscr{N}$ of *μ-negligible* functions. We define

$$\mathrm{L}^1_{\mathrm{loc},\mathbf{R}}(X, \mu) = \mathscr{L}^1_{\mathrm{loc},\mathbf{R}}(X, \mu)/\mathscr{N}$$

and $\mathrm{L}^1_{\mathrm{loc},C}(X, \mu)$ analogously. The seminorms $p_K(g)$ depend only on the class $\tilde{g}$ of g: if we put $p_K(\tilde{g}) = p_K(g)$, then we obtain seminorms defining the topologies of these spaces, which are therefore metrizable and locally convex.

(13.13.5) *If g is locally μ-integrable, then so is $|g|$, and we have $|g \cdot \mu| = |g| \cdot \mu$.*

The first assertion follows immediately from (13.13.1) and (13.7.4). To prove the second, note first that if $f \geqq 0$ is a function belonging to $\mathscr{K}_{\mathbf{R}}(X)$ and if $u \in \mathscr{K}_{\mathbf{C}}(X)$ is such that $|u| \leqq f$, then we have

$$\left| \int ug \, d\mu \right| \leqq \int |u| \cdot |g| \, d\mu \leqq \int f|g| \, d\mu$$

(13.10.3), and therefore (13.3.2.1) $|g \cdot \mu| \leqq |g| \cdot \mu$. To prove the reverse inequality, let L denote the (compact) support of f. Let A be the set of all $x \in L$ such that $g(x) \neq 0$; the set A is integrable ((13.9.9) and (13.9.2)) and therefore (13.9.1) there exists an increasing sequence of compact subsets K_n of A such that $A - \bigcup_n K_n$ is μ-negligible. Hence, by (13.8.4), for any $\varepsilon > 0$ there exists an integer n such that $\int_{A-K_n} f|g| \, d\mu \leqq \varepsilon$. The same reasoning, using the measurability of g and (13.8.4), shows that there exists a compact subset K_n' of K_n such that $g \mid K_n'$ is *continuous* and $\int_{A-K_n'} f|g| \, d\mu \leqq 2\varepsilon$. Now consider on K_n' the *continuous* function $x \mapsto |g(x)|/g(x)$; by applying (3.18.2) and the Tietze–Urysohn theorem (4.5.1) to the real and imaginary parts of this function, we see that there exists a function $w \in \mathscr{K}_{\mathbf{C}}(X)$ such that $w(x) = |g(x)|/g(x)$ for all $x \in K_n'$. The function $v = w \cdot \inf(1, 1/|w|)$ (with the convention $1/0 = +\infty$) is then *continuous* on X, hence belongs to $\mathscr{K}_{\mathbf{C}}(X)$, and is such that $v(x) = |g(x)|/g(x)$ in K_n' and $|v(x)| \leqq 1$ throughout X. Hence $|fv| \leqq f$ and

$$\int fvg \, d\mu = \int_{K_{n'}} fvg \, d\mu + \int_{A-K_{n'}} fvg \, d\mu;$$

but

$$\int_{K_{n'}} fvg \, d\mu = \int_{K_{n'}} f|g| \, d\mu,$$

and

$$\left| \int_{A-K_{n'}} fvg \, d\mu \right| \leqq \int_{A-K_{n'}} f|g| \, d\mu \leqq 2\varepsilon,$$

so that finally

$$\left| \int fvg \, d\mu \right| \geqq \int f|g| \, d\mu - 4\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof.

PROBLEMS

1. Show that the locally convex metrizable space $L^1_{loc,\mathbf{R}}(X, \mu)$ (resp. $L^1_{loc,\mathbf{C}}(X, \mu)$) is complete (in other words, is a Fréchet space).

2. Let $\mathscr{H}$ be a vector subspace of $\mathscr{L}^1_{loc,\mathbf{R}}$ containing the subspace $\mathscr{N}$ of μ-negligible functions. Suppose that the quotient space $H = \mathscr{H}/\mathscr{N}$ is endowed with the structure of a real *Hilbert* space, in which the scalar product is denoted by $(\tilde{f} \mid \tilde{g})$ and the norm $|\tilde{f}|$. Suppose moreover that, for every compact subset K of X, there exists a constant $a_K \geqq 0$ such that

(A) $$\int_K |u| \, d\mu \leqq a_K \, |u|$$

for all $u \in \mathscr{H}$ (cf. Section 15.11, Problem 26). (We shall also use the notation $(u \mid v)$ and $|u|$ for $(\tilde{u} \mid \tilde{v})$ and $|\tilde{u}|$ when u and v belong to $\mathscr{H}$.) Let $\mathscr{L}^\infty_{\mathbf{C}}$ denote the set of bounded measurable functions on X with compact support.

(a) Given any function $f \in \mathscr{L}^\infty_{\mathbf{C}}$, show that there exists a function $U^f \in \mathscr{H}$ such that $(U^f \mid u) = \int uf \, d\mu$ for all $u \in \mathscr{H}$; and that the class of U^f in H is uniquely determined by the class of f. The function U^f is called the *potential* of f. Show that the set of classes of potentials U^f is dense in the Hilbert space H (use (6.3.2)). If f, g are two elements of $\mathscr{L}^\infty_{\mathbf{C}}$, then $(U^f \mid U^g) = \int gU^f \, d\mu = \int fU^g \, d\mu$.

(b) As f runs through the functions $\geqq 0$ belonging to $\mathscr{L}^\infty_{\mathbf{C}}$, the set of potentials U^f is a convex cone in $\mathscr{H}$. We denote by $\mathscr{P}$ the *closure* of this cone (with respect to the topology defined by the seminorm $|\tilde{u}|$ on $\mathscr{H}$). The elements of $\mathscr{P}$ are called *pure potentials*. Let P denote the image of $\mathscr{P}$ in H. For each element $\tilde{u} \in H$ let $\tilde{v}$ be the projection of $\tilde{u}$ on P (Section 12.15, Problem 3(a)). Show that we have $|\tilde{v}|^2 = (\tilde{v} \mid \tilde{u})$, that $v(x) \geqq u(x)$ almost everywhere, and that $\tilde{v}$ is the only element of P satisfying these conditions. Also we have $|\tilde{v}| \leqq |\tilde{u}|$.

Deduce that, for each $\tilde{u} \in H$, the projection of 0 on the closed convex set of points $\tilde{v} \in H$ such that $v(x) \geqq u(x)$ almost everywhere belongs to P.

(c) Deduce from (b) that an element $v \in \mathscr{H}$ belongs to $\mathscr{P}$ if and only if $(v|w) \geqq 0$ for all $w \in \mathscr{H}$ such that $w(x) \geqq 0$ almost everywhere (consider the difference between $\tilde{v}$ and its projection on $\mathscr{P}$). Equivalently, $|\tilde{v} + \tilde{w}| \geqq |\tilde{v}|$ for all $w \in \mathscr{H}$ such that $w(x) \geqq 0$ almost everywhere.

(d) Suppose that for all $u \in \mathscr{H}$ we have $|u| \in \mathscr{H}$ and $|(|u|)^\sim| \leqq |\tilde{u}|$. Show that every pure potential u is $\geqq 0$ almost everywhere (use (c)). If u, v are two pure potentials, show that $\inf(u, v)$ is a pure potential. (Among the elements of $\mathscr{H}$ which majorize $\inf(u, v)$, consider an element w such that $|\tilde{w}|$ is a minimum; then w is a pure potential, by (b), and we have $(u + w \mid u - w) \leqq (u + w \mid |u - w|)$. By calculating $|(\inf(u, w))^\sim|^2$, deduce that $|(\inf(u, w))^\sim| \leqq |\tilde{w}|$; likewise that $|(\inf(v, w))^\sim| \leqq |\tilde{w}|$; and hence that $w = \inf(u, w) = \inf(v, w)$ almost everywhere.)

(e) With the same hypotheses as in (d), show that if $f \in \mathscr{L}^\infty_{\mathbf{C}}$ is $\geqq 0$ almost everywhere, and if $u \in \mathscr{P}$ is such that $U^f(x) \leqq u(x)$ almost everywhere *in the set of points* $x \in X$ *such that* $f(x) > 0$, then $U^f(x) \leqq u(x)$ almost everywhere *in* X ("principle of domination"). (Observe that $v = \inf(U^f, u)$ is a pure potential, that $(U^f \mid U^f - v) = 0$ and $(v \mid U^f - v) \geqq 0$, and deduce that $v = U^f$ almost everywhere.)

(f) Suppose that the hypotheses in (d) are satisfied and also that for all $u \in \mathscr{H}$ we have $\inf(u, 1) \in \mathscr{H}$ and $|(\inf(u, 1))^\sim| \leqq |\tilde{u}|$. Show then that if u is a pure potential,

then so is $\inf(u, 1)$ (same method as in (d)). Deduce that, if u and v are pure potentials, then $\inf(u, v+1)$ is a pure potential, by remarking that $v + \inf(u, 1)$ is a pure potential and using (d). Finally show that, if $f \in \mathscr{L}^{\infty}_{\mathscr{C}}$ is $\geqq 0$ almost everywhere and if $u \in \mathscr{P}$ is such that $U^f(x) \leqq u(x) + 1$ almost everywhere *in the set of points* $x \in X$ *such that* $f(x) > 0$, then $U^f(x) \leqq u(x) + 1$ almost everywhere *in* X ("complete maximum principle": same method as in (e)).

14. INTEGRATION WITH RESPECT TO A POSITIVE MEASURE WITH BASE μ

(13.14.1) *Let g be a locally μ-integrable function which is $\geqq 0$ on* X, *and let $\nu = g \cdot \mu$. Then if $f \geqq 0$ is any $\overline{\mathbf{R}}$-valued function on* X, *we have*

$$\int^* f\, d\nu = \int^* fg\, d\mu \tag{13.14.1.1}$$

where, on the right-hand side, the value of fg is by definition taken to be 0 *at every point $x \in X$ at which one of $f(x)$, $g(x)$ vanishes (even if the other factor is $+\infty$* (13.11)).

The proof consists of several steps.

(13.14.1.2) Suppose first of all that $f \in \mathscr{I}$. Then (12.7.8) there exists an increasing sequence (f_n) of functions belonging to $\mathscr{K}_{\mathbf{R}}(X)$ such that $f = \sup_n f_n$. In view of the convention about products, this implies that the sequence $(f_n g)$ is increasing and that fg is equivalent to $\sup_n(f_n g)$, because g is almost everywhere finite (with respect to μ). Moreover, the functions $f_n g$ are μ-integrable (13.13.1), and $\int f_n g\, d\mu = \int f_n\, d\nu$. Hence it follows from (13.5.7) that

$$\int^* f\, d\nu = \sup_n \int f_n\, d\nu = \sup_n \int f_n g\, d\mu = \int^* fg\, d\mu.$$

(13.14.1.3) *Every μ-negligible set* N *is also ν-negligible.*

Suppose first of all that N is *relatively compact*. Then ((3.18.2) and (13.7.9)) there exists a decreasing sequence of *relatively compact* open sets U_n containing N, such that $\inf_n \mu(U_n) = 0$. Since $g\varphi_{U_n}$ is μ-integrable by virtue of (13.13.1), we have $\int g\varphi_{U_n}\, d\mu = \nu(N_n)$ by (13.14.1.2) and

(13.7.7). But, by (13.8.4), if $N' \supset N$ is the intersection of the U_n, then $\inf_n \int g\varphi_{U_n}\, d\mu = \int g\varphi_{N'}\, d\mu = 0$; and since $\nu(N) \leqq \nu(U_n)$ for all n, we have $\nu(N) = 0$.

Now let N be any μ-negligible set, and (K_n) a denumerable covering of X by compact sets. Then the sets $N \cap K_n$ are ν-negligible, by what has just been proved, and hence so is their union N.

(13.14.1.4) Suppose now that $K = \mathrm{Supp}(f)$ is *compact* and that $f|K$ is continuous, with values in $\mathbf{R}$ (and therefore bounded (3.17.10)). Then there exists a decreasing sequence (U_n) of relatively compact neighborhoods of K such that $K = \bigcap_n U_n$ (3.18.2); also, by virtue of the Tietze–Urysohn theorem (4.5.1), there exists for each n a function $f_n \in \mathscr{K}_{\mathbf{R}}(X)$, with support contained in U_n, which extends f and is such that $\|f_n\| = \|f\|$. Hence we have $\int f_n\, d\nu = \int f_n g\, d\mu$ for all n. Bearing in mind (13.13.1), it follows from (13.8.4) that f is ν-integrable and fg is μ-integrable, and that $\int f\, d\nu = \int fg\, d\mu$.

(13.14.1.5) *The set* A *of points* $x \in X$ *such that* $g(x) = 0$ *is* ν*-negligible.*

Since g is μ-measurable (13.13.1), A is μ-measurable (13.9.9), and hence is the union of a sequence (K_n) of compact sets and a μ-negligible set N. By virtue of (13.14.1.4) applied to $f = \varphi_{K_n}$, we have $\nu(K_n) = \int g\varphi_{K_n}\, d\mu = 0$, and $\nu(N) = 0$ by (13.14.1.3); hence $\nu(A) = 0$.

(13.14.1.6) Consider now the case where $\mathrm{Supp}(f) = K$ is compact and $f|K$ is *lower semicontinuous* on K. Then it follows from (12.7.8) that there exists an increasing sequence of finite real-valued functions u_n which are continuous and $\geqslant 0$ on K and such that $f|K = \sup_n u_n$. Let f_n be the function which is equal to u_n on K and zero on $X - K$, so that $f = \sup_n f_n$. By virtue of (13.5.7) and (13.14.1.4), we have

$$\int^* f\, d\nu = \sup_n \int^* f_n\, d\nu = \sup_n \int^* f_n g\, d\mu = \int^* fg\, d\mu$$

since fg is equivalent (with respect to μ) to $\sup_n f_n g$, by virtue of the convention about products and the fact that g is finite almost everywhere with respect to μ.

End of the Proof

(13.14.1.7) For every function $v \in \mathscr{I}$ such that $f \leqq v$, we have $fg \leqq vg$, and $\int^* v\,dv = \int^* vg\,d\mu$ by (13.14.1.2), so that $\int^* fg\,d\mu \leqq \int^* vg\,d\mu = \int^* v\,dv$. Hence, by definition of the upper integral, we have $\int^* fg\,d\mu \leqq \int^* f\,dv$. Hence it remains to establish the opposite inequality

$$\int^* f\,dv \leqq \int^* fg\,d\mu. \tag{13.14.1.8}$$

Let $h \in \mathscr{I}$ be such that $h \geqq fg$. Then it is enough to show that

$$\int^* f\,dv \leqq \int^* h\,d\mu. \tag{13.14.1.9}$$

The set $X - A$ is the union of a denumerable increasing sequence of compact sets H_n and a μ-negligible set N, such that $g \mid H_n$ is continuous, finite and >0 for all n. We define a mapping u of X into $\overline{\mathbf{R}}$ as follows: $u = h/g$ in the union of the sets H_n, and $u(x) = +\infty$ in N and in A. In each of the sets H_n we have $ug = h$, and by virtue of (13.14.1.6)

$$\int_{H_n}^* u\,dv = \int_{H_n}^* ug\,d\mu = \int_{H_n}^* h\,d\mu.$$

But $\nu(N) = 0$ by virtue of (13.14.1.3), and $\nu(A) = 0$ by virtue of (13.14.1.5), hence, as $h \geqq 0$, (13.5.7)

$$\int^* u\,dv = \sup_n \int_{H_n}^* u\,dv = \sup_n \int_{H_n}^* h\,d\mu \leqq \int^* h\,d\mu.$$

Since $f \leqq u$, we obtain the required inequality (13.14.1.9). Q.E.D.

(13.14.2) *Let g be a locally μ-integrable function which is $\geqq 0$ on* X, *let* S *be the μ-measurable set of points $x \in X$ such that $g(x) > 0$, and let $\nu = g \cdot \mu$. Let f be a mapping of* X *into* $\overline{\mathbf{R}}$. *Then the following conditions (with the conventions of* (13.11) *for products) are equivalent:*

(a) *f is ν-measurable;*

(b) *$f\varphi_S$ is μ-measurable;*

(c) *fg is μ-measurable.*

We may suppose g to be finite. With the conventions we have made, we have $fg = (f\varphi_S)(g\varphi_S)$, and since the two factors on the right-hand side never

take the values 0 and $\pm\infty$ simultaneously, it follows from (13.9.8.1) that (b) implies (c). Conversely, it is immediately seen that the function g' which is equal to g^{-1} on S and vanishes on $X - S$ is μ-measurable: for there exists a partition of S (resp. $X - S$) into compact sets L_n (resp. M_n) and a μ-negligible set P (resp. Q) such that the restriction of g to each L_n is continuous, and it follows that the restrictions of g' to L_n and M_n are continuous. Furthermore, we have $(fg)g' = f\varphi_S$, hence by the same argument it follows that (c) implies (b). Hence it is enough to prove that (a) and (b) are equivalent.

Suppose first that $f\varphi_S$ is μ-measurable. The set $\complement S$ is ν-negligible, by virtue of (13.14.1.5); also the hypothesis implies the existence of a partition of S into a sequence of compact sets L_n and a μ-negligible set N, such that $f \mid L_n$ is continuous for each n. Also N is ν-negligible, by (13.14.1.3); hence f is ν-measurable.

Conversely, suppose that f is ν-measurable. Then there exists a partition of X consisting of a sequence of compact sets H_n and a ν-negligible set N, such that $f \mid H_n$ is continuous for all n. Also there exists a partition of S consisting of a μ-negligible set L and a sequence of compact sets K_n such that $g \mid K_n$ is continuous (and >0) for all n. For all n, we have $\inf_{x \in K_n} g(x) = a_n > 0$. Finally, there exists also a partition of $X - S$ into a μ-negligible set L′ and a sequence of compact sets K'_n. It is clear that the restriction of $f\varphi_S$ to each of the sets $H_n \cap K_m$ and K'_m is continuous, and it is therefore enough to prove that $N \cap S$ is μ-negligible. Now if $\mu^*(N \cap K_n) > 0$, then we should have $\int_{N \cap K_n} g\, d\mu \geqq a_n \mu^*(N \cap K_n) > 0$, which is absurd because

$$0 = \nu(N \cap S) = \int_{N \cap S} g\, d\mu$$

by virtue of (13.14.1). Hence each set $N \cap K_n$ is μ-negligible, therefore so is $N \cap S$, and hence finally $f\varphi_S$ is μ-measurable.

(13.14.3) *Let g be a locally μ-integrable function which is $\geqq 0$ on X, and let $\nu = g \cdot \mu$. Then a mapping $f: X \to \overline{\mathbf{R}}$ is ν-integrable if and only if (with the conventions of* (13.11)*) fg is μ-integrable; and in that case we have*

$$\int f\, d\nu = \int fg\, d\mu. \tag{13.14.3.1}$$

For f to be ν-integrable it is necessary and sufficient that f^+ and f^- should be (13.7.4); and since we have $(fg)^+ = f^+g$, $(fg)^- = f^-g$, with the conventions of 13.11 about products, it is enough to prove (13.14.3) when $f \geqq 0$. The first assertion is now a consequence of (13.14.1), (13.14.2), and (13.9.13); and the relation (13.14.3.1) is just (13.14.1.1).

(13.14.4) *With the hypotheses of* (13.14.3), *the measure* ν *is bounded if and only if* g *is* μ*-integrable*; *and* $\nu = 0$ *if and only if* g *is* μ*-negligible.*

(13.14.5) *Let* g_1, g_2 *be two mappings of* X *into* $\overline{\mathbf{R}}$, *such that* $g_1 \geqq 0$ *and locally* μ*-integrable. Then* g_2 *is locally* $(g_1 \cdot \mu)$*-integrable if and only if (with the product convention of* (13.11)) $g_2 g_1$ *is locally* μ*-integrable, and in that case we have*

$$g_2 \cdot (g_1 \cdot \mu) = (g_2 g_1) \cdot \mu. \tag{13.14.5.1}$$

By considering g_2^+ and g_2^-, we reduce immediately to the case where $g_2 \geqq 0$. To say that g_2 is locally $(g_1 \cdot \mu)$-integrable signifies (13.13.1) that, for every $f \in \mathscr{K}_{\mathbf{R}}(X)$, the function $g_2 f$ is $(g_1 \cdot \mu)$-integrable, or equivalently (13.14.3), that $g_2 g_1 f$ is μ-integrable; which in turn means that $g_2 g_1$ is locally μ-integrable. Also, if we put $\nu = g_1 \cdot \mu$ and $\lambda = g_2 \cdot (g_1 \cdot \mu)$, we have $\int f \, d\lambda = \int f g_2 \, d\nu = \int f g_2 g_1 \, d\mu$ by (13.14.3); hence the relation (13.14.5.1).

PROBLEMS

1. Let X be a locally compact space, μ a positive measure on X, and (f_n) a sequence of functions belonging to $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$. Write $\mu_n = f_n \cdot \mu$ and $\mu_n(A) = \int_A f_n \, d\mu$ for any μ-measurable set A.

(a) Show that if X is not compact it can happen that the sequence $(\tilde{f}_n)$ is unbounded in L^1, but that the sequence (μ_n) is vaguely bounded.

(b) Suppose that, for every subset A of X consisting of a single point, and for every open subset A of X such that the measure induced by μ on the frontier of A has finite support, the sequence $(\mu_n(A))$ is bounded. Show that the sequence $(\tilde{f}_n)$ is bounded in L^1. (Show first that every point $x_0 \in X$ has an open neighborhood U such that the sequence of numbers $|\mu_n|(U)$ is bounded. To do this, argue by contradiction, by showing that otherwise it would be possible to define a strictly increasing sequence of integers (n_k), a decreasing sequence (U_k) of open neighborhoods of x_0, and a sequence (W_k) of μ-quadrable (Section 13.9, Problem 7) open sets, with the following properties: $\bar{U}_k \subset U_{k-1}$, $\mu(U_k - \{x_0\}) \leqq 1/k$, $|\mu_{n_i}|(U_k - \{x_0\}) \leqq 1$ for $i < k$, $\bar{W}_k \subset U_k - \bar{U}_{k+1}$, and finally

$$|\mu_{n_k}(W_k)| > k + \sum_{i<k} |\mu_{n_k}(W_i)|.$$

Consider the union W of the W_k, and obtain a contradiction. Then show by a similar argument that there exists a compact subset K of X such that the sequence $(|\mu_n|(X - K))$ is bounded.)

(c) Deduce from (b) that if the sequence $(\mu_n(A))$ is bounded for every open set A in X, then the sequence $(\tilde{f}_n)$ is bounded in L^1.

(d) On the interval $[0, 1]$, take μ_n to be the measure defined by a mass n at the point

0 and a mass $-n$ at the point $1/n$. Then the sequence $(\mu_n(A))$ is bounded for every open set A which is $|\mu_n|$-quadrable for all n, but the sequence of norms $\|\mu_n\|$ is unbounded. Again, if we take μ_n to be the measure defined by a mass n at the point $1/n$ and a mass $-n$ at the point $1/(n+1)$, then the sequence $(\mu_n(A))$ is bounded for every interval $A \subset [0, 1]$ (and therefore also for every finite subset A of $[0, 1]$), but the sequence of norms $\|\mu_n\|$ is not bounded.

2. For each integer $n \geqq 1$, let E_n be an at most denumerable closed subset of $I =]0, 1[$, and let $\mathscr{I}_n$ be the family of component intervals of $I - E_n$. Suppose that the maximum length d_n of the intervals $J \in \mathscr{I}_n$ tends to 0 as $n \to +\infty$. Let λ be Lebesgue measure on I and let A be a λ-measurable subset of I. Suppose that there exists a number $k \in]0, 1[$ such that, for all n and all $J \in \mathscr{I}_n$, we have $\lambda(A \cap J) < k\lambda(J)$. Show that $\lambda(A) = 0$. (By using (13.7.9), show that $\lambda(A) \leqq \varepsilon + k(\lambda(A) + \varepsilon)$ for all $\varepsilon > 0$.)

3. For each integer $n > 0$, the *Farey series* of order n is the set F_n of all rational numbers which, when expressed in their lowest terms p/q, are such that $0 \leqq p \leqq q \leqq n$, and arranged in increasing order. The distance between two consecutive terms of F_n is therefore $\leqq 1/n$.

(a) Show that if two rational numbers $r = p/q$ and $r' = p'/q'$ are such that $qp' - pq' = \pm 1$, then for every pair of integers (p'', q'') there exist integers x, y such that $p'' = px + p'y$ and $q'' = qx + q'y$. The fraction p''/q'' belongs to the closed interval with endpoints r, r' if and only if x, y are of the same sign.

(b) Deduce from (a) that if $r = p/q$ and $r' = p'/q'$ are two rational numbers belonging to the interval $[0, 1]$, such that $q > 0$, $q' > 0$ and $qp' - pq' = \pm 1$, then r and r' are consecutive elements of the Farey series $F_{\sup(q, q')}$. Moreover the smallest of the integers m such that the *open* interval with endpoints r, r' contains a point of F_m is the integer $q + q'$, and this open interval contains only one point of $F_{q+q'}$, namely $(p + p')/(q + q')$.

(c) Conversely, if r and r' are consecutive terms of F_n, show that $qp' - pq' = \pm 1$ (induction on n).

4. For each $x \in I =]0, 1]$, put $\rho(x) = (1/x) - [1/x]$, where $[t]$ denotes the integral part of the real number t. If $\rho^n(x) = \rho(\rho^{n-1}(x))$ is defined and nonzero, put $q_{n+1}(x) = [1/\rho_n(x)]$ (with $q_1(x) = [1/x]$).

(a) For every n such that $\rho^n(x)$ is defined, show that

$$x = \frac{A_{n-1}(x)\rho^n(x) + A_n(x)}{B_{n-1}(x)\rho^n(x) + B_n(x)}$$

where $A_n(x)$ and $B_n(x)$ are integers $\geqq n - 1$. The fractions $A_{n-1}(x)/B_{n-1}(x)$ and $A_n(x)/B_n(x)$ are two consecutive terms of a Farey series (Problem 3) and x belongs to the closed interval with these as endpoints. Deduce that $\rho^n(x)$ is defined for all $n \geqq 1$ if and only if x is irrational.

The (finite or infinite) sequence of numbers $q_n(x)$ is called the *continued fraction expansion* of x, and the fractions $A_n(x)/B_n(x)$ are the *convergents* to x. The $A_n(x)$ and $B_n(x)$ are constant on the complement of the denumerable closed set E_n of points x such that $\rho_{n+1}(x)$ is not defined, and on each of the component intervals of $I - E_n$ the function ρ_n is monotone and varies from 0 to 1.

(b) Let λ be Lebesgue measure on I and let A be a λ-measurable subset of I consisting entirely of irrational numbers. Suppose that φ_A is λ-*invariant* with respect to ρ (Section

13.9, Problems 13 and 24). Show that $\lambda(A)$ is either 0 or 1. (Suppose that $d = \lambda(A) \in]0, 1[$. Show that, for each component interval J of $I - E_n$, we have

$$\lambda(A \cap J) \leqq (2d/(1+d))\lambda(J).$$

For this purpose use (a) above and Problem 1(c) of Section 13.21 to show that

$$\int_J \varphi_A(x)\,dx = \int_0^1 \varphi_A\left(\frac{pt+q}{rt+s}\right)\frac{dt}{(rt+s)^2} = \int_0^1 \varphi_A(t)\,\frac{dt}{(rt+s)^2} \leqq \int_0^d \frac{dt}{(rt+s)^2}$$

for suitably chosen integers p, q, r, s with $ps - qr = \pm 1$. Use Problem 2 to complete the proof.)

(c) Show that the measure $\mu = g \cdot \lambda$, where $g(x) = 1/(1+x)$, is *invariant with respect to* ρ. Deduce from (b) and the ergodic theorem (Section 13.9, Problems 13 and 24) that, for every λ-integrable function f on I and almost all irrational $x \in I$, we have

$$\lim_{n\to\infty} \frac{1}{n}\left(f(x) + f(\rho(x)) + \cdots + f(\rho^{n-1}(x))\right) = \frac{1}{\log 2}\int_0^1 \frac{f(t)}{1+t}\,dt.$$

(Gauss–Kuzmin formula).

(d) Deduce from (c) that, for each integer $p \geqq 1$, if $\nu_n(x, p)$ is the number of indices $k \leqq n$ such that $q_k(x) = p$ (x irrational), then

$$\lim_{n\to\infty} \frac{\nu_n(x,p)}{n} = \frac{1}{\log 2}\log\frac{(p+1)^2}{p(p+2)}$$

for almost all irrational $x \in I$.

(e) Deduce from (c) that

$$\lim_{n\to\infty} (q_1(x)q_2(x)\cdots q_n(x))^{1/n} = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2+2n}\right)^{\log n/\log 2}$$

for almost all irrational x.

15. THE LEBESGUE–NIKODYM THEOREM AND THE ORDER RELATION ON $M_R(X)$

(13.15.1) *Let μ and ν be two positive measures on a locally compact space* X, *such that $\nu \leqq \mu$. Then there exists a locally μ-integrable function g such that $\nu = g \cdot \mu$.*

We distinguish two cases.

(I) Suppose first that ν is *bounded*. Since the function 1 is then ν-integrable, it follows from (13.11.2.2) that for every $f \in \mathscr{K}_{\mathbf{R}}(X)$

(13.15.1.1) $$|\nu(f)|^2 \leqq \nu(1)\nu(f^2) \leqq \nu(1)\mu(f^2).$$

This shows first of all that if f is μ-negligible then $\nu(f) = 0$. Passing to the quotient by the subspace $\mathscr{N}$ of μ-negligible functions, the linear form ν on $\mathscr{K}_{\mathbf{R}}(X)$ therefore defines a linear form $\tilde{f} \mapsto \nu(\tilde{f})$ on the subspace $\tilde{\mathscr{K}}_{\mathbf{R}}(X)$ of $L^2_{\mathbf{R}}(X, \mu)$ which is the canonical image of $\mathscr{K}_{\mathbf{R}}(X)$ in $L^2_{\mathbf{R}}(X, \mu)$. The inequality (13.5.1.1) shows that this linear form is *continuous* (5.5.1) on the subspace $\tilde{\mathscr{K}}_{\mathbf{R}}(X)$; since the latter is *dense* in $L^2_{\mathbf{R}}(X, \mu)$ (13.11.6), the linear form $\tilde{f} \mapsto \nu(\tilde{f})$ extends by continuity to a linear form on the *Hilbert* space $L^2_{\mathbf{R}}(X, \mu)$ (5.5.4). Hence there exists a function $g \in \mathscr{L}^2_{\mathbf{R}}(X, \mu)$ such that $\nu(f) = \mu(gf)$ for all $f \in \mathscr{K}_{\mathbf{R}}(X)$ (6.3.2). Since g is locally μ-integrable (13.13), it follows that $\nu = g \cdot \mu$.

(II) *General case.* There exists a partition of X into a sequence of compact sets K_n and a μ-negligible set N (13.9.2). If we put $M = X - N = \bigcup_n K_n$, then φ_M is locally μ-integrable and $1 - \varphi_M$ is μ-negligible, hence (13.14.4) $\varphi_M \cdot \mu = \mu$. Put $\mu_n = \varphi_{K_n} \cdot \mu$ and $\nu_n = \varphi_{K_n} \cdot \nu$. Since φ_{K_n} is the lower envelope of a decreasing sequence of functions $\geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$, the relation $\nu \leqq \mu$ implies $\nu_n \leqq \mu_n$ for all n; furthermore, ν_n is bounded (13.14.4). Hence, by (I) above, there exists a locally μ_n-integrable function g_n such that $\nu_n = g_n \cdot \mu_n = (g_n \varphi_{K_n}) \cdot \mu = |g_n \varphi_{K_n}| \cdot \mu$ ((13.13.5) and (13.14.5)), and the function $g_n \varphi_{K_n}$ is locally μ-integrable. Let g denote the function which is equal to $|g_n \varphi_{K_n}|$ on each K_n and is zero on N. Then g is the sum of the series $\sum_{n=1}^{\infty} |g_n \varphi_{K_n}|$. If f is any function $\geqq 0$ in $\mathscr{K}_{\mathbf{R}}(X)$, and m is any integer, then

$$\int f\left(\sum_{n=1}^{m} |g_n \varphi_{K_n}|\right) d\mu = \sum_{n=1}^{m} \int f |g_n \varphi_{K_n}| \, d\mu = \sum_{n=1}^{m} \int f \, d\nu_n$$

$$= \sum_{n=1}^{m} \int f \varphi_{K_n} \, d\nu \leqq \int f \, d\nu,$$

and therefore ((13.8.4) and (13.13.1)) the function g is locally μ-integrable. Also, since $f = \sum_{n=1}^{\infty} f\varphi_{K_n}$ almost everywhere (with respect to μ), we have (13.8.4) $\int fg \, d\mu = \int f\varphi_M \, d\nu$. To show that $\nu = g \cdot \mu$, it remains to be proved that N is also ν-*negligible*. But by definition (13.5.5), for each $\varepsilon > 0$ there exists a function $h \in \mathscr{I}$ such that $\varphi_N \leqq h$ and $\mu(h) \leqq \varepsilon$. Since h is the upper envelope of an increasing sequence of functions $\geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$, the inequality $\nu \leqq \mu$ implies that $\nu^*(h) \leqq \mu(h) \leqq \varepsilon$, and the proof is complete.

The following lemma will be generalized later (13.15.8):

(13.15.2) *If $(\mu_j)_{1 \leqq j \leqq r}$ is any finite sequence of complex measures on* X, *there exists a positive measure λ such that each μ_j is a measure with base λ.*

By writing each μ_j as a linear combination of four positive measures (13.3.6), we may assume that the measures μ_j are all *positive*. Then we take $\lambda = \sum_{j=1}^{r} \mu_j$, and apply the result of (13.15.1) to each $\mu_j \leqq \lambda$.

(13.15.3) (i) *With respect to the order relation on* $\mathrm{M_R(X)}$, *any two real measures* μ, ν *have a least upper bound* $\sup(\mu, \nu)$ *and a greatest lower bound* $\inf(\mu, \nu)$. *For each real measure* μ *put* $\mu^+ = \sup(\mu, 0)$ *and* $\mu^- = \sup(-\mu, 0)$; *then we have*

$$\text{(13.15.3.1)} \quad \inf(\mu^+, \mu^-) = 0, \qquad \mu = \mu^+ - \mu^-, \qquad |\mu| = \mu^+ + \mu^- = \sup(\mu, -\mu)$$

and, for any two real measures μ, ν,

$$\text{(13.15.3.2)} \quad \begin{cases} \mu + \nu = \sup(\mu, \nu) + \inf(\mu, \nu), \\ \sup(\mu, \nu) = \frac{1}{2}(\mu + \nu + |\mu - \nu|), \\ \inf(\mu, \nu) = \frac{1}{2}(\mu + \nu - |\mu - \nu|). \end{cases}$$

(ii) *Let* λ *be a positive measure and let* g_1, g_2 *be real-valued locally* λ*-integrable functions on* X. *If* $\mu_1 = g_1 \cdot \lambda$ *and* $\mu_2 = g_2 \cdot \lambda$, *then*

$$\text{(13.15.3.3)} \quad \begin{cases} \sup(\mu_1, \mu_2) = \sup(g_1, g_2) \cdot \lambda, \\ \inf(\mu_1, \mu_2) = \inf(g_1, g_2) \cdot \lambda \end{cases}$$

and in particular, for any real-valued locally λ*-integrable function* g,

$$\text{(13.15.3.4)} \qquad (g \cdot \lambda)^+ = g^+ \cdot \lambda, \qquad (g \cdot \lambda)^- = g^- \cdot \lambda.$$

We have $g_1 \cdot \lambda \leqq g_2 \cdot \lambda$ *if and only if* $g_1(x) \leqq g_2(x)$ *almost everywhere with respect to* λ.

(iii) *Let* (g_n) *be an increasing sequence of locally* λ*-integrable real-valued functions. Then the increasing sequence of measures* $g_n \cdot \lambda$ *is bounded above in* $\mathrm{M_R(X)}$ *if and only if the function* $\sup_n g_n$ *is locally* λ*-integrable, and in that case we have*

$$\text{(13.15.3.5)} \qquad \sup_n (g_n \cdot \lambda) = \left(\sup_n g_n\right) \cdot \lambda.$$

To prove (i), we remark that the measures μ, ν can be written in the form $\mu = \mu_1 - \mu_2$, $\nu = \nu_1 - \nu_2$, where μ_1, μ_2, ν_1 and ν_2 are positive (13.3.6). Applying (13.15.2) to the four measures μ_1, μ_2, ν_1, ν_2 we see (by using (13.13.2)) that $\mu = u \cdot \lambda$ and $\nu = v \cdot \lambda$, where λ is a positive measure and u, v

are locally λ-integrable real-valued functions. Hence the assertions of (i) are consequences of those of (ii).

To prove (13.15.3.3), we reduce to the case where $\mu_2 = 0$; for if $\sup(\mu_1 - \mu_2, 0)$ exists and is equal to $(g_1 - g_2)^+ \cdot \lambda$, it follows immediately (13.3) that we shall have

$$\sup(\mu_1, \mu_2) = \mu_2 + \sup(\mu_1 - \mu_2, 0) = (g_2 + (g_1 - g_2)^+) \cdot \lambda = \sup(g_1, g_2) \cdot \lambda.$$

This reduces us to proving (13.15.3.4). For this, we shall begin by proving the last assertion of (ii), or equivalently that the relation $g \cdot \lambda \geqq 0$ implies $g(x) \geqq 0$ almost everywhere with respect to λ. Let N be the set of points $x \in X$ such that $g(x) < 0$. If we put $\nu = g^- \lambda$, it follows from (13.14.1.5) that $\nu(X - N) = 0$; so if we can show that $\nu(N) = 0$, it will follow that $\nu = 0$, and this will establish our assertion (13.14.4). Clearly it is enough to prove that $\nu(N \cap K) = 0$ for each compact subset K of X. If U is any relatively compact open set containing $N \cap K$, then we have $\int_U g^- \, d\lambda \leqq \int_U g^+ \, d\lambda$. For by hypothesis $\int (g^+ - g^-) f \, d\lambda \geqq 0$ for every function $f \geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$, and it is therefore sufficient to remark that $\int_U g^- \, d\lambda = \sup \int g^- f \, d\lambda$, $\int_U g^+ \, d\lambda = \sup \int g^+ f \, d\lambda$, the supremum being taken over all functions $f \in \mathscr{K}_{\mathbf{R}}(X)$ such that $0 \leqq f \leqq \varphi_U$ (13.5.1). Since $g^+(x) = 0$ for $x \in N \cap K$, we have

$$\int_{N \cap K} g^+ \, d\lambda = \inf_U \int_U g^+ \, d\lambda = 0;$$

consequently, $\int_{N \cap K} g^- \, d\lambda = \inf_U \int_U g^- \, d\lambda = 0$, which shows that $\nu(N \cap K) = 0$ (13.14.3).

Now let ρ be a measure such that $\rho \geqq 0$ and $\rho \geqq g \cdot \lambda$. Putting $\lambda + \rho = \sigma$, we have (13.15.1) $\rho = u \cdot \sigma$ and $\lambda = v \cdot \sigma$, where u and v are locally σ-integrable and $u \geqq 0$ and $v \geqq 0$ almost everywhere with respect to σ. It follows that $u \geqq gv$ almost everywhere with respect to σ, hence that $u \geqq (gv)^+ = g^+ v$ almost everywhere with respect to σ; but this implies that

$$\rho = u \cdot \sigma \geqq (g^+ v) \cdot \sigma = g^+ \cdot \lambda$$

by virtue of (13.14.5).

Finally, we have to prove (iii). If the sequence $(g_n \cdot \lambda)$ is bounded above in $M_{\mathbf{R}}(X)$, then for each function $f \geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$ we have

$$\sup_n \int f g_n \, d\lambda < +\infty;$$

consequently (13.8.1) the function $fg = \sup_n fg_n$ is λ-integrable, and

$$\int fg\, d\lambda = \sup_n \int fg_n\, d\lambda.$$

Hence g is locally λ-integrable (13.13.1) and $g \cdot \lambda = \sup_n (g_n \cdot \lambda)$ (13.4.4).

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(13.15.3.6) We may therefore say that, for a positive measure ρ, the mapping $\tilde{g} \mapsto g \cdot \rho$ is a *linear bijection* of the vector space $L^1_{loc,\mathbf{R}}(X, \rho)$ of equivalence classes (relative to ρ) of locally ρ-integrable real-valued functions onto the vector space of *real measures with base* ρ, and moreover that this bijection is *order-preserving*. This is the fundamental fact that we shall use repeatedly from now on until the end of Section 13.19.

We remark also that the mapping $\tilde{g} \mapsto g \cdot \rho$, considered as a mapping into $M(X)$, is continuous when $L^1_{loc,\mathbf{R}}(X, \rho)$ is endowed with the topology defined in (13.13.4) and $M(X)$ is endowed with the vague topology.

(13.15.4) *In the space* $M_{\mathbf{R}}(X)$, *every subset* H *which is bounded above has a least upper bound* ν, *and there exists an increasing sequence* (μ_n) *of elements of* H *such that* $\nu = \sup_n (\mu_n)$.

If $f \geqq 0$ belongs to $\mathscr{K}_{\mathbf{R}}(X)$, then the set of numbers $\mu(f)$, where $\mu \in H$, has a finite least upper bound by hypothesis, and we shall show that $\sup_{\mu \in H} \mu(f)$ is equal to $\nu(f)$. For this purpose consider an increasing sequence (K_n) of compact subsets of X which cover X and are such that $\mathscr{K}_{\mathbf{R}}(X)$ is the union of the $\mathscr{K}_{\mathbf{R}}(X; K_n)$ (3.18.3), and let $(g_{mn})_{m \geqq 1}$ be a dense sequence in the Banach space $\mathscr{K}_{\mathbf{R}}(X; K_n)$ ((7.4.4) and (3.10.9)). For each pair (m, n), there exists a sequence $(\mu_{mnp})_{p \geqq 1}$ of elements of H such that $\sup_p (\mu_{mnp}(g^+_{mn})) = \sup_{\mu \in H} \mu(g^+_{mn})$. Now let λ_r denote

$$\sup_{m \leqq r,\, n \leqq r,\, p \leqq r} (\mu_{mnp})$$

(which exists by (13.15.3)). The sequence (λ_r) is increasing and bounded above in $M_{\mathbf{R}}(X)$, hence has a least upper bound ν_0 which is also its limit in the vague topology (13.4.4). If $\mu \in H$, we have $\mu(g^+_{mn}) \leqq \nu_0(g^+_{mn})$ for each pair (m, n); hence, by continuity and because $|f^+ - g^+_{mn}| \leqq |f - g_{mn}|$, it follows that $\mu(f) \leqq \nu_0(f)$ for all $f \geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X; K_n)$ and for all n; hence $\mu \leqq \nu_0$.

On the other hand, if $\rho \geqq \mu$ for all $\mu \in H$, then in particular $\rho \geqq \nu_0$, by the definition of ν_0; hence ν_0 is the least upper bound ν, and we have

$$\nu(f) = \sup_{\mu \in H} \mu(f)$$

for all $f \geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(X)$.

(13.15.5) (Lebesgue–Nikodym Theorem) *Let μ, ν be two positive measures on* X. *Then the following conditions are equivalent*:

(a) *ν is a measure with base μ.*

(b) *Every μ-negligible set is ν-negligible.*

(b′) *Every compact μ-negligible set is ν-negligible.*

(c) *For every function $f \geqq 0$ which is both μ-integrable and ν-integrable, and for every real number $\varepsilon > 0$, there exists $\delta > 0$ such that the relations $0 \leqq h \leqq f$ and $\int^* h\, d\mu \leqq \delta$ imply $\int^* h\, d\nu \leqq \varepsilon$.*

(c′) *For every compact subset* K *of* X *and every real number $\varepsilon > 0$, there exists $\delta > 0$ such that the relations $A \subset K$ and $\mu^*(A) \leqq \delta$ imply $\nu^*(A) \leqq \varepsilon$* ("absolute continuity" of ν with respect to μ).

(d) $\nu = \sup_n (\inf(\nu, n\mu))$.

The fact that (a) implies (b) follows immediately from (13.14.1) and (13.6.3).

Clearly (b) implies (b′). Conversely, suppose that (b′) is satisfied, and let N be a μ-negligible set. Since X is a denumerable union of compact sets K_n, it is enough to show that each of the sets $N \cap K_n$ is ν-negligible, and we may therefore assume that N is *relatively compact*. Since N then has a compact neighborhood (3.18.2), it follows from (13.7.9) and (13.8.7, (i)) that we may restrict ourselves to the case where N is a denumerable intersection of relatively compact open sets; but in this case N is ν-integrable ((13.7.7) and (13.8.7, (i)), hence is the union of a sequence (H_n) of compact sets and a ν-negligible set P. Since by hypothesis $\nu(H_n) = 0$ for all n, it follows that N is ν-negligible.

To prove that (b) implies (d) and that (d) implies (a), we remark that we may express μ and ν in the form $\mu = u \cdot \lambda$ and $\nu = v \cdot \lambda$, where λ is a positive measure and u, v are finite, $\geqq 0$ and locally λ-integrable ((13.15.2) and (13.15.3)). If we put $\nu' = \sup_n(\inf(v, nu))$ and $\nu'' = \nu - \nu'$, then we have $\nu' \geqq 0$, $\nu'' \geqq 0$ and $\sup_n(\inf(\nu, n\mu)) = \nu' \cdot \lambda$ (13.15.3). Hence to show that (b) implies (d) it is enough to show that (b) implies that $\nu'' \cdot \lambda = 0$. Now the set A of points $x \in X$ at which $\nu''(x) > 0$ is contained in the set of points x at which

$u(x) = 0$, and therefore is μ-negligible (13.14.1), hence ν-negligible by hypothesis. On the other hand, $v'' = v\varphi_A$, hence ((13.14.1) and (13.6.3)) the relation $v(A) = 0$ is equivalent to v'' being λ-negligible, and therefore implies that $v'' \cdot \lambda = 0$ (13.14.4). To show that (d) implies (a), it is enough to remark that the condition (d) signifies that $v'' \cdot \lambda = 0$, and hence (13.14.4) the set A is λ-negligible. Hence if we put $g(x) = v(x)/u(x)$ at points where $u(x) > 0$, and $g(x) = 0$ elsewhere, we shall have $v(x) = g(x)u(x)$ at all $x \notin A$, hence almost everywhere with respect to λ. Consequently (13.14.5), g is locally μ-integrable, and we have $\nu = g \cdot \mu$.

It remains to establish the equivalence of (b), (c), and (c'). Clearly (c') is a consequence of (c), applied to $f = \varphi_K$. Also (c') implies that $\nu^*(A) = 0$ for every relatively compact μ-negligible set A; since every μ-negligible set is the union of relatively compact μ-negligible sets, this proves that (c') implies (b). To prove that (b) implies (c) we shall argue by contradiction. Suppose therefore that there exists a function $f_0 \geqq 0$ which is μ-integrable and ν-integrable, and a real number $a > 0$ such that, for each integer $n > 0$, there exists a function g_n for which $0 \leqq g_n \leqq f_0$, $\int^* g_n \, d\mu \leqq 2^{-n}$ and $\int^* g_n \, d\nu > a$. By virtue of (13.5.5), we may replace g_n by $\inf(f_0, g_n')$ for a suitably chosen function $g_n' \in \mathscr{I}$, without disturbing the above properties, hence (13.9.13) we may assume that g_n is μ-integrable and ν-integrable. Now let

$$h = \limsup_{n \to \infty} g_n = \inf_n h_n,$$

where

$$h_n = \sup_{p>0} g_{n+p} \leqq \sum_{p=1}^{\infty} g_{n+p}.$$

Since $g_n \leqq f_0$ for all n, it follows that h_n is μ-integrable and ν-integrable for all n (13.8.2), and we have

$$\int h_n \, d\mu \leqq \sum_{p=1}^{\infty} \int g_{n+p} \, d\mu \leqq 2^{-n}$$

(13.5.8). Hence $\int h \, d\mu = 0$ (13.8.1), and the hypothesis (b) therefore implies that h is ν-negligible (13.6.3). But we have $\int h \, d\nu = \lim_{n \to \infty} \int h_n \, d\nu \geqq a$ (13.8.1), which gives the required contradiction. Hence (b) implies (c), and the proof is complete.

(13.15.6) *Let μ and ν be two positive measures on* X. *Then the following conditions are equivalent*:

(a) *The negligible sets are the same for μ and for ν.*

(b) $\nu = g \cdot \mu$, *where* g *is locally* μ*-integrable and* $g(x) > 0$ *almost everywhere with respect to* μ.

If (a) is satisfied, it follows from (13.15.5) that $\nu = g \cdot \mu$ and $\mu = h \cdot \nu$, where g (resp. h) is positive and locally integrable with respect to μ (resp ν), Hence (13.4.5) the function hg is locally μ-integrable, and we have $\mu = (hg) \cdot \mu$, which implies (13.15.3) that hg is equivalent (relative to μ) to the function $\mathbf{1}$, so that $g(x) > 0$ and $h(x) = 1/g(x)$ almost everywhere with respect to μ. Conversely, suppose that $\nu = g \cdot \mu$, with $g(x) > 0$ almost everywhere with respect to μ. Since $(1/g(x))g(x)$ is equal to 1 almost everywhere with respect to μ, the function $1/g(x)$ is locally ν-integrable, and we have $\mu = (1/g) \cdot \nu$ (13.14.5), so that the condition (a) is satisfied.

If μ and ν satisfy the equivalent conditions of (13.15.6) they are said to be *equivalent* positive measures on X. Clearly we have here an equivalence relation on the set of positive measures on X. The notion of a *measurable function* is the same for two equivalent measures (13.9.4).

(13.15.7) *If* μ *is any positive measure on* X, *there exists a continuous function* h *such that* $h(x) > 0$ *for all* $x \in X$ *and such that the measure* $\nu = h \cdot \mu$ (*which is equivalent to* μ *by virtue of* (13.15.6)) *is bounded.*

Let (U_n) be an increasing sequence of relatively compact open sets in X with the properties of (3.18.3), and for each n let f_n be a continuous function on X with values in $[0, 1]$, such that $f_n(x) = 1$ for all $x \in U_n$ and $f_n(x) = 0$ for all $x \in X - U_{n+1}$ (4.5.2). Let (a_n) be a sequence of real numbers >0 such that $\sum_{n=1}^{\infty} a_n < +\infty$. Then the series $h = \sum_{n=1}^{\infty} a_n f_n$ is normally convergent in X (7.1), hence h is continuous on X (7.2.1), and $h(x) > 0$ for all $x \in X$. If $\nu = h \cdot \mu$, then $\nu^*(1) = \int^* h \, d\mu \leqq \sum_n a_n \int f_n \, d\mu$ ((13.14.1) and (13.5.8)). If for example we take

$$a_n = \begin{cases} 2^{-n}\left(\int f_n \, d\mu\right)^{-1} & \text{if } \int f_n \, d\mu > 1, \\ 2^{-n} & \text{otherwise,} \end{cases}$$

then we have $\sum_n a_n < +\infty$ and $\nu^*(1) < +\infty$. This completes the proof.

(13.15.8) *Let* (μ_n) *be any sequence of positive measures on* X. *Then there exists a bounded positive measure* ν *on* X *such that the relation* $\nu(N) = 0$ *is equivalent to* $\mu_n(N) = 0$ *for all* n (which implies (13.15.5) that each μ_n is a measure with base ν). *Furthermore, if* ν' *is another positive measure on* X *having this property, then* ν *and* ν' *are equivalent.*

The last assertion follows from the definition of equivalent measures. To prove the first assertion, we may, by virtue of (13.15.7), assume that the measures μ_n are *bounded*. By multiplying each μ_n by a suitable positive real number, we may also assume that $\mu_n(1) \leqq 2^{-n}$.

If $\nu_n = \sup(\mu_1, \mu_2, \ldots, \mu_n)$, then by virtue of (13.15.3.2) we have $\nu_n(f) \leqq \sum_{k=1}^{n} \mu_k(f) \leqq 1$ for every $f \in \mathscr{K}_{\mathbf{R}}(X)$ such that $0 \leqq f \leqq 1$, and every n. Hence (13.4.4) the increasing sequence (ν_n) converges vaguely to its least upper bound ν in $M_{\mathbf{R}}(X)$, and we have $\nu(f) \leqq 1$ for every $f \in \mathscr{K}_{\mathbf{R}}(X)$ such that $0 \leqq f \leqq 1$. Hence $\nu^*(1) \leqq 1$ and so ν is bounded. We may therefore write $\mu_n = g_n \cdot \nu$, where g_n is locally ν-integrable and $\geqq 0$; hence $\nu_n = \left(\sup_{1 \leqq k \leqq n} g_k\right) \cdot \nu$ (13.15.3.3) and consequently $\nu = \left(\sup_n g_n\right) \cdot \nu$ (13.15.3.5). It is now clear (13.15.5) that the relation $\nu(N) = 0$ implies that $\mu_n(N) = 0$ for all n; conversely, if $\mu_n(N) = 0$ for all n, then each of the functions $g_n \varphi_N$ is ν-negligible (13.14.1), hence so is $\left(\sup_n g_n\right)\varphi_N$ by (13.6.2), and so N is ν-negligible.

(13.15.9) *Let (μ_n) be an increasing sequence of positive measures having a least upper bound ν in $M_{\mathbf{R}}(X)$. For a function f to be ν-integrable it is necessary and sufficient that f should be μ_n-integrable for all n, and that*

$$\sup_n \int |f| \, d\mu_n < +\infty.$$

In that case we have

$$\int f \, d\nu = \lim_{n \to \infty} \int f \, d\mu_n .$$

We may write $\mu_n = g_n \cdot \nu$, where g_n is $\geqq 0$ and locally ν-integrable, and the sequence (g_n) is increasing. Moreover, we have $\nu = \left(\sup_n g_n\right) \cdot \nu$ by (13.15.3), hence $\sup_n g_n = 1$ almost everywhere with respect to ν. If f is ν-integrable, then so is fg_n for all n, by virtue of (13.9.6), the inequality $|fg_n| \leqq |f|$ (which is true almost everywhere with respect to ν), and (13.9.13). Hence by (13.14.3) f is μ_n-integrable for all n. Conversely, if f is μ_n-integrable for all n, then the functions fg_n are ν-integrable (13.14.3) and we have $f = \lim_{n \to \infty} fg_n$ almost everywhere with respect to ν, so that f is ν-measurable (13.9.11). Also we have $|f| = \sup_n |fg_n|$ almost everywhere with respect to ν. Hence the result follows from (13.9.13) and (13.8.1), and the last assertion of the proposition follows from the dominated convergence theorem (13.8.4).

16. APPLICATIONS: I. INTEGRATION WITH RESPECT TO A COMPLEX MEASURE

(13.16.1) *Let μ, ν be two positive measures on* X.

(i) *We have* $(\mu + \nu)^* = \mu^* + \nu^*$.

(ii) *For a function f to be $(\mu + \nu)$-integrable* (resp. $(\mu + \nu)$-*measurable*) *it is necessary and sufficient that f should be μ-integrable and ν-integrable* (resp. μ-*measurable and* ν-*measurable*), *and in that case we have*

$$\text{(13.16.1.1)} \qquad \int f\, d(\mu + \nu) = \int f\, d\mu + \int f\, d\nu.$$

We may write $\mu = g \cdot (\mu + \nu)$ and $\nu = h \cdot (\mu + \nu)$, where g and h are $\geqq 0$ and locally $(\mu + \nu)$-integrable ((13.15.1) and (13.15.3)). Since $\mu + \nu = (g + h) \cdot (\mu + \nu)$, it follows that $g(x) + h(x) = 1$ almost everywhere with respect to $\mu + \nu$. Put $\lambda = \mu + \nu$. To prove (i), we have to show that

$$\int^* f\, d\lambda = \int^* fg\, d\lambda + \int^* fh\, d\lambda$$

for all functions $f \geqq 0$. Clearly $\int^* f\, d\lambda \leqq \int^* fg\, d\lambda + \int^* fh\, d\lambda$. Also there is no loss of generality in assuming that $\int^* f\, d\lambda < +\infty$, in which case there exists a decreasing sequence (u_n) of λ-integrable functions belonging to $\mathscr{I}$, such that $f \leqq u_n$ for all n and $\int^* f\, d\lambda = \inf_n \int u_n\, d\lambda$. Since $\int u_n\, d\lambda = \int u_n g\, d\lambda + \int u_n h\, d\lambda$, we obtain the desired inequality by passing to the limit. Assertion (ii) then follows from (13.14.2) and (13.14.3).

Now let λ be a *complex* measure on X. A complex-valued function f on X is said to be λ-*integrable* if it is $|\lambda|$-integrable. If we put $\lambda_1 = \mathscr{R}\lambda$ and $\lambda_2 = \mathscr{I}\lambda$, it follows from (13.3.7) and (13.16.1) that f is λ-integrable if and only if f is λ_1-integrable and λ_2-integrable. Since by (13.15.3.1) we have $|\lambda_1| = \lambda_1^+ + \lambda_1^-$ and $|\lambda_2| = \lambda_2^+ + \lambda_2^-$, it follows that f is λ-integrable if and only if f is integrable with respect to each of the four positive measures λ_1^+, λ_1^-, λ_2^+, λ_2^-.

The *integral of f with respect to* λ is then defined to be the complex number

$$\text{(13.16.2)} \qquad \int f\, d\lambda = \int f_1\, d\lambda_1^+ - \int f\, d\lambda_1^- + i \int f\, d\lambda_2^+ - i \int f\, d\lambda_2^-.$$

The mapping $f \mapsto \int f\,d\lambda$ so defined is clearly a **C**-*linear form* on the complex vector space $\mathscr{L}^1_{\mathbf{C}}(X, |\lambda|)$ which extends the given **C**-linear form λ on $\mathscr{K}_{\mathbf{C}}(X)$.

(13.16.3) *If λ is any complex measure on* X, *we have $\lambda = h \cdot |\lambda|$, where h is a locally $|\lambda|$-integrable function such that $|h(x)| = 1$ almost everywhere with respect to $|\lambda|$.*

Let λ_1, λ_2 denote the real measures $\mathscr{R}\lambda$, $\mathscr{I}\lambda$, respectively. Then $|\lambda_1| \leqq |\lambda|$ and $|\lambda_2| \leqq |\lambda|$ **(13.3.7)**, so that λ_1 and λ_2 are measures with base $|\lambda|$, and therefore the same is true of λ. If $\lambda = h \cdot |\lambda|$, it follows that $|\lambda| = |h| \cdot |\lambda|$ **(13.13.5)**, which implies that $|h(x)| = 1$ almost everywhere with respect to $|\lambda|$ **(13.15.3)**.

It follows now from the definition **(13.16.2)** and from **(13.14.3)** that, for any λ-integrable function f, we have

$$\int f\,d\lambda = \int fh\,d|\lambda| \tag{13.16.4}$$

and hence, by virtue of **(13.16.3)** and **(13.10.3)**,

$$\left|\int f\,d\lambda\right| \leqq \int |f|\,d|\lambda|. \tag{13.16.5}$$

A function is said to be λ-*negligible* (resp. λ-*measurable*) if it is $|\lambda|$-negligible (resp. $|\lambda|$-measurable). Likewise for sets.

If λ and μ are two complex measures and if f is both λ-integrable and μ-integrable, then f is $(\lambda + \mu)$-integrable, by virtue of **(13.16.1)** and the inequality $|\lambda + \mu| \leqq |\lambda| + |\mu|$ **(13.3.8)**.

If λ is a complex measure, a function g is said to be *locally λ-integrable* if it is locally $|\lambda|$-integrable. In that case, for every function $f \in \mathscr{K}_{\mathbf{C}}(X)$, the function gf is λ-integrable, and as in **(13.13)** we see that $f \mapsto \int gf\,d\lambda$ is a measure, denoted by $g \cdot \lambda$. If $\lambda = h \cdot |\lambda|$ **(13.16.3)**, we have $\int fg\,d\lambda = \int fgh\,d|\lambda|$, or equivalently $g \cdot \lambda = (gh) \cdot |\lambda|$, so that the study of measures of the form $g \cdot \lambda$ is immediately reduced to the case where λ is a positive measure. In particular, it follows from **(13.13.4)** that

$$|g \cdot \lambda| = |g| \cdot |\lambda|. \tag{13.16.6}$$

17. APPLICATIONS: II. DUAL OF L^1

(13.17.1) *Let μ be a positive measure on X. For each function $g \in \mathscr{L}^\infty_{\mathbf{R}}(X, \mu)$ (resp. $g \in \mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$), the mapping $\theta_g : f \mapsto \int fg\, d\mu$ is a continuous linear form on the space $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$ (resp. $\mathscr{L}^1_{\mathbf{C}}(X, \mu)$) with respect to the topology defined by the norm N_1. The corresponding continuous linear form $\tilde{\theta}_g : \tilde{f} \mapsto \mu(\tilde{f}\tilde{g})$ on the Banach space $L^1_{\mathbf{R}}(X, \mu)$ (resp. $L^1_{\mathbf{C}}(X, \mu)$) has norm (5.7.1) equal to $N_\infty(g)$. Conversely, every continuous linear form on $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$ (resp. $\mathscr{L}^1_{\mathbf{C}}(X, \mu)$) is of the form $f \mapsto \int fg\, d\mu$, where g is a function belonging to $\mathscr{L}^\infty_{\mathbf{R}}(X, \mu)$ (resp. $\mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$), and the equivalence class of g is uniquely determined.*

We shall give the proof for $\mathscr{L}^1_{\mathbf{C}}$. The fact that θ_g is a continuous linear form on $\mathscr{L}^1_{\mathbf{C}}$, and the inequality $\|\tilde{\theta}_g\| \leqq N_\infty(g)$, are consequences of (13.12.5). To prove that $\|\tilde{\theta}_g\| = N_\infty(g)$, we may assume that g is not μ-negligible (in other words, that $N_\infty(g) > 0$), since the assertion is trivial otherwise.

Let a be any real number such that $0 < a < N_\infty(g)$. Then there exists a compact subset K of X, with measure $\mu(K) > 0$, such that $g \mid K$ is continuous and $|g(x)| > a$ for all $x \in K$ ((13.9.9) and (13.9.4)). For each $\varepsilon > 0$ there exists a finite covering (U_i) of K by open sets in K, such that the oscillation of g in each U_i is $\leqq \varepsilon$ (3.16.5). Hence there exists a partition of K into a finite number of integrable sets A_k such that the oscillation of g in each A_k is $\leqq \varepsilon$ (13.9.12.1). At least one of these sets, which we shall denote by A, has measure $\mu(A) > 0$. Let b be one of the values taken by g in A. We have $|b| > a$ and $|g(x) - b| \leqq \varepsilon$ for all $x \in A$. Now put

$$f = (\bar{b}/(|b|\mu(A)))\varphi_A;$$

clearly $f \in \mathscr{L}^1_{\mathbf{C}}$ and $N_1(f) = 1$. Also we have

$$\left| \int \frac{\bar{b}}{|b|} g\varphi_A\, d\mu \right| = \left| |b|\mu(A) + \int \frac{\bar{b}}{|b|}(g - b)\varphi_A\, d\mu \right|$$

$$\geqq (|b| - \varepsilon)\mu(A) \geqq (a - \varepsilon)\mu(A),$$

so that $\left|\int fg\, d\mu\right| \geqq a - \varepsilon$. This proves our assertion, because $\varepsilon > 0$ and $a < N_\infty(g)$ were arbitrary.

Conversely, let u be a continuous linear form on $\mathscr{L}^1_{\mathbf{C}}$. Then there exists a real number $c > 0$ such that $|u(f)| \leqq c \cdot N_1(f)$ for all $f \in \mathscr{L}^1_{\mathbf{C}}$ (12.14.1). If K is any compact subset of X, it is known (13.11.6) that the topology induced on $\mathscr{K}_{\mathbf{C}}(X; K)$ by that of $\mathscr{L}^1_{\mathbf{C}}(X, \mu)$ is coarser than the topology defined by the norm $\|f\|$. Hence the linear form u is continuous on each of the Banach

spaces $\mathscr{K}_{\mathbf{C}}(X; K)$ (with respect to the norm $\|f\|$); in other words, the restriction of μ to $\mathscr{K}_{\mathbf{C}}(X)$ is a (complex) *measure* ν. Furthermore, it follows from the inequality $|u(f)| \leqq c \cdot N_1(f)$, where $f \in \mathscr{K}_{\mathbf{C}}(X)$, and from (13.3.2.1) that $|\nu|(|f|) \leqq c \cdot N_1(f) = c \cdot \mu(|f|)$. Hence we deduce from (13.15.1) and (13.15.3) that there exists a locally μ-integrable function g_0 such that $|g_0(x)| \leqq c$ almost everywhere with respect to μ, and such that $|\nu| = g_0 \cdot \mu$. If $\nu = h \cdot |\nu|$ (13.16.3) we have therefore $\nu = g \cdot \mu$, where $g = g_0 h$ (13.14.5), and it is clear that $g \in \mathscr{L}^{\infty}_{\mathbf{C}}(\mu)$. This being so, the linear forms u and θ_g agree on the subspace $\mathscr{K}_{\mathbf{C}}(X)$, which is dense in $\mathscr{L}^1_{\mathbf{C}}$ (13.11.6); since they are both continuous on $\mathscr{L}^1_{\mathbf{C}}$, it follows that they are equal (3.15.2). Finally, the fact that $\theta_{g_1} = \theta_{g_2}$ implies $\tilde{g}_1 = \tilde{g}_2$ follows from (13.15.3). Q.E.D.

It follows that the mapping $\tilde{g} \mapsto \tilde{\theta}_g$ is a linear *isometry* of the Banach space $L^{\infty}_{\mathbf{C}}$ onto the *dual* of the Banach space $L^1_{\mathbf{C}}$ (12.15).

We remark also that it follows from (13.12.5) that, for each function $f \in \mathscr{L}_{\mathbf{C}}(X, \mu)$, the linear form $g \mapsto \int fg \, d\mu$ on $\mathscr{L}^{\infty}_{\mathbf{C}}(X, \mu)$ is continuous with respect to the topology defined by the seminorm N_{∞}. But in general there exist linear forms on $\mathscr{L}^{\infty}_{\mathbf{C}}(X, \mu)$ which are continuous with respect to this topology but *are not* of the above type.

PROBLEMS

1. (a) Let X be a locally compact space, μ a positive measure on X. Let f be a μ-measurable mapping of X into $\bar{\mathbf{R}}$ and p a real number such that $1 \leqq p \leqq +\infty$. Show that $N_p(f) = \sup \int^* |fg| \, d\mu$, where g runs through the set of functions belonging to $\mathscr{K}_{\mathbf{R}}(X)$ such that $N_q(g) \leqq 1$, where $q = p/(p-1)$ (and $q = +\infty$ if $p = 1$). State and prove an analogous result for a μ-measurable mapping of X into $\mathbf{C}$.
(b) Suppose that $1 < p < +\infty$. For each function $g \in \mathscr{L}^q_{\mathbf{R}}(X, \mu)$ (resp. $g \in \mathscr{L}^q_{\mathbf{C}}(X, \mu)$), let $\theta_{\tilde{g}}$ denote the linear form $\tilde{f} \mapsto \mu(f\tilde{g})$ on $L^p_{\mathbf{R}}(X, \mu)$ (resp. $L^p_{\mathbf{C}}(X, \mu)$). Show that $\tilde{g} \mapsto \theta_{\tilde{g}}$ is an isometry of $L^q_{\mathbf{R}}(X, \mu)$ (resp. $L^q_{\mathbf{C}}(X, \mu)$) onto the Banach space (5.7.3) dual to the Banach space $L^p_{\mathbf{R}}(X, \mu)$ (resp. $L^p_{\mathbf{C}}(X, \mu)$). (Imitate the proof of (13.17.1), using (a).)

2. Let X be a *compact* space, μ a positive measure on X, and (f_n) an orthonormal sequence in $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$. Write

$$K_n(s, t) = \sum_{k=1}^{n} f_k(s)\overline{f_k(t)}, \qquad H_n(s) = \int_X |K_n(s, t)| \, d\mu(t)$$

(the "nth Lebesgue function" of the system (f_n), cf. Section 11.6, Problem 2). For any $g \in \mathscr{L}^2_{\mathbf{C}}(X, \mu)$, let $s_n(g)$ denote the function

$$x \mapsto \sum_{k=1}^{n} (g|f_k) f_k(x) = \int K_n(x, y) g(y) \, d\mu(y).$$

(a) Let $x_0 \in X$. Show that, for every complex-valued continuous function g on X the partial sums $s_n(g)$ are bounded (by a number depending on g and on x_0) if and only if the set of numbers $H_n(x_0)$ is bounded (use Problem 1(a) and the Banach–Steinhaus theorem). (This result generalizes Problem 2(a) of Section **11.6.**)

(b) For *every* complex-valued continuous function g on X the partial sums $s_n(g)$ converge *uniformly* to g if and only if the following two conditions are satisfied: (1) every complex-valued continuous function on X can be approximated uniformly on X by linear combinations of the f_n; (2) there exists a constant a such that $|H_n(x)| \leqq a$ for all $x \in X$ and all n. (To prove that (2) is necessary, observe that if $s_n(g)$ converges uniformly to g, then for any increasing sequence of integers (n_k) and any sequence (x_k) of points of X, the sequence of numbers $\int K_{n_k}(x_k, y)g(y)\,d\mu(y)$ is bounded (by a number depending on g); now use the Banach–Steinhaus theorem.)

In particular, for the Haar orthonormal system (Section **8.7**, Problem 7), the Lebesgue functions are uniformly bounded.

3. Let X be a locally compact space, μ a positive measure on X, and p a real number satisfying $1 \leqq p < +\infty$; let $q = p/(p-1)$. Let g be a complex-valued measurable function such that fg is integrable for all $f \in \mathscr{L}^p_{\mathbf{C}}(X, \mu)$. Show that $g \in \mathscr{L}^q_{\mathbf{C}}(X, \mu)$. (Show that the mapping $\tilde{f} \mapsto \tilde{fg}$ of L^p into L^1 is continuous, by using the closed graph theorem (**12.16.11**).) (Cf. Section **12.16**, Problem 24.)

4. Let U be a continuous endomorphism of $L^1_{\mathbf{C}}(X, \mu)$ which satisfies the conditions of Section **13.11**, Problem 16. Its transpose ${}^tU$ is therefore a continuous endomorphism of the Banach $L^\infty_{\mathbf{C}}(X, \mu)$ with norm $\leqq 1$ (Section **12.15**, Problem 4) and such that the relation $\tilde{f} \geqq 0$ implies ${}^tU \cdot \tilde{f} \geqq 0$. For any $f \in \mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$, we denote by ${}^tU \cdot f$ any function in the class ${}^tU \cdot \tilde{f}$.

(a) Let (f_n) be a sequence of functions in $\mathscr{L}^\infty_{\mathbf{C}}$ which converges almost everywhere to f and is such that the sequence of norms $N_\infty(f_n)$ is bounded. Show that the sequence $({}^tU \cdot f_n)$ converges almost everywhere to ${}^tU \cdot f$. (Consider the sequence of functions $g_n = \sup_p |f_{n+p} - f_n|$, and use the relation $\int ({}^tU \cdot g_n)h\,d\mu = \int g_n(U \cdot h)\,d\mu$, which is valid for any function $h \in \mathscr{L}^1_{\mathbf{C}}$.)

(b) Let $\Psi \geqq 0$ be a function in $\mathscr{L}^\infty_{\mathbf{C}}$. With the notation of Section **13.11**, Problem 16, show that if ${}^tU \cdot \Psi \leqq \Psi$ almost everywhere, then for each function $f \in \mathscr{L}^1_{\mathbf{R}}$, we have $\int_{E(f)} f\Psi\,d\mu \geqq 0$. (Consider the measure $\Psi \cdot \mu$ and apply Section **13.11**, Problem 16(d).)

(c) With the notation of Section **13.11**, Problem 18, show that ${}^tU \cdot \varphi_{x_0} \leqq \varphi_{x_0}$ almost everywhere. Deduce that, for each function $f \in \mathscr{L}^1_{\mathbf{R}}$ which vanishes almost everwhere in X_0, the function $U \cdot f$ vanishes almost everywhere in X_0.

5. With the notation of Section **13.11**, Problem 18, suppose that the set X_0 is *negligible*. (This will always be the case if μ is bounded and $U \cdot 1 = 1$.)

(a) Show that if $f \in \mathscr{L}^\infty_{\mathbf{R}}$ is such that ${}^tU \cdot f \leqq f$ almost everywhere, then ${}^tU \cdot f = f$ almost everywhere. (Majorize the integral of the function

$$(f - {}^tU \cdot f)(\Phi + U \cdot \Phi + \cdots + U^n \cdot \Phi).)$$

(b) Let $\mathscr{J}$ be the set of all functions $h \in \mathscr{L}^\infty_{\mathbf{R}}$ such that ${}^tU \cdot h = h$ almost everywhere (i.e., the set of functions whose classes $\tilde{h}$ are fixed by ${}^tU$). Show that the constant functions belong to $\mathscr{J}$ and that if (h_n) is any sequence in $\mathscr{J}$ such that the sequence $(N_\infty(h_n))$ is bounded, then $\sup_n h_n \in \mathscr{J}$. (Use (a) above, and Problem 4(a).) Deduce

that for h to belong to $\mathscr{J}$, it is necessary and sufficient that for each $\alpha \in \mathbf{R}$ the characteristic function of the set $h^{-1}(]\alpha, +\infty[)$ should belong to $\mathscr{J}$.

(c) Show that if $h \in \mathscr{J}$ and $f \in \mathscr{L}^1_{\bar{\mathbf{R}}}$, then $U \cdot (hf) = h(U \cdot f)$ almost everywhere. (Reduce to the case where $h = \varphi_A$, and show that the integral of $(1 - \varphi_A)(U \cdot (\varphi_A f))$ is 0.)

(d) Show that the set $\mathscr{V}$ of functions of the form $h\Phi + g - U \cdot g$, where $h \in \mathscr{J}$ and $g \in \mathscr{L}^1_{\bar{\mathbf{R}}}$, is dense in $\mathscr{L}^1_{\bar{\mathbf{R}}}$. (Use Section 12.15, Problem 4(f); show that if $u \in \mathscr{L}^\infty_{\bar{\mathbf{R}}}$ is such that $\int u(g - U \cdot g)\, d\mu = 0$ for all $g \in \mathscr{L}^1_{\bar{\mathbf{R}}}$, then u necessarily belongs to $\mathscr{J}$, and deduce that $\int u^2 \Phi\, d\mu = 0$.)

(e) If $f = h\Phi + g - U \cdot g \in \mathscr{V}$, then the sequence $(R_n(f, \Phi))$ converges to h almost everywhere, and we have $N_1(h\Phi) \leqq N_1(f)$. (For the latter assertion, consider the function $u = \operatorname{sgn} h$ and evaluate $\int uf\, d\mu$, bearing in mind that $u \in \mathscr{J}$.) Deduce that there exists a continuous endomorphism R_Φ of $L^1_{\bar{\mathbf{R}}}$, of norm $\leqq 1$, whose image is the closure in $L^1_{\bar{\mathbf{R}}}$ of the vector subspace formed by the classes $\tilde{h}\tilde{\Phi}$ where $h \in \mathscr{J}$, and such that $R_\Phi \cdot \tilde{f}$ is equal to $\tilde{h}\tilde{\Phi}$ if $f = h\Phi + g - U \cdot g \in \mathscr{V}$.

(f) For each function $f \in \mathscr{L}^1_{\bar{\mathbf{R}}}$, we denote by $R_\Phi \cdot f$ any function in the class $R_\Phi \cdot \tilde{f}$. Deduce from (e) that the sequence $(R_n(f, \Phi))$ converges to $\Phi^{-1}(R_\Phi \cdot f)$ almost everywhere. Furthermore, for each $\alpha \in \mathbf{R}$, the set of points $x \in X$ such that $(R_\Phi \cdot f)(x) \geqq \alpha\Phi(x)$ is such that its characteristic function belongs to $\mathscr{J}$, and for each function $h \in \mathscr{J}$ we have $\int h(R_\Phi \cdot f)\, d\mu = \int hf\, d\mu$. (If $f_1 \in \mathscr{V}$ is such that $N_1(f - f_1) \leqq \varepsilon$, show that the set of $x \in X$ such that $R^*(f - f_1, \Phi)(x) \geqq \varepsilon^{1/2}$, or $R_\Phi(f - f_1)(x) \geqq \varepsilon^{1/2}\Phi(x)$, has measure $\leqq 2\varepsilon^{1/2}$ with respect to $\Phi \cdot \mu$, by using Section 13.11, Problem 18(c).)

(g) Deduce that for any two functions $f, g \in \mathscr{L}^1_{\bar{\mathbf{R}}}$ such that $g \geqq 0$, the sequence $(R_n(f, g))$ converges almost everywhere to a finite limit in the set B of points $x \in X$ such that $(U^n \cdot g)(x) > 0$ for at least one value of the integer $n \geqq 0$ (Chacón–Ornstein theorem).

6. (a) Let $\sigma_1 < \sigma_2$ be two real numbers, let B be the vertical strip $\sigma_1 \leqq \mathscr{R}s \leqq \sigma_2$ in $\mathbf{C}$, and let f be a function holomorphic in $\mathring{B}$ and continuous in B. Suppose also that

(i) $|f(s)| \leqq M$ for $s = \sigma_1 + it$ and $s = \sigma_2 + it$, where $t \in \mathbf{R}$ is arbitrary;

(ii) there exist two constants $a > 0$, $A > 0$ such that $|f(\sigma + it)| \leqq A \cdot e^{a|t|}$ for all $\sigma + it \in B$.

Show that under these assumptions we have $|f(s)| \leqq M$ throughout B. (Use the Phragmén–Lindelöf principle as in Section 9.5, Problem 17, by taking $g(s) = e^{s^2}$.)

(b) If f is continuous and bounded in B and holomorphic in $\mathring{B}$, and if we put $L(\sigma) = \sup_{t \in \mathbf{R}} |f(\sigma + it)|$ for all $\sigma \in [\sigma_1, \sigma_2]$, then we have

$$(*) \qquad L(\sigma)^{\sigma_2 - \sigma_1} \leqq L(\sigma_1)^{\sigma_2 - \sigma} L(\sigma_2)^{\sigma - \sigma_1}$$

(three-line theorem). (Proceed as in Section 9.5, Problem 10, by considering the function $s \mapsto e^{\alpha s} f(s)$ for a suitable $\alpha \in \mathbf{R}$. Hadamard's three-circle theorem is a particular case of $(*)$.)

7. Let X, Y be two locally compact spaces and let μ (resp. ν) be a positive measure on X (resp. Y). Let $E \subset L^1_{\mathbf{C}}(X, \mu)$ be the space of classes of μ-integrable step functions, and let U be a linear mapping of E into the space $L^1_{\mathrm{loc},\mathbf{C}}(Y, \nu)$ (Section 13.13). For each pair of numbers p, q (finite or not) belonging to the interval $[1, +\infty]$ of $\bar{\mathbf{R}}$, the

mapping U is said to be of *type* (p, q) if $U(E) \subset L^q_c(Y, \nu)$ and if U is continuous in the topology induced on E by that of $L^p_c(X, \mu)$, U being considered as a mapping of E into $L^q_c(Y, \nu)$. Let $\|U\|_{p,q}$ denote the norm of this linear mapping. If $p \neq +\infty$, then U may be extended by continuity to a linear mapping, with norm $\|U\|_{p,q}$, of $L^p_c(X, \mu)$ into $L^q_c(Y, \nu)$. The same is true if $p = +\infty$, provided that μ is bounded.

Show that if U is of type (p_0, q_0) and also of type (p_1, q_1), then U is also of type (p, q) where

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for all $t \in [0, 1]$ (with the convention that $1/\infty = 0$, $1/0 = +\infty$); moreover, we have

$$\|U\|_{p,q} \leqq \|U\|^{1-t}_{p_0,q_0} \cdot \|U\|^{t}_{p_1,q_1}$$

(M. Riesz–Thorin interpolation theorem). (Use Problem 1, by majorizing the integral $\int |(U \cdot f)g| \, d\nu$ for $f \in E$ and $g \in L^{q'}_c(Y, \nu)$, where $\frac{1}{q} + \frac{1}{q'} = 1$; for this purpose, put $f = |f|u$, $g = |g|v$ with $|u| = |v| = 1$, and $f_\zeta = |f|^\zeta u$, $g_\zeta = |g|^\zeta v$ for all complex numbers ζ. Then choose appropriately two affine-linear functions

$$\zeta \to a\zeta + b, \qquad \zeta \to c\zeta + d,$$

and apply the three-line theorem (Problem 6) to the holomorphic function

$$\int (U \cdot f_{a\zeta+b}) g_{c\zeta+d} \, d\mu.)$$

18. CANONICAL DECOMPOSITIONS OF A MEASURE

Two (complex) measures μ, ν on X are said to be *disjoint* if $\inf(|\mu|, |\nu|) = 0$. A measure μ is said to be *concentrated on a set* M, or *carried by* M, if $X - M$ is $|\mu|$-*negligible*, or equivalently if $|\mu| = \varphi_M \cdot |\mu|$ (13.15.3); every measure with base $|\mu|$ is then also concentrated on M (13.15.5).

(13.18.1) *For two measures μ, ν to be disjoint it is necessary and sufficient that there should exist two disjoint subsets* M, N *of* X *such that μ is concentrated on* M *and ν is concentrated on* N; *and the sets* M, N *can be chosen to be universally measurable.*

We may restrict ourselves to the case where μ and ν are positive, and then we can write $\mu = g \cdot \rho$ and $\nu = h \cdot \rho$, where ρ is a positve measure and g, h are locally ρ-integrable functions $\geqq 0$ ((13.15.2) and (13.15.3)). We have then $\inf(\mu, \nu) = \inf(g, h) \cdot \rho$, and hence $\inf(\mu, \nu) = 0$ if and only if $\inf(g, h)$ is ρ-negligible (13.15.3). If M_0 and N_0 are the sets of points $x \in X$ such that $g(x) \neq 0$ and $h(x) \neq 0$, respectively, then $\inf(g, h)$ is ρ-negligible if and only

if $M_0 \cap N_0$ is ρ-negligible. Putting $M = M_0 - (M_0 \cap N_0)$ and $N = N_0 - (M_0 \cap N_0)$, this condition is equivalent to the relations $g = \varphi_M g$ and $h = \varphi_N h$ almost everywhere with respect to ρ (13.15.3). But $g = \varphi_M g$ almost everywhere with respect to ρ if and only if $\mu = \varphi_M \cdot \mu$ (13.15.3) and (13.14.5)), or equivalently if and only if μ is concentrated on M. This proves the first assertion. The second follows by replacing M and N by universally measurable sets $M' \subset M$ and $N' \subset N$ such that $|\mu|(M - M') = 0$ and $|\nu|(N - N') = 0$ (13.9.3).

In particular, if μ is any real measure, then μ^+ and μ^- are concentrated on two *disjoint* μ-measurable sets, each of which is both μ^+-measurable and μ^--measurable ((13.15.3) and (13.16.1)).

(13.18.2) (i) *If λ is a measure disjoint from each of two measures μ, ν, then λ is disjoint from $\mu + \nu$.*

(ii) *If* H *is a set of positive measures disjoint from a measure ν, and if* H *is bounded above, then $\mu = \sup$ H (13.15.4) is disjoint from ν.*

(i) We may assume that λ, μ, ν are all positive and of the forms $\lambda = f \cdot \rho$, $\mu = g \cdot \rho$, $\nu = h \cdot \rho$, where ρ is a positive measure and f, g, h are positive. The result then follows from the inequality $\inf(f, g + h) \leqq \inf(f, g) + \inf(f, h)$.

(ii) We may assume that $\nu \geqq 0$ and that H consists of an increasing sequence (μ_n) (13.15.4), and we may write $\nu = h \cdot \rho$, $\mu_n = g_n \cdot \rho$, $\mu = \left(\sup_n g_n\right) \cdot \rho$, where ρ is a positive measure and f and the g_n are positive functions. Then the result follows from the formula $\inf\left(f, \sup_n g_n\right) = \sup_n \left(\inf(f, g_n)\right)$.

(13.18.3) *If two measures μ, ν are disjoint, then $|\mu + \nu| = |\mu| + |\nu|$.*

Write $\mu = g \cdot \rho$ and $\nu = h \cdot \rho$, where ρ is a positive measure and $\inf(|g|, |h|) = 0$; then use the relation $|g + h| = |g| + |h|$.

(13.18.4) (Lebesgue's decomposition theorem) *Let μ be a positive measure on* X. *Then every complex measure ν on* X *is uniquely expressible in the form $\nu = \nu' + \nu''$, where ν' has base μ and ν'' is disjoint from μ. If $\nu \geqq 0$, then ν' and ν'' are positive, and $\nu' = \sup_n(\inf(\nu, n\mu))$.*

If $\nu_1' + \nu_1'' = \nu_2' + \nu_2''$, where ν_1'' and ν_2'' are disjoint from μ, and ν_1' and ν_2' are measures with base μ, then we have $\nu_1'' - \nu_2'' = \nu_2' - \nu_1'$, and since $\nu_2' - \nu_1'$ is disjoint from μ (13.18.2), it can be a measure with base μ only if it is zero, by virtue of (13.18.1). This establishes the uniqueness of the decomposition. For

the existence, write $\mu = f \cdot \rho$ and $\nu = g \cdot \rho$, where ρ is a positive measure and $f \geqq 0$, and let M be the ρ-measurable set of points $x \in X$ such that $f(x) > 0$, so that μ is concentrated on M (13.14.1). Then the measures $\nu' = \varphi_M g \cdot \rho$ and $\nu'' = \nu - \nu' = (1 - \varphi_M)g \cdot \rho$ satisfy the conditions of the theorem. For ν'' is disjoint from μ by virtue of (13.18.1); also if A is a μ-negligible set, then $A \cap M$ is ρ-negligible ((13.14.1) and (13.6.3)), hence A is ν'-negligible (13.14.1) and consequently ν' is a measure with base μ (13.15.5). It is clear that if ν is positive then so are ν' and ν''; the formula $\nu' = \sup_n(\inf(\nu, n\mu))$ then follows from the corresponding formula $\varphi_M g = \sup_n(\inf(g, nf))$, which is valid almost everywhere with respect to ρ.

A measure μ on X is said to be *diffuse* if $\mu(\{x\}) = 0$ for all $x \in X$. For example, Lebesgue measure on **R** is diffuse. With respect to a diffuse measure μ every *denumerable* set is *μ-negligible*; equivalently, a diffuse measure is concentrated on the complement of any denumerable set. On a denumerable discrete space, the only diffuse measure is therefore $\mu = 0$. The sum of two diffuse measures is diffuse (13.16.1). The least upper bound of any set of diffuse positive measures which is bounded above is diffuse, by (13.15.9) and (13.15.4).

(13.18.5) *If μ is any measure on* X, *the set* A *of points $x \in X$ such that $|\mu|(\{x\}) > 0$ is at most denumerable.*

Since X is the union of a sequence of compact sets K_n, it is enough to show that each of the sets $A \cap K_n$ is at most denumerable. For this, it is enough to show that, for each integer $m \geqq 1$, the set A_{mn} of points $x \in A \cap K_n$ such that $|\mu|(\{x\}) \geqq 1/m$ is *finite*; now this is immediate, for if $B \subset A_{mn}$ consists of p points, then we have $p/m \leqq |\mu|(B) \leqq |\mu|(K_n)$.

A measure μ on X is said to be *atomic* if it is *concentrated on an at most denumerable set*. From this definition and from (13.18.1) it follows immediately that an atomic measure and a diffuse measure are always *disjoint*. The sum of two atomic measures is atomic. The least upper bound of a set of atomic positive measures which is bounded above is atomic ((13.15.4) and (13.15.9)).

(13.18.6) *Every measure μ can be expressed uniquely in the form $\mu_1 + \mu_2$, where μ_1 is a diffuse measure and μ_2 an atomic measure.*

The uniqueness of the decomposition follows from the fact that an atomic measure and a diffuse measure are disjoint. To establish the existence of the decomposition, it is enough to consider the at most denumerable set

D of points $x \in X$ such that $|\mu|(\{x\}) > 0$ (13.18.5). This set is clearly μ-measurable, and if we write $\mu = h \cdot |\mu|$ (13.16.3), the measure $\mu_2 = (\varphi_D h) \cdot |\mu|$ is concentrated on D (13.14.1), and $\mu_1 = \mu - \mu_2$ is diffuse. For we have $|\mu_2| = \varphi_D \cdot |\mu|$ ((13.13.4) and (13.16.3)), hence $|\mu_1| = |\mu| - |\mu_2|$; for each $x \in D$, we have $|\mu|(\{x\}) = |\mu_2|(\{x\})$ by construction, and for each $x \notin D$ we have $|\mu|(\{x\}) = |\mu_2|(\{x\}) = 0$ by the definition of D; hence, by (13.16.1), $|\mu_1|(\{x\}) = 0$ for all $x \in X$.

Example

(13.18.7) Let λ be Lebesgue measure on $\mathbf{R}$. From (13.18.4) and (13.18.6) it follows that every measure μ on $\mathbf{R}$ is uniquely of the form $\mu_1 + \mu_2 + \mu_3$, where $\mu_1 = g \cdot \lambda$ is a measure with base λ, μ_2 is an atomic measure and μ_3 is a *diffuse measure disjoint from* λ and therefore concentrated on a λ-negligible set (necessarily nondenumerable).

Remarks

(13.18.8) (i) If μ is an atomic measure, the denumerable set of points $x \in X$ such that $|\mu|(\{x\}) > 0$ is the *smallest* set on which μ is concentrated. On the other hand, for a diffuse measure $\nu \neq 0$, there exists no smallest set on which ν is concentrated: in other words, there exists no largest ν-negligible set, because every set consisting of a single point is ν-negligible.

(ii) Let μ be an atomic measure concentrated on a denumerable set D, and let (a_n) be the sequence consisting of the distinct points of D arranged in some order. Put $|\mu|(\{a_n\}) = \gamma_n > 0$. If $D_n = \{a_1, \ldots, a_n\}$, then $\varphi_D = \sup_n \varphi_{D_n}$; since $|\mu| = \varphi_D \cdot |\mu|$, then by (13.14.1) and (13.5.7) we have

$$|\mu|^*(f) = \sup_n |\mu|^*(f\varphi_{D_n}) = \sum_{n=1}^{\infty} \gamma_n f(a_n)$$

for any function $f \geqq 0$ (the sum on the right is either a finite real number or $+\infty$). This implies in particular that, for any compact set K, we have $\sum_{a_n \in K} \gamma_n < +\infty$. Furthermore, since every set consisting of a single point is μ-measurable and since $X - D$ is μ-negligible, it follows that *every* mapping of X into a topological space is μ-measurable. The μ-integrable functions are therefore those for which $\sum_{n=1}^{\infty} \gamma_n |f(a_n)| < +\infty$ (13.9.13), and we have

$-\sum_{n=1}^{\infty} \gamma_n f(a_n)$. Since $\mu = h \cdot |\mu|$ with $|h(a_n)| = 1$ for all n (13.16.3), we $\varphi_{\{a_n\}}) = \beta_n$ with $|\beta_n| = \gamma_n$, and $\int f\, d\mu = \sum_{n=1}^{\infty} \beta_n f(a_n)$ for every μ-integ- nction f.

versely, consider an *arbitrary* denumerable sequence (a_n) of points of a sequence (γ_n) of real numbers >0 such that $\sum_{a_n \in K} \gamma_n < +\infty$ for all t subsets K of X. We have already seen (13.1.3) that

$$f \mapsto \nu(f) = \sum_{n=1}^{\infty} \gamma_n f(a_n)$$

itive measure on X. With the same notation as above, it is clear that $(\varphi_{D_n} \cdot \nu)$, and since $\varphi_{D_n} \cdot \nu$ is concentrated on D_n and hence on D, it that ν is concentrated on D. Also we have $\nu(\{a_n\}) = \gamma_n$. For if K is a neighborhood of a_n, then for each $\varepsilon > 0$ there exists an integer m such $\sum_{K,\ p \geqq m} \gamma_p \leqq \varepsilon$. If $f: X \to [0, 1]$ is a continuous mapping which takes e 1 at a_n, and the value 0 on $X - K$ and at points a_k such that $k \neq n$ $\geqq m$ (4.5.2), then we have $\nu(\{a_n\}) \leqq \nu(f) \leqq \gamma_n + \varepsilon$; on the other hand choose the neighborhood K such that $\nu(K) \leqq \nu(\{a_n\}) + \varepsilon$ (13.7.9), and ri $\gamma_n \leqq \nu(f) \leqq \nu(\{a_n\}) + \varepsilon$. Since ε was arbitrary, our assertion is

MS

, σ be two atomic measures on a locally compact space X, and let M, N be the lest sets carrying $|\rho|$, $|\sigma|$, respectively. Show that ρ, σ are disjoint if and only if $N = \varnothing$. Hence give an example of an atomic measure ν on the interval $I = [0, 1]$ such that I is the support (13.19) of ν^+ and of ν^-.

Let μ be a positive measure on a locally compact space X. If $A \subset X$ is universally urable and not μ-negligible, show that there exists a positive measure ν carried by ch that $\nu \neq 0$ and $\nu \leqq \mu$. (Observe that A contains a compact set K which is not gligible.)

Let M be a universally measurable subset of X. For a positive measure μ to be ed by M it is necessary and sufficient that ν should be disjoint from every positive ure carried by $X - M$ (use (a)).

f M is closed in X, show that the vector subspace of M(X) consisting of the ures carried by M is vaguely closed in M(X).

Show that Lebesgue measure on $I = [0, 1]$ is the vague limit of a sequence of ic measures carried by a fixed denumerable subset D of I. The subspace of M(I) sting of measures carried by D is therefore not vaguely closed.

3. (a) Let μ be a positive measure on a locally compact space X, and let A be a μ-integrable set such that $\mu(A) > 0$. Suppose that, for each μ-integrable subset B of A, we have either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Show that there exists a point $a \in A$ such that $\mu(\{a\}) = \mu(A)$. (Consider the intersection of the compact sets $K \subset A$ such that $\mu(K) = \mu(A)$: show that it is not empty, has measure equal to $\mu(A)$ and consists of a single point.)

(b) Suppose that μ is a diffuse measure. For each μ-integrable set A such that $\mu(A) > 0$ and each $\varepsilon > 0$, show that there exists a μ-integrable subset B of A such that $0 < \mu(B) \leqq \varepsilon$ (use (a) to show that there exists an integrable subset C of A such that $0 < \mu(C) \leqq \frac{1}{2}\mu(A)$). Deduce that, as B runs through the set of μ-integrable subsets of A, the set of values of $\mu(B)$ is the closed interval $[0, \mu(A)]$. (For each real number b such that $0 < b < \mu(A)$, let c be the least upper bound of the measures of measurable subsets C of A such that $\mu(C) \leqq b$. Show that $c = b$ by using the preceding result, and then show that there exists an increasing sequence (C_n) of measurable subsets of A such that $\lim_{n\to\infty} \mu(C_n) = b$.)

4. (a) Let ν be a positive atomic measure on a locally compact space X, and let A be a ν-integrable subset of X. Show that the set of values of $\nu(B)$, as B runs through the set of ν-integrable subsets of A, is closed in **R**. (Let P be the smallest set carrying ν. Assuming that $A \cap P$ is infinite, and arranging the points of $A \cap P$ in a sequence (x_n), consider the mapping φ of the product space $\{0, 1\}^{\mathbf{N}}$ into **R** defined by $\varphi(\varepsilon_n) = \sum_n \varepsilon_n \nu(\{x_n\})$, and show that φ is continuous.)

(b) Deduce from (a) and Problem 3(b) that if μ is any positive measure on X and if A is a μ-integrable subset of X, then the set of values of $\mu(B)$, as B runs through the set of μ-integrable subsets of A, is closed in **R**. Extend this result to the situation where μ is any real measure on X.

(c) Deduce from Problem 3(b) that if μ is a diffuse real measure on X, then the set of values of $\mu(A) = \mu^+(A) - \mu^-(A)$, where A runs through the set of $|\mu|$-integrable subsets of X, is a closed (possibly unbounded) interval of **R**.

(d) Give an example of an atomic positive measure ν on a locally compact, non-compact space X, such that the set of values of $\nu(A)$, where A runs through the set of ν-integrable subsets of X, is not closed in **R**. (Take ν so that $\inf_{x \in X} \nu(\{x\}) > 0$.)

5. (a) Let X be a compact space and $\mu \neq 0$ a diffuse positive measure on X. Let (f_n) be a *total* orthonormal sequence in $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$. Show that $\sum_n |f_n(x)|^2 = +\infty$ for almost all $x \in X$. (Argue by contradiction, using Problem 3(b), Bessel's inequality, and the Cauchy–Schwarz inequality for series: show that there would exist a measurable set B with measure >0 and arbitrarily small, contained in the set of points x such that $\sum_n |f_n(x)|^2 < +\infty$, such that, if $c_n = (\varphi_B \mid f_n)$, the series $\sum_n c_n f_n(x)$ converges almost everywhere in B to a function $\leqq \frac{1}{2}$; then observe that by virtue of the hypothesis that the sequence (f_n) is total, together with Parseval's identity, we have

$$\lim_{n\to\infty} \int_B |1 - s_n(x)|^2 \, d\mu(x) = 0, \qquad \text{where} \quad s_n = \sum_{k=1}^{n} c_k f_k .$$

Hence obtain a contradiction.)

(b) Under the same hypotheses, show that there exists a sequence (b_n) of scalars such that $\lim_{n\to\infty} b_n = 0$ and $\sum_n |b_n|^2 |f_n(x)|^2 = +\infty$ almost everywhere. (Use Egoroff's theorem

to obtain an increasing sequence of sets $A_m \subset X$ such that $\mu(X - A_m) \to 0$ as $m \to \infty$ and in which the partial sums $s_n(x) = \sum_{k=1}^{n} |f_k(x)|^2$ tend uniformly to $+\infty$. Then remark that the series with general term $|f_n(x)|^2/s_n(x)$ is divergent almost everywhere (Section 5.3, Problem 6).

(c) Under the same hypotheses show that for almost all $x_0 \in X$ there exists a function $g \in \mathscr{L}^2_{\mathbf{C}}(X, \mu)$ (depending on x_0) such that the series with general term $(g|f_n)f_n(x_0)$ has sum equal to $+\infty$ (use Problem 23 of Section 12.16).

6. (a) Let $\theta : \mathbf{R} \to \mathbf{R}$ be increasing and continuous on the right. Show that there exists a unique positive measure ν on $\mathbf{R}$ such that $\nu(]a, b]) = \theta(b) - \theta(a)$ for every half-open interval $]a, b]$. (This measure is called the "Stieltjes measure defined by θ", and we write $\int f\, d\theta$ in place of $\int f\, d\nu$.) Conversely, every positive measure on $\mathbf{R}$ may be obtained in this way, and two functions θ_1, θ_2 on $\mathbf{R}$ (both of them increasing and continuous on the right) define the same measure if and only if $\theta_2 - \theta_1$ is constant. Under what conditions is the measure ν diffuse? What is then the image of ν under θ?

(b) Let K be the Cantor set (Section 4.2, Problem 2). Show that there exists a diffuse positive measure ν on $\mathbf{R}$ with support K and total mass equal to 1 (Section 4.2, Problem 2(d)). Deduce that there exists on $\mathbf{R}$ a *diffuse* positive measure, disjoint from Lebesgue measure, with support equal to $I = [0, 1]$. (In each component interval J of $I - K$ choose a measure proportional to the image of ν under an affine linear mapping of I onto J; then proceed by induction.)

(c) Deduce from (a) that if μ is a positive measure on $\mathbf{R}_+$ and f a function belonging to $\mathscr{L}^1_{\mathrm{loc},\,\mathbf{C}}(\mathbf{R}_+, \mu)$, and (t_n) a dense sequence in $\mathbf{R}$ such that $\int_{[0,\, t_n]} f\, d\mu = 0$ for all n, then f is μ-negligible.

7. Let X be a compact space and μ a diffuse positive measure on X with total mass equal to 1.

(a) Construct a family $(U(t))_{0 \leqslant t \leqslant 1}$ of μ-quadrable open sets, such that $U(0) = \varnothing$, $U(1) = X$, $\overline{U(t)} \subset U(t')$ whenever $t < t'$, and $\mu(U(t)) = t$ for all t. (First define the $U(t)$ for t of the form $k/2^n$, by induction on n. Use Problem 7(d) of Section 13.9 to prove that if V, W are two quadrable open sets in X such that $\bar{V} \subset W$ and $\mu(V) < \mu(W)$, then there exists a quadrable open set U such that $\bar{V} \subset U \subset \bar{U} \subset W$ and

$$\tfrac{1}{3}\mu(W - V) < \mu(U - V) < \tfrac{2}{3}\mu(W - V).$$

Also use the result of Problem 3(b).)

(b) Deduce from (a) that there exists a continuous mapping π of X onto $I = [0, 1]$ such that the image of μ under π is Lebesgue measure λ on I.

(c) Show that there exists a μ-negligible subset N of X, a λ-negligible subset M of I and a *homeomorphism* π_0 of $I - M$ onto $X - N$ such that, for every λ-measurable subset A of $I - M$, the set $\pi_0(A)$ is μ-measurable and of measure $\mu(\pi_0(A)) = \lambda(A)$. (Use Problem 7(d) of Section 13.9 to show that for each integer $n > 0$ there exists a finite partition of X consisting of a μ-negligible set and quadrable open sets of diameter $\leqslant 1/n$ (with respect to a distance defining the topology of X) and of measure $\leqslant 1/n$. Proceeding by induction on n and passing to the limit, obtain a homeomorphism f of $I - D$ onto a subset of X of measure 1, where D is a denumerable subset of I.)

8. Let X, Y be two compact spaces, U a continuous linear mapping of the Banach space $\mathscr{C}_{\mathbf{R}}(X)$ into the Banach space $\mathscr{C}_{\mathbf{R}}(Y)$, and $\pi : X \to Y$ a continuous mapping.

Assume that

$$\inf_{\pi(x)=y} f(x) \leqq (U \cdot f)(y) \leqq \sup_{\pi(x)=y} f(x)$$

for all $y \in Y$ and all $f \in \mathscr{C}_{\mathbf{R}}(X)$. Show that under these conditions there exists a vaguely continuous mapping $\sigma: y \mapsto \sigma_y$ of Y into $M_+(X)$ such that σ_y is carried by $\pi^{-1}(y)$ for all $y \in Y$, and such that $(U \cdot f)(\pi(x)) = \langle f, \sigma_{\pi(x)} \rangle$ for all $x \in X$. Consider the converse. (Consider the transpose ${}^tU : M_{\mathbf{R}}(Y) \mapsto M_{\mathbf{R}}(X)$ of U (12.15.3).)

9. Let X be a compact space, let μ be a *diffuse* positive measure on X such that $\mu(X) = 1$, and let $u : X \to X$ be a bijection such that both u and u^{-1} are μ-measurable, μ is u-invariant (Section 13.9, Problem 24) and u is *ergodic* with respect to μ (Section 13.9, Problems 13 and 24). Show that for each $\varepsilon > 0$ and each integer $n > 0$, there exists a μ-measurable subset A of X such that (i) the sets $u^j(A)$ $(0 \leqq j \leqq n-1)$ are pairwise disjoint, and (ii) the complement of $\bigcup_{j=0}^{n-1} u^j(A)$ has measure $\leqq \varepsilon$ (Rokhlin's theorem). (Choose a μ-measurable set B such that $0 < \mu(B) < \varepsilon/n$, and construct the corresponding "Kakutani skyscraper" consisting of the sets $u^p(E_m)$ for $m \geqq 1$ and $0 \leqq p < m$ (Section 13.9, Problem 14(d)). Show that we may take A to be the union of the sets $u^{jn}(E_m)$ for all integers $j > 0$ and integers m such that $m \geqq (j+1)n$.)

19. SUPPORT OF A MEASURE. MEASURES WITH COMPACT SUPPORT

(13.19.1) *If μ is any measure on* X, *the union of all μ-negligible open sets is μ-negligible* (and hence is the *largest* μ-negligible open set).

For if U is this union, it follows from (13.1.9) that the measure induced by μ on U is zero.

The complement in X of the largest μ-negligible open set is called the *support* of μ, and is denoted by $\mathrm{Supp}(\mu)$. To say that $x \in \mathrm{Supp}(\mu)$ signifies that $|\mu|^*(V) > 0$ for *every* neighborhood V of x, or equivalently (13.5.1) that $|\mu|(|f|) > 0$ for every $f \in \mathscr{K}_{\mathbf{C}}(X)$ such that $f(x) \neq 0$; or equivalently again, that for each neighborhood V of x there exists a function $f \in \mathscr{K}_{\mathbf{C}}(X)$ with support contained in V, such that $\mu(f) \neq 0$. If $\mathrm{Supp}(\mu) = X$, the only *μ-negligible continuous* function is the constant 0.

From the definition we have $\mathrm{Supp}(\mu) = \mathrm{Supp}(|\mu|)$, and it is clear that $\mathrm{Supp}(a\mu) = \mathrm{Supp}(\mu)$ for all scalars $a \neq 0$. More generally, if g is any locally $|\mu|$-integrable function, then we have $\mathrm{Supp}(g \cdot \mu) \subset \mathrm{Supp}(g) \cap \mathrm{Supp}(\mu)$; for if we put $\nu = g \cdot |\mu|$, and if an open set U does not intersect $\mathrm{Supp}(g)$ or does not intersect $\mathrm{Supp}(\mu)$, then $|\nu|^*(U) = 0$ (13.14.1).

(13.19.2) (i) *If μ and ν are positive measures, then*

$$\mathrm{Supp}(\mu + \nu) = \mathrm{Supp}(\mu) \cup \mathrm{Supp}(\nu).$$

(ii) *If* H *is any majorized family of positive measures and* $\nu = \sup H$ (13.15.4), *then* Supp(ν) *is the closure of the union of the supports of the measures* $\mu \in H$.

This follows immediately from (13.16.1) and (13.15.9), applied to the characteristic function of a relatively compact open set (having regard to (13.15.4)).

The complement of Supp(μ) can also be defined as the largest of the *interiors* of μ-negligible sets. Hence Supp(μ) is the intersection of the *closures* of *all* sets on which the measure μ is concentrated (13.18). But it should be realized that two *disjoint* measures can have the same support: for example, an atomic measure on **R**, concentrated on a denumerable dense set, and such that each point of this set has nonzero measure (13.18.8), has the same support as Lebesgue measure.

(13.19.3) *For a measure* μ *on* X *to be such that every continuous complex-valued function on* X *is* μ*-integrable, it is necessary and sufficient that* Supp(μ) *should be compact. The mapping* $f \mapsto \mu(f)$ *is then a continuous linear form on the Fréchet space* $\mathscr{C}_{\mathbf{C}}(X)$ (12.14.6), *and conversely every continuous linear form on* $\mathscr{C}_{\mathbf{C}}(X)$ *is of this type.*

If μ has compact support S, then X − S is μ-negligible, hence $f\varphi_S$ is equal to f almost everywhere. Since $f\varphi_S$ is measurable (13.9.6) and $|f\varphi_S|$ is bounded above by a multiple of φ_S (3.17.10), it follows that $f\varphi_S$, and hence also f, is integrable.

Let us show conversely that if $S = \mathrm{Supp}(\mu)$ is not compact, then there exist continuous real-valued functions $f \geqq 0$ such that $|\mu|^*(f) = +\infty$. By hypothesis (3.18.3) there exists an increasing sequence of relatively compact open sets U_n in X such that $\bar{U}_n \subset U_{n+1}$ and the U_n cover X. Since X is not compact, we have $U_n \neq X$ for all n. We now define inductively a sequence (a_n) of points of X and a sequence (V_n) of relatively compact open sets, as follows: $a_1 \in S$; V_1 is a neighborhood of a_1; a_{n+1} belongs to the intersection of S and the complement of the union of $\bar{U}_n$ and the $\bar{V}_k$ with $k \leqq n$ (by hypothesis, this intersection is not empty); V_{n+1} is a relatively compact neighborhood of a_{n+1} such that $\bar{V}_{n+1}$ does not intersect $\bar{U}_n$ nor any $\bar{V}_k$ with $k \leqq n$. For each n, let g_n be a continuous mapping of X into [0, 1] which takes the value 1 at a_n and the value 0 throughout $X - V_n$ (4.5.2). The hypothesis implies that $|\mu|(g_n) > 0$. Put $f_n = g_n/|\mu|(g_n)$, so that $|\mu|(f_n) = 1$. Then the function $f = \sum_{n=1}^{\infty} f_n$ is everywhere finite and continuous (because every point $x \in X$

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belongs to some U_n, which intersects only finitely many of the V_k), and we have $|\mu|^*(f) = +\infty$, because $|\mu|^*(f) \geqq \sum_{k=1}^{n} |\mu|(f_k)$ for all n.

If $S = \mathrm{Supp}(\mu)$ is compact, the first part of the proof shows that $|\mu|(f) \leqq |\mu|(S) \cdot \sup_{x \in S} |f(x)|$ for all functions $f \in \mathscr{C}_{\mathbf{C}}(X)$, and therefore μ is a continuous function on the Fréchet space $\mathscr{C}_{\mathbf{C}}(X)$. Conversely, let λ be a continuous linear form on this space, so that there exists a compact set $K \subset X$ and a constant $c > 0$ such that $|\lambda(f)| \leqq c \cdot \sup_{x \in K} |f(x)|$ for all $f \in \mathscr{C}_{\mathbf{C}}(X)$ (12.14.6). If L is any compact subset of X, we then have $|\lambda(f)| \leqq c \cdot \|f\|$ for all $f \in \mathscr{K}_{\mathbf{C}}(X; L)$, and hence the restriction μ of λ to $\mathscr{K}_{\mathbf{C}}(X)$ is a *measure* on X. Moreover, if h is a continuous mapping of X into $[0, 1]$, with compact support and such that $h(x) = 1$ for all $x \in K$ ((3.18.2) and (4.5.2)), then $\lambda(fh) = \lambda(f)$ for all functions $f \in \mathscr{C}_{\mathbf{C}}(X)$, because $f - fh$ is zero on K. From this it follows immediately that the support of μ is contained in $\mathrm{Supp}(h)$; hence every $f \in \mathscr{C}_{\mathbf{C}}(X)$ is μ-integrable, and $\mu(f) = \mu(fh) = \lambda(fh) = \lambda(f)$. Q.E.D.

(13.19.4) If $\pi : X \to X'$ is a homeomorphism and μ is a measure on X, we have $\mathrm{Supp}(\pi(\mu)) = \pi(\mathrm{Supp}(\mu))$. This follows directly from the definitions.

PROBLEM

Let μ be a positive measure on X, and A a μ-measurable set. Let $i(A)$ be the set of points $x \in X$ such that there exists a compact neighborhood V of x in X for which $\mu(V \cap (X - A)) = 0$. Show that $i(A)$ is open and that $\mu(i(A) \cap (X - A)) = 0$. (Consider $\mathrm{Supp}(\varphi_{X-A} \cdot \mu)$.)

20. BOUNDED MEASURES

For every (complex) measure μ on X, we define

(13.20.1) $$\|\mu\| = \sup_{\|f\| \leqq 1,\, f \in \mathscr{K}_{\mathbf{C}}(X)} |\mu(f)|$$

which is a finite real number, or $+\infty$.

(13.20.2) *We have* $\|\mu\| = |\mu|^*(1)$. *Hence* $\|\mu\|$ *is finite if and only if the positive measure* $|\mu|$ *is bounded* (13.9), *and in that case*

$$\|\mu\| = |\mu|(1) = |\mu|(X)$$

is the total mass of X *with respect to* μ.

For every function $f \in \mathscr{K}_{\mathbf{C}}(X)$ we have (13.3.3)

$$|\mu(f)| \leqq |\mu|(|f|) \leqq \|f\| \cdot |\mu|^*(1)$$

with the usual convention about products in $[0, +\infty]$ (13.11), hence $\|\mu\| \leqq |\mu|^*(1)$. Conversely, for each positive real number $a < |\mu|^*(1)$, there exists a function $g \in \mathscr{K}_{\mathbf{R}}(X)$ such that $0 \leqq g \leqq 1$ and $|\mu|(g) > a$ (13.5.1), and hence a function $f \in \mathscr{K}_{\mathbf{C}}(X)$ such that $|f| \leqq g \leqq 1$ and $|\mu(f)| > a$ (13.3.2.1). This completes the proof.

If $\|\mu\|$ is finite, the measure μ is said to be *bounded*. If μ is a positive measure, this definition coincides with that given in (13.9). For a measure μ to be bounded it is necessary and sufficient that $|\mu|$ should be bounded, and we have $\| |\mu| \| = \|\mu\|$. The set $M^1_{\mathbf{C}}(X)$ (also denoted by $M^1(X)$) of bounded complex measures on X is a *vector subspace* of the space $M_{\mathbf{C}}(X)$. We write $M^1_{\mathbf{R}}(X) = M^1_{\mathbf{C}}(X) \cap M_{\mathbf{R}}(X)$ for the space of bounded real measures on X. The definition (13.20.1) shows that it comes to the same thing to say that the bounded measures on X are the *continuous linear forms* on the space $\mathscr{K}_{\mathbf{C}}(X)$, endowed with the norm $\|f\|$ (5.5.1), and that $\|\mu\|$ is the usual *norm* (5.7.1) on the *dual* $M^1_{\mathbf{C}}(X)$ of this normed space (12.15). Moreover, $M^1_{\mathbf{C}}(X)$ is *complete* with respect to this norm (5.7.3).

If μ is any measure on X and g a locally μ-integrable function, then the measure $g \cdot \mu$ (13.16) is bounded if and only if g is μ-integrable, and we have

$$\|g \cdot \mu\| = N_1(g) = \int |g| \, d|\mu|. \tag{13.20.3}$$

This follows immediately from the relation $|g \cdot \mu| = |g| \cdot |\mu|$ (13.16.6).

It is clear that every measure μ with compact support is bounded, because if $S = \mathrm{Supp}(\mu)$ then $|\mu|^*(1) = |\mu|(S)$.

If μ is a bounded positive measure, then we have

$$\mathscr{L}^\infty_{\mathbf{C}}(X, \mu) \subset \mathscr{L}^2_{\mathbf{C}}(X, \mu) \subset \mathscr{L}^1_{\mathbf{C}}(X, \mu), \tag{13.20.4}$$

because by (13.9.17) every bounded measurable function is integrable, and the Cauchy–Schwarz inequality (13.11.2.2) applied to $g = 1$, together with (13.9.13), shows that every $f \in \mathscr{L}^2_{\mathbf{C}}(X, \mu)$ is integrable and that

$$N_1(f) \leqq N_2(f) \cdot \mu(X)^{1/2}.$$

This shows also that the canonical injection $\mathscr{L}^2_{\mathbf{C}}(X, \mu) \to \mathscr{L}^1_{\mathbf{C}}(X, \mu)$ is continuous with respect to the seminorms on these two spaces; the same is true

for the injection $\mathscr{L}^\infty_{\mathbf{C}}(X, \mu) \to \mathscr{L}^2_{\mathbf{C}}(X, \mu)$ (with respect to the corresponding seminorms) because it follows from (13.12.5) that $N_2(f) \leqq N_\infty(f) \cdot \mu(X)^{1/2}$ for all $f \in \mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$.

(13.20.5) In particular (13.9.17), if μ is a bounded (complex) measure, every function $f \in \mathscr{C}^\infty_{\mathbf{C}}(X)$ is μ-integrable, and we have $\left|\int f\, d\mu\right| \leqq \|\mu\| \cdot \|f\|$ by (13.16.5). In other words, $f \mapsto \int f\, d\mu$ is a *continuous linear form* on the Banach space $\mathscr{C}^\infty_{\mathbf{C}}(X)$. But it should be noted that, in general, there exist continuous linear forms on this space which are not of this type.

(13.20.6) The space $M^1_{\mathbf{C}}(X)$ is also the dual of the *closure* $\mathscr{C}^0_{\mathbf{C}}(X)$ of $\mathscr{K}_{\mathbf{C}}(X)$ in the Banach space $\mathscr{C}^\infty_{\mathbf{C}}(X)$ (12.15). A function f belonging to $\mathscr{C}^0_{\mathbf{C}}(X)$ may be characterized by the following property: *for each* $\varepsilon > 0$, *there exists a compact subset* K *of* X *such that* $|f(x)| \leqq \varepsilon$ *for all* $x \in X - K$. For if $f \in \mathscr{C}^0_{\mathbf{C}}(X)$, then for each $\varepsilon > 0$ there exists by definition a function $g \in \mathscr{K}_{\mathbf{C}}(X)$ such that $\|f - g\| \leqq \varepsilon$; if K is the support of g, then $|f(x)| \leqq \varepsilon$ for all $x \notin K$. Conversely, suppose that f has the above property, and let h be a continuous mapping of X into [0, 1], with compact support and equal to 1 on K ((3.18.2) and (4.5.2)); it is clear that $\|f - fh\| \leqq \varepsilon$ and that $fh \in \mathscr{K}_{\mathbf{C}}(X)$, hence $f \in \mathscr{C}^0_{\mathbf{C}}(X)$. The functions belonging to $\mathscr{C}^0_{\mathbf{C}}(X)$ are called (complex-valued) continuous functions *which tend to* 0 *at infinity*. (When $X = \mathbf{R}$, they are indeed the continuous functions f such that $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0$.) We put $\mathscr{C}^0_{\mathbf{R}}(X) = \mathscr{C}^0_{\mathbf{C}}(X) \cap \mathscr{C}_{\mathbf{R}}(X)$ for the corresponding space of real-valued functions.

When X is *compact*, we have

$$\mathscr{C}^\infty_{\mathbf{C}}(X) = \mathscr{C}^0_{\mathbf{C}}(X) = \mathscr{K}_{\mathbf{C}}(X) = \mathscr{C}_{\mathbf{C}}(X).$$

PROBLEMS

1. The space $M^1_{\mathbf{R}}(X)$ of bounded real measures on a locally compact space X can be considered as a space of linear forms on each of the following vector spaces:

(1) the space $E_1 = \mathscr{K}_{\mathbf{R}}(X)$ of continuous functions with compact support;
(2) the space $E_2 = \mathscr{C}^0_{\mathbf{R}}(X)$ of continuous functions which tend to 0 at infinity;
(3) the space $E_3 = \mathscr{C}^\infty_{\mathbf{R}}(X)$ of bounded continuous functions;
(4) the space E_5 of linear combinations of (upper or lower) semicontinuous bounded functions;
(5) the space $E_6 = \mathscr{U}_{\mathbf{R}}(X)$ of universally measurable bounded functions.

Moreover, if ν is any positive measure on X, the space $M^1_{\mathbf{R}}(X, \nu)$ of bounded measures *with base* ν (which may be identified with the space $L^1_{\mathbf{R}}(X, \nu)$ by virtue of 13.14.4)) can be considered as a space of linear forms on the vector space $E_{4,\nu}$ of bounded functions which are *continuous almost everywhere with respect to* ν.

Let $\mathscr{T}_i$ denote the weak topology on $M_{\mathbf{R}}^1(X)$ corresponding to the vector space E_i ($i \neq 4$), and $\mathscr{T}_{4,\nu}$ the weak topology on $M_{\mathbf{R}}^1(X, \nu)$ (or on $L_{\mathbf{R}}^1(X, \nu)$) corresponding to $E_{4,\nu}$ (cf. (12.15)).

The topology $\mathscr{T}_i$ on $M_{\mathbf{R}}^1(X)$ is coarser than $\mathscr{T}_j$ if $i < j$. The topology induced by $\mathscr{T}_3$ on $M_{\mathbf{R}}^1(X, \nu)$ is coarser than $\mathscr{T}_{4,\nu}$, which in turn is coarser than the topology induced by $\mathscr{T}_5$. (Consider the lower semicontinuous regularization of a function which is continuous almost everywhere with respect to ν (Section 12.7, Problem 8).)

(a) Let (μ_n) be a sequence of bounded real measures on a locally compact but not compact space X. Give an example in which (μ_n) tends to 0 vaguely (i.e., for the topology $\mathscr{T}_1$) but does not converge for the topology $\mathscr{T}_2$. A sequence (μ_n) which vaguely converges to 0 also converges to 0 with respect to $\mathscr{T}_2$ if and only if the sequence of norms $(\|\mu_n\|)$ is bounded (use the Banach–Steinhaus theorem).

(b) Give an example of a sequence (μ_n) which converges to 0 for the topology $\mathscr{T}_2$ but not for $\mathscr{T}_3$. A sequence (μ_n) of bounded real measures which converges vaguely to a measure μ also converges to μ for the topology $\mathscr{T}_3$ if and only if, for each $\varepsilon > 0$, there exists a compact subset K of X such that $|\mu_n|(X - K) \leqq \varepsilon$ for all n. (To show that the condition is necessary, argue by contradiction, by using the method of Section 13.14, Problem 1.)

(c) Give an example of a sequence (μ_n) in $M_{\mathbf{R}}^1(X, \nu)$ which converges to 0 for the topology $\mathscr{T}_3$ but does not converge to 0 for the topology $\mathscr{T}_{4,\nu}$ (take $X = [0, 1]$). A sequence (μ_n) of measures belonging to $M_{\mathbf{R}}^1(X, \nu)$ which converges to 0 for the topology $\mathscr{T}_3$ also converges to 0 with respect to $\mathscr{T}_{4,\nu}$ if and only if it satisfies the following condition:

($C_{4,\nu}$) For each ν-negligible compact subset K of X and each $\varepsilon > 0$, there exists an open neighborhood U of K such that $|\mu_n|(U) \leqq \varepsilon$ for all n.

(To prove that the condition is sufficient, reduce to the case where X is compact and apply ($C_{4,\nu}$) to the set K_ε of points $x \in X$ at which the oscillation of an almost everywhere continuous function is $\geqq \varepsilon$. To prove that the condition is necessary, argue by contradiction as in (b) above.)

(d) Show that a sequence (μ_n) in $M_{\mathbf{R}}^1(X)$ which is a Cauchy sequence for one of the topologies $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ is convergent for this topology. (For $\mathscr{T}_3$, use (b) and argue by contradiction: form a sequence of measures which tends to 0 with respect to $\mathscr{T}_3$ without satisfying the condition given in (b).)

(e) A sequence (μ_n) of measures belonging to $M_{\mathbf{R}}^1(X, \nu)$ is a Cauchy sequence with respect to the topology $\mathscr{T}_{4,\nu}$ if and only if it satisfies the condition ($C_{4,\nu}$) (argue by contradiction as in (d)); the sequence (μ_n) then converges with respect to $\mathscr{T}_{4,\nu}$ to a measure *belonging to* $M_{\mathbf{R}}^1(X, \nu)$.

(f) Let (μ_n) be a sequence of measures belonging to $M_{\mathbf{R}}^1(X, \nu)$. For each subset A of X which is either finite or else open and ν-quadrable (Section 13.9, Problem 7), suppose that the sequence $(\mu_n(A))$ has a finite limit; show that (μ_n) is then a Cauchy sequence for the topology $\mathscr{T}_{4,\nu}$. (Use Problem 1 of Section 13.14, and argue by contradiction.) Show that the hypothesis relative to finite sets A cannot be omitted.

(g) Let (μ_n) be a sequence of measures belonging to $M_{\mathbf{R}}^1(X, \nu)$ which converges in the topology $\mathscr{T}_3$ to a measure μ *belonging to* $M_{\mathbf{R}}^1(X, \nu)$. If also $\lim_{n\to\infty} \|\mu_n\| = \|\mu\|$, then the sequence (μ_n) converges to μ in the topology $\mathscr{T}_{4,\nu}$ (use (e)). Give an example of a sequence (μ_n) of *positive* measures belonging to $M_{\mathbf{R}}^1(X, \nu)$, where the space X is compact, such that (μ_n) converges vaguely to a measure which *does not belong to* $M_{\mathbf{R}}^1(X, \nu)$.

2. The notation is the same as in Problem 1. Let (μ_n) be a sequence of measures belonging to $M_{\mathbf{R}}^1(X)$. Then there exists a positive measure ν on X such that $\mu_n \in M_{\mathbf{R}}^1(X, \nu)$ for all n.

(a) Show that the following properties are equivalent:

(α) The sequence (μ_n) is convergent for the topology $\mathscr{T}_6$.

(β) For each *closed* subset A of X, the sequence $(\mu_n(A))$ has a finite limit.

(γ) The sequence (μ_n) converges for the topology $\mathscr{T}_3$ and satisfies the foll condition:

(C_5) For each compact subset K of X and each $\varepsilon > 0$, there exists an neighborhood U of K such that $|\mu_n|(U - K) \leqq \varepsilon$ for all n.

(δ) The sequence (μ_n) converges for the topology $\mathscr{T}_3$ and satisfies the foll condition:

(C_6) For each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each universally meas set A satisfying $\nu(A) \leqq \delta$, we have $|\mu_n|(A) \leqq \varepsilon$ for all n.

(To show that (β) implies (γ), use Problem 1 of Section 13.14 to show th sequence $\|\mu_n\|$ is bounded. To establish (C_5), argue by contradiction: first co the case in which the sequence (μ_n) tends to 0 for the topology $\mathscr{T}_3$, and then p the general case as in Problem 1(d). Next show that (γ) implies that the seq (μ_n) is convergent for the topology $\mathscr{T}_5$, and in particular implies (β). To sho (δ) implies (α), use the definition of measurable functions. To show that (α) i (δ), argue by contradiction. Finally, to show that (γ) implies (α), consider first th where $\lim_{n\to\infty} \|\mu_n\| = \|\mu\|$, where μ is the limit of (μ_n) for the topology $\mathscr{T}_5$, and ar contradiction to prove that (γ) implies (δ). To pass to the general case, reduce situation where $\mu = 0$, and argue by contradiction: we may assume that there a function $f \in E_6$ such that the sequence $(\mu_n(f))$ has a limit $\neq 0$. On the other ha can pick a subsequence (μ_{n_k}) of the sequence (μ_n) such that the sequence $(\mu^+_{n_k}$ hence also $(\mu^-_{n_k})$, converges vaguely. Hence arrive at a contradiction, by observin these two sequences also satisfy (C_5).)

(b) Give an example of a sequence (μ_n) of measures belonging to $M^1_{\mathbf{R}}(X, \nu)$ converges for the topology $\mathscr{T}_{4,\nu}$ but not for the topology $\mathscr{T}_5$ (cf. Section 13.18, Pr 2(d)).

(c) Take $X = [0, 1]$. Show that the topologies $\mathscr{T}_5$ and $\mathscr{T}_6$ on $M^1_{\mathbf{R}}(X)$ are d (cf. Section 12.15, Problem 2(c), and Section 13.11, Problem 3).

(d) Extend the results of Problems 1 and 2 to bounded complex measures.

3. Let X be a locally compact space, μ a positive measure on X. Let (g_n) be a seque μ-integrable functions such that (1) the sequence (g_n) converges in measure (S 13.12, Problem 2) to a function g; (2) the sequence of measures $(g_n \cdot \mu)$ converg the topology $\mathscr{T}_6$ (Problem 2). Show that under these conditions the functio μ-integrable and that the sequence (g_n) tends to g in $\mathscr{L}^1_{\mathbf{R}}(X, \mu)$.

4. Let X be a compact space, μ a positive measure on X, and (f_n) an orthonormal seq in $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$ consisting of functions which are *uniformly bounded* on X.

(a) Show that, for each function $g \in \mathscr{L}^1_{\mathbf{C}}(X, \mu)$, the sequence of numbers $(g$ $\int g(x)\overline{f_n(x)}\, d\mu(x)$ tends to 0. (Reduce to the case where g is bounded and belongs to $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$.)

(b) Let $x_0 \in X$. For the series $\sum_n (g \mid f_n) f_n(x_0)$ to converge for *every* fu $g \in \mathscr{L}^\infty_{\mathbf{C}}(X, \mu)$, it is necessary and sufficient that (in the notation of Section Problem 2) the sequence of bounded measures $K_n(x_0, \cdot) \cdot \mu$ should converge f topology $\mathscr{T}_6$ (Problem 2) to a bounded measure $h_{x_0} \cdot \mu$; hence in particular we $f_n(x_0) = \int h_{x_0}(x) f_n(x)\, d\mu(x)$ for all n.

(c) Show that there exists a measurable subset A of X with measure >0, such that for each $x_0 \in A$ there exists a function $g \in \mathscr{L}^\infty_C(X, \mu)$ (depending on x_0) for which the series with general term $(g \mid f_n)f_n(x_0)$ does not converge. (Take A to be the set of points $x \in X$ for which the sequence $(f_n(x))$ does not converge to 0, and use Problem 7(a) of Section 13.11. Then argue by contradiction, using (a) and (b).)

5. Let S be a closed subset of $\mathbf{R}$. Given a sequence $(c_n)_{n \geq 0}$ of real numbers, show that there exists a positive measure μ on $\mathbf{R}$ with support contained in S and such that $\int x^n \, d\mu(x) = c_n$ for all $n \geq 0$ if and only if, for every polynomial $P(X) = \sum_{k=0}^{n} \xi_k X^k$ such that $P(x) \geq 0$ for all $x \in S$, we have $\sum_{k=0}^{n} \xi_k c_k \geq 0$. (As in Problem 2 of Section 13.3, show that there exists a positive measure μ with support S which extends to a linear form u defined on $\mathscr{K}_{\mathbf{R}}(\mathbf{R})$ and on the space of polynomials on $\mathbf{R}$, and such that $u(P) = \sum_{k=0}^{n} \xi_k c_k$ for each polynomial $P(X) = \sum_{k=0}^{n} \xi_k X^k$. Then prove that each power x^n is μ-integrable and that $\int x^n \, d\mu(x) = c_n$. For this purpose, remark that for each integer $n \geq 0$ and each $\varepsilon > 0$ there exists a number $R > 0$ such that $|f_{n,R}(x)| \leq \varepsilon x^{2n+2}$, where $f_{n,R}$ is the function equal to 0 for $|x| < R$ and to x^n for $|x| \geq R$.)

Particular cases: (1) $S = \mathbf{R}$ ("Hamburger's moment problem"): the condition is that the quadratic forms $\sum_{j,k=0}^{n} c_{j+k} \xi_j \xi_k$ should be positive for all n (observe that every polynomial $P(x)$ which is ≥ 0 on $\mathbf{R}$ is the sum of two squares $(P_1(x))^2 + (P_2(x))^2$).

(2) $S = [0, +\infty[$ ("Stieltjes' moment problem"): the condition is that the quadratic forms $\sum_{j,k=0}^{n} c_{j+k} \xi_j \xi_k$ and $\sum_{j,k=0}^{n} c_{j+k+1} \xi_j \xi_k$ should be positive for all $n \geq 0$ (remark that every polynomial $P(x)$ which is ≥ 0 on $[0, +\infty[$ can be written in the form $(P_1(x))^2 + (P_2(x))^2 + x((P_3(x))^2 + (P_4(x))^2)$.

6. Let ν be a positive measure on X and (f_n) a sequence of functions belonging to $\mathscr{L}^1_{\mathbf{R}}(X, \nu)$ such that the sequence of measures $(f_n \cdot \nu)$ converges to $f \cdot \nu$ (where $f \in \mathscr{L}^1_{\mathbf{R}}(X, \nu)$) for the topology $\mathscr{T}_6$. Show that $f(x) \leq \limsup_{n \to \infty} f_n(x)$ almost everywhere with respect to ν. (Use (13.8.3).)

7. Let μ, ν be two bounded complex measures on X. Show that the following conditions are equivalent: (1) μ and ν are disjoint; (2) $\|\mu \pm \nu\| = \|\mu\| + \|\nu\|$; (3) $\|\mu + \nu\| + \|\mu - \nu\| = 2(\|\mu\| + \|\nu\|)$.

21. PRODUCT OF MEASURES

(13.21.1) *Let* X, Y *be two locally compact spaces,* λ *a measure on* X *and* μ *a measure on* Y. *Then there exists a unique measure* ν *on the product space* $X \times Y$ *such that, for each pair of functions* $f \in \mathscr{K}_C(X)$ *and* $g \in \mathscr{K}_C(Y)$, *we have*

$$(13.21.1.1) \qquad \int f(x)g(y) \, d\nu(x, y) = \left(\int f(x) \, d\lambda(x)\right)\left(\int g(y) \, d\mu(y)\right).$$

(1) *Uniqueness.* Every compact subset of $X \times Y$ is contained of the form $L \times M$, where L and M are relatively compact open subse and Y respectively ((3.20.17) and (3.18.2)). We shall show that, f function $h \in \mathscr{K}_{\mathbf{C}}(X \times Y)$ with support contained in $L \times M$, the v $\nu(h)$ is well-determined. This will follow from

(13.21.1.2) *If* $L \subset X$ *and* $M \subset Y$ *are relatively compact open sets, the function* $h \in \mathscr{K}_{\mathbf{C}}(X \times Y)$ *with support contained in* $L \times M$ *is a cluste in the Banach space* $\mathscr{K}_{\mathbf{C}}(X \times Y; \bar{L} \times \bar{M})$, *of the set of functions of t* $(x, y) \mapsto \sum_{i,k} f_i(x) g_k(y)$, *where* (f_i) *is a finite sequence of functions in with support contained in* L, *and* (g_k) *is a finite sequence of functions in with support contained in* M.

Assuming this for the moment, by hypothesis there exists a numbe such that $|\nu(u)| \leqq c \cdot \|u\|$ for all $u \in \mathscr{K}_{\mathbf{C}}(X \times Y)$ with support conta $L \times M$. On the other hand, for each $\varepsilon > 0$ there exist two finite seque functions $f_i \in \mathscr{K}_{\mathbf{C}}(X)$, $g_k \in \mathscr{K}_{\mathbf{C}}(Y)$ such that

$$|h(x, y) - \sum_{i,k} f_i(x) g_k(y)| \leqq \varepsilon$$

for all $(x, y) \in L \times M$, the support of the left-hand side being conta $L \times M$. Using (13.21.1.1), it follows that

$$|\nu(h) - \sum_{i,k} \lambda(f_i) \mu(g_k)| \leqq c \cdot \varepsilon,$$

and since ε is arbitrary, this establishes the uniqueness of ν.

To prove (13.21.1.2), let ε be a positive real number. Then for ea $(x, y) \in L \times M$ there exists a compact neighborhood $U \subset L$ (resp. V of x in X (resp. of y in Y) such that the oscillation of h in $U \times V$ (3.20.1). The projections $S \subset L$ and $T \subset M$ of Supp(h) are compact for each $x \in S$ there exists a finite number of points $y_j(x)$ $(1 \leqq j \leqq n(x)$ and for each j a compact neighborhood $U_j(x) \subset L$ of x in X and a co neighborhood $V_j(y_j(x)) \subset M$ of $y_j(x)$ in Y, such that the oscillation $U_j(x) \times V_j(y_j(x))$ is $\leqq \varepsilon$, and such that the interiors of the sets V cover T. The set $U'(x) = \bigcap_j U_j(x)$ is a compact neighborhood of x Hence there exists a finite number of points x_i $(1 \leqq i \leqq m)$ in S such t interiors of the sets $U'(x_i)$ cover S. Put $A_i = U'(x_i)$, and let $(B_k)_{1 \leqq k \leqq p}$ family of sets obtained as follows: for each point $y \in T$, let $W(y)$ be th section of the (finitely many) interiors of sets $V_j(y_j(x_i))$ which contain sets $W(y)$ are open and nonempty, and form a finite open covering of T we denote by $(B_k)_{1 \leqq k \leqq p}$. Notice also that, by construction, the oscilla h in each set $A_i \times B_k$ is $\leqq \varepsilon$. Now let $(f_i)_{1 \leqq i \leqq m}$ (resp. $(g_k)_{1 \leqq k \leqq p}$) be cont mappings of X (resp. Y) into [0, 1], such that Supp(f_i) $\subset A_i$ and Supp(g

for each pair of indices i, k, and such that $\sum_i f_i(x) \leqq 1$ for all $x \in X$, $\sum_k g_k(y) \leqq 1$ for all $y \in Y$, and $\sum_i f_i(x) = 1$ for all $x \in S$, $\sum_k g_k(y) = 1$ for all $y \in T$ (12.6.4). Then we have

$$h(x, y) = \sum_{i,k} h(x, y) f_i(x) g_k(y)$$

for all $x \in X$ and all $y \in Y$. If y_k is any point of B_k, the hypotheses therefore imply that

$$\begin{aligned} &\left| h(x, y) - \sum_{i,k} h(x_i, y_k) f_i(x) g_k(y) \right| \\ &\quad = \left| \sum_{i,k} \Big(h(x, y) - h(x_i, y_k) \Big) f_i(x) g_k(y) \right| \\ &\quad \leqq \varepsilon \sum_{i,k} f_i(x) g_k(y) \leqq \varepsilon \end{aligned}$$

for all $(x, y) \in A \times B$, where $A = \bigcup_i A_i$ and $B = \bigcup_k B_k$, and hence also for all $(x, y) \in X \times Y$ (since the expression on the left vanishes outside $A \times B$).

(2) *Existence.* First we shall prove the following lemma:

(13.21.1.3) *Let* L *be a compact subset of* X *and* M *a compact subset of* Y. *If* $h \in \mathscr{K}_c(X \times Y)$ *is such that* $\mathrm{Supp}(h) \subset L \times M$, *then the function* $g(y) = \int h(x, y)\, d\lambda(x)$ *is continuous on* Y, *and* $\mathrm{Supp}(g) \subset M$.

For each $y \in Y$, the function $x \mapsto h(x, y)$ belongs to $\mathscr{K}_c(X; L)$ and is identically zero if $y \notin M$. On the other hand, because h is uniformly continuous (3.16.5), for each $\varepsilon > 0$ and each $y \in Y$ there exists a neighborhood W of y in Y such that the relations $x \in X$ and $y' \in W$ imply that $|h(x, y') - h(x, y)| \leqq \varepsilon$. Finally, there exists a number $a_L > 0$ such that $|\lambda(u)| \leqq a_L \|u\|$ for all $u \in \mathscr{K}_c(X; L)$. We deduce that

$$|g(y') - g(y)| = \left| \int (h(x, y') - h(x, y))\, d\lambda(x) \right| \leqq a_L \varepsilon$$

for all $y' \in W$, and the lemma is proved.

We return to the proof of (13.21.1). For each function $h \in \mathscr{K}_c(X \times Y)$, the number $\nu(h) = \mu(g)$ (which by abuse of notation is also written in the form $\mu\Big(\int h(x, y)\, d\lambda(x)\Big)$) is defined. Furthermore, with the notation used above, there exist by hypothesis two numbers a_L and b_M such that, for each

function $u \in \mathscr{K}(X; L)$ (resp. $v \in \mathscr{K}(Y; M)$), we have $|\lambda(u)| \leqq a_L \|u\|$ (resp. $|\mu(v)| \leqq b_M \|v\|$). Hence, for each function $h \in \mathscr{K}(X \times Y; L \times M)$, we have

$$\left| \int h(x, y)\, d\lambda(x) \right| \leqq a_L \|h\|$$

for all $y \in Y$, and by virtue of (13.21.1.3) and the definition of $\nu(h)$,

$$|\nu(h)| \leqq a_L b_M \|h\|.$$

Since every compact subset of $X \times Y$ is contained in the product of its projections on X and on Y, the proof of (13.21.1) is complete.

The number $\nu(h) = \mu\left(\int h(x, y)\, d\lambda(x)\right)$ is also denoted by

$$\int d\mu(y) \int h(x, y)\, d\lambda(x).$$

Since we can clearly interchange the roles of X and Y, it follows that

$$\text{(13.21.2)} \qquad \int h(x, y)\, d\nu(x, y) = \int d\lambda(x) \int h(x, y)\, d\mu(y) = \int d\mu(y) \int h(x, y)\, d\lambda(x)$$

for all $h \in \mathscr{K}_C(X \times Y)$.

By reason of this formula we write $\iint h\, d\lambda\, d\mu$ or $\iint h(x, y)\, d\lambda(x)\, d\mu(y)$ instead of $\int h(x, y)\, d\nu(x, y)$, and we may also interchange λ and μ in these notations. The measure ν is called the *product* of λ and μ, and is denoted by $\lambda \otimes \mu$. It is clear that the mapping $(\lambda, \mu) \mapsto \lambda \otimes \mu$ of $M_C(X) \times M_C(Y)$ into $M_C(X \times Y)$ is *bilinear*: in other words, we have

$$(\lambda_1 + \lambda_2) \otimes (\mu_1 + \mu_2) = \lambda_1 \otimes \mu_1 + \lambda_1 \otimes \mu_2 + \lambda_2 \otimes \mu_1 + \lambda_2 \otimes \mu_2$$

and

$$(a\lambda) \otimes \mu = \lambda \otimes (a\mu) = a(\lambda \otimes \mu)$$

for any scalar a.

Furthermore, if λ and μ are *real* (resp. *positive*) measures, then so is $\lambda \otimes \mu$.

(13.21.3) *Let λ, μ be two positive measures on X, Y respectively, and $\nu = \lambda \otimes \mu$ their product. For each function $h \in \mathscr{S}(X \times Y)$, the function*

$$x \mapsto \int^* h(x, y)\, d\mu(y)$$

belongs to $\mathscr{I}(X)$, *and we have*

$$\int^* h\, dv = \lambda^*\left(\int^* h(x, y)\, d\mu(y)\right) \tag{13.21.3.1}$$

(with an abuse of notation analogous to those above).

By (12.7.8) there exists an increasing sequence (h_n) of functions belonging to $\mathscr{K}_{\mathbf{R}}(X \times Y)$ such that $h = \sup_n h_n$. For each n, it follows from (13.21.1.3) that the function $f_n(x) = \int h_n(x, y)\, d\mu(y)$ belongs to $\mathscr{K}_{\mathbf{R}}(X)$; hence $f = \sup_n f_n$ belongs to $\mathscr{I}(X)$, and moreover we have $f(x) = \int^* h(x, y)\, d\mu(y)$ for all $x \in X$, by virtue of (13.5.2). Since $\nu(h_n) = \lambda(f_n)$ by definition, another application of (13.5.2) completes the proof.

From now on, up to and including (13.21.16), we shall assume that the measures $\lambda \in M(X)$ and $\mu \in M(Y)$ are *positive*, and we shall write $\nu = \lambda \otimes \mu$.

(13.21.4) *If* h *is any mapping of* $X \times Y$ *into* $\overline{\mathbf{R}}$, *then*

$$\int^* h\, dv \geqq \lambda^*\left(\int^* h(x, y)\, d\mu(y)\right). \tag{13.21.4.1}$$

Let $u \in \mathscr{I}(X \times Y)$ be such that $h \leqq u$. Then for each $x \in X$ we have $\int^* h(x, y)\, d\mu(y) \leqq \int^* u(x, y)\, d\mu(y)$, and consequently

$$\lambda^*\left(\int^* h(x, y)\, d\mu(y)\right) \leqq \lambda^*\left(\int^* u(x, y)\, d\mu(y)\right) = \nu^*(u)$$

by (13.21.3). Hence the inequality (13.21.4.1) follows from the definition of $\nu^*(h)$ (13.15.5).

We shall write $\iint^* h\, d\lambda\, d\mu$ or $\iint^* h(x, y)\, d\lambda(x)\, d\mu(y)$ instead of $\nu^*(h)$, and $\int^* d\lambda(x) \int^* h(x, y)\, d\mu(y)$ in place of $\lambda^*\left(\int^* h(x, y)\, d\mu(y)\right)$. Similarly, we shall use the notations $\iint_* h\, d\lambda\, d\mu$ and $\iint_* h(x, y)\, d\lambda(x)\, d\mu(y)$ for lower integrals. Thus the inequality (13.21.4.1) takes the form

$$\iint^* h(x, y)\, d\lambda(x)\, d\mu(y) \geqq \int^* d\lambda(x) \int^* h(x, y)\, d\mu(y) \tag{13.21.4.2}$$

with equality if $h \in \mathscr{I}(X \times Y)$. There is of course an analogous inequality (resp. equality) obtained by interchanging the roles of X and Y, and an

inequality in the reverse direction (with equality when $h \in \mathscr{S}(X \times$ lower integrals:

(13.21.4.3) $$\iint_* h(x, y)\, d\lambda(x)\, d\mu(y) \leqq \int_* d\lambda(x) \int_* h(x, y)\, d\mu(y).$$

(13.21.5) *If* N *is a* ν*-negligible subset of* $X \times Y$, *then the set of poir such that the section* $N(x) \subset Y$ *of* N *is not* μ*-negligible is* λ*-negli other words, we have* $\mu(N(x)) = 0$ *almost everywhere with respect to* λ

This follows immediately from (13.21.4.2) applied to $h = \varphi_N$.

(13.21.6) *If* h *is a* ν*-measurable mapping of* $X \times Y$ *into a topological then the set of points* $x \in X$ *for which the partial mapping* $y \mapsto h(x,$ μ*-measurable is* λ*-negligible* (*in other words, the mapping* $y \mapsto h(x,$ *measurable almost everywhere with respect to* λ).

By hypothesis, there exists a partition of $X \times Y$ consisting of a ν-n set N and a sequence $(K_n)_{n \geqq 1}$ of compact sets, such that each of the tions $h \mid K_n$ is continuous (13.9). Let M be the λ-negligible set c $x \in X$ at which the section $N(x)$ of N is not μ-negligible (13.21.5). I $x \notin M$, Y admits a partition consisting of compact sets $K_n(x)$ ($n \geqq$ the μ-negligible set $N(x)$, such that the restriction of $y \mapsto h(x, y)$ of the sets $K_n(x)$ is continuous. Hence the result.

It should be noted that it can happen that for each $x \in X$ the $y \mapsto f(x, y)$ is μ-measurable, and for each $y \in Y$ the function $x \mapsto f$ λ-measurable, *but that* f *is not* ν*-measurable.*

(13.21.7) (Lebesgue–Fubini Theorem) *Let* λ, μ *be positive measu and* Y, *respectively, and* $\nu = \lambda \otimes \mu$ *their product. For each* ν*-integra ping* h *of* $X \times Y$ *into* $\overline{\mathbf{R}}$, *the set of points* $x \in X$ *such that the partial* $y \mapsto h(x, y)$ *is not* μ*-integrable is* λ*-negligible*; *the function* $x \mapsto \int h(x,$ *which is defined almost everywhere with respect to* λ, *is* λ*-integrable*;

(12.21.7.1) $$\iint h(x, y)\, d\lambda(x)\, d\mu(y) = \int d\lambda(x) \int h(x, y)\, d\mu(y).$$

It follows from (13.21.4.1) that the function $x \mapsto \int^* |h(x, y)|\, d\mu(y)$ on the complement of a λ-negligible set N_1 (13.6.4). On the other h set N_2 of points $x \in X$ such that $y \mapsto h(x, y)$ is not μ-measurable is λ-n

(13.21.6). Hence, from (13.9.13), it follows that for each $x \notin N = N_1 \cup N_2$ the mapping $y \mapsto h(x, y)$ is μ-integrable, and therefore the function

$$x \mapsto \int h(x, y)\, d\mu(y)$$

is defined almost everywhere with respect to λ. The fact that it is λ-integrable, and the relation (13.21.7.1), then follow from (13.21.4.2), and (13.21.4.3).

Interchanging the roles of X and Y we have also, under the hypotheses of (13.21.7),

$$\text{(13.21.7.2)} \qquad \iint h(x, y)\, d\lambda(x)\, d\mu(y) = \int d\mu(y) \int h(x, y)\, d\lambda(x).$$

But here again it needs to be said that the right-hand sides of (13.21.7.1) and (13.21.7.2) can be defined and equal *without h being ν-integrable* (even if h is ν-measurable) (Problem 3).

(13.21.8) *Let $h \geqq 0$ be a ν-measurable function. Then the mapping*

$$x \mapsto \int^* h(x, y)\, d\mu(y)$$

is λ-measurable, and

$$\text{(13.21.8.1)} \qquad \iint^* h(x, y)\, d\lambda(x)\, d\mu(y) = \int^* d\lambda(x) \int^* h(x, y)\, d\mu(y).$$

Let (K_n) be an increasing sequence of compact subsets of $X \times Y$ which cover $X \times Y$ (3.18.3). Then we have $h = \sup_n h_n$, where $h_n = \inf(h, n\varphi_{K_n})$, and h_n is *ν-integrable* ((13.9.7) and (13.9.13)); on the other hand (13.5.7), $\nu^*(h) = \sup_n \nu(h_n)$. By (13.6.2) and (13.21.7), there exists a λ-negligible set N such that, for all $x \notin N$, all the functions $y \mapsto h_n(x, y)$ are μ-integrable; also (13.21.7) the functions $x \mapsto \int h_n(x, y)\, d\mu(y)$ are λ-integrable and we have $\nu(h_n) = \int d\lambda(x) \int h_n(x, y)\, d\mu(y)$ for all n. Hence it follows from (13.5.7) that

$$\int^* h(x, y)\, d\mu(y) = \sup_n \int h_n(x, y)\, d\mu(y)$$

for all $x \notin N$. Consequently the function $x \mapsto \int^* h(x, y)\, d\mu(y)$ is λ-measurable, by (13.9.11), and the relation (13.21.8.1) follows by another application of (13.5.7).

(13.21.9) *A ν-measurable function h is ν-integrable if and only if*

$$\int^* d\lambda(x) \int^* |h(x, y)|\, d\mu(y) < +\infty.$$

This follows from (13.21.8) and (13.9.13).

(13.21.10) *Let* A *be a ν-measurable set in* $X \times Y$.

(i) *The set* M *of all $x \in X$ such that the section* $A(x)$ *is not μ-measurable is λ-negligible; the function $x \mapsto \mu^*(A(x))$ is λ-measurable; and*

$$\nu^*(A) = \int^* \mu^*(A(x))\, d\lambda(x).$$

In particular, if $A(x)$ *is μ-negligible except on a λ-negligible set of values of x, then* A *is ν-negligible.*

(ii) *If* A *is ν-integrable, then the set of all $x \in X$ such that* $A(x)$ *is not μ-integrable is λ-negligible; the function $x \mapsto \mu(A(x))$ (which is defined almost everywhere with respect to λ) is λ-integrable; and* $\nu(A) = \int \mu(A(x))\, d\lambda(x)$.

These assertions are particular cases of (13.21.6), (13.21.8) and (13.21.7).

(13.21.11). *Let f* (resp. g) *be a mapping of* X (resp. Y) *into* $[0, +\infty]$. *With the convention of* (13.11) *for products, we have*

$$\text{(13.21.11.1)} \qquad \iint^* f(x)g(y)\, d\lambda(x)\, d\mu(y) = \left(\int^* f(x)\, d\lambda(x)\right)\left(\int^* g(y)\, d\mu(y)\right).$$

By virtue of (13.21.4), we have

$$\iint^* f(x)g(y)\, d\lambda(x)\, d\mu(y) \geqq \int^* d\lambda(x) \int^* f(x)g(y)\, d\mu(y).$$

On the other hand, for each $x \in X$ we have

$$\int^* f(x)g(y)\, d\mu(y) = f(x) \int^* g(y)\, d\mu(y)$$

and

$$\begin{aligned} \int^* d\lambda(x) \int^* f(x)g(y)\, d\mu(y) &= \int^* \left(\int^* g(y)\, d\mu(y)\right) f(x)\, d\lambda(x) \\ &= \left(\int^* f(x)\, d\lambda(x)\right)\left(\int^* g(y)\, d\mu(y)\right) \end{aligned}$$

with the product convention referred to above. Hence it is enough to prove the inequality

$$\text{(13.21.11.2)} \qquad \iint^* f(x)g(y)\,d\lambda(x)\,d\mu(y) \leqq \left(\int^* f(x)\,d\lambda(x)\right)\left(\int^* g(y)\,d\mu(y)\right).$$

Now this inequality is clearly valid when the right-hand side is equal to $+\infty$. So consider the case in which each of the factors on the right is *finite*. Then there exist two decreasing sequences (f_n), (g_n) such that $f_n \in \mathscr{F}(X)$, $g_n \in \mathscr{F}(Y)$, $f \leqq f_n$, $g \leqq g_n$ for all n, and

$$\int^* f\,d\lambda = \lim_{n\to\infty} \int^* f_n\,d\lambda, \qquad \int^* g\,d\mu = \lim_{n\to\infty} \int^* g_n\,d\mu.$$

By reason of our conventions and of (12.7.5), the function $(x, y) \mapsto f_n(x)g_n(y)$ belongs to $\mathscr{F}(X \times Y)$, and we have $f(x)g(y) \leqq f_n(x)g_n(y)$ for all n and all $(x, y) \in X \times Y$. But, by virtue of (13.21.3),

$$\begin{aligned}\left(\int^* f_n(x)\,d\lambda(x)\right)\left(\int^* g_n(y)\,d\mu(y)\right) &= \int^* d\lambda(x) \int^* f_n(x)g_n(y)\,d\mu(y)\\ &= \iint^* f_n(x)g_n(y)\,d\lambda(x)\,d\mu(y)\\ &\geqq \iint^* f(x)g(y)\,d\lambda(x)\,d\mu(y),\end{aligned}$$

from which the desired inequality follows by letting n tend to $+\infty$.

Finally, to deal with the case where (for example) f is λ-negligible, it is enough (by virtue of our conventions) to prove that the function $(x, y) \mapsto f(x)g(y)$ is ν-negligible. This is a consequence of the following proposition:

(13.21.12) *If* N *is any* λ*-negligible subset of* X, *then the set* $N \times Y$ *is* ν*-negligible.*

For since Y is the union of a denumerable sequence (L_n) of compact sets, it is enough to show that each of the sets $N \times L_n$ is ν-negligible. But since we have proved that (13.21.11.1) is valid whenever both of the factors on the right-hand side are *finite*, we may take $f = \varphi_N$ and $g = \varphi_{L_n}$ in this formula.

(13.21.13) *Let* E, F, G *be three topological spaces and* $u : E \times F \to G$ *a continuous mapping. Let f* (resp. *g*) *be a* λ*-measurable mapping of* X *into* E (resp. *a* μ*-measurable mapping of* Y *into* F). *Then* $(x, y) \mapsto u(f(x), g(y))$ *is a* ν*-measurable mapping of* $X \times Y$ *into* G.

By (13.9.6) it is enough to show that $(x, y) \mapsto f(x)$ is a ν-measurable mapping of $X \times Y$ into E. There exists a partition of X consisting of a sequence (K_n) of compact sets and a λ-negligible set N such that the restriction of f to each K_n is continuous. For each compact subset L of Y, the restriction of the mapping $(x, y) \mapsto f(x)$ to each of the compact sets $K_n \times L$ is continuous, and the set $N \times L$ is ν-negligible (13.21.12), whence the result follows (13.9.4).

Given two mappings $f: X \to \overline{\mathbf{R}}$ and $g: Y \to \overline{\mathbf{R}}$, we denote by $f \otimes g$ the mapping $(x, y) \mapsto f(x)g(y)$ of $X \times Y$ into $\overline{\mathbf{R}}$ (with the convention of (13.11) for products in $\overline{\mathbf{R}}$). Similarly for mappings into **C**.

(13.21.14) *If f* (resp. g) *is a λ-integrable* (resp. *μ-integrable*) *mapping of* X (resp. Y) *into* $\overline{\mathbf{R}}$ *or* **C**, *then the function $f \otimes g$ is ν-integrable, and we have*

$$\text{(13.21.14.1)} \qquad \iint f(x)g(y)\, d\lambda(x)\, d\mu(y) = \left(\int f(x)\, d\lambda(x)\right)\left(\int g(y)\, d\mu(y)\right).$$

By linearity we reduce to the case where f and g are mappings into $\overline{\mathbf{R}}$. By virtue of (13.21.12), the set of points $(x, y) \in X \times Y$ at which $f(x)$ or $g(y)$ is infinite is ν-negligible, hence it follows from (13.21.13) and (13.9.6) that $f \otimes g$ is *ν-measurable*. The fact that $f \otimes g$ is *ν-integrable* then follows from (13.21.11) and (13.9.13). Finally, the formula (13.21.14.1) is a consequence of the Lebesgue–Fubini theorem.

(13.21.15) *Let* A *be a subset of* X, *and* B *a subset of* Y. *Then*

(i) $\nu^*(A \times B) = \lambda^*(A)\mu^*(B)$ (*with the product convention of* (13.11)).

(ii) *If* A *is λ-measurable and* B *is μ-measurable, then* $A \times B$ *is ν-measurable.*

(iii) *If* A *is λ-integrable and* B *is μ-integrable, then* $A \times B$ *is ν-integrable, and* $\nu(A \times B) = \lambda(A)\mu(B)$.

These assertions are particular cases of (13.21.11), (13.21.13) and (13.21.14).

(13.21.16) *If f* (resp. g) *is a locally λ-integrable mapping of* X (resp. *locally μ-integrable mapping of* Y) *into* $\overline{\mathbf{R}}$ *or* **C**, *then $f \otimes g$ is locally ν-integrable, and*

$$\text{(13.21.16.1)} \qquad (f \otimes g) \cdot (\lambda \otimes \mu) = (f \cdot \lambda) \otimes (g \cdot \mu).$$

Since the set of points (x, y) at which $f(x)$ or $g(y)$ is infinite is ν-negligible (13.21.12), it follows from (13.21.13) and (13.9.8.1) that $f \otimes g$ is ν-measurable. Moreover, for each compact subset K (resp. L) of X (resp. Y) the function

$(f \otimes g)\varphi_{K \times L} = (f\varphi_K) \otimes (g\varphi_L)$ is integrable, by virtue of (13.21.14), hence $f \otimes g$ is locally ν-integrable (13.13.1). Also, for each $u \in \mathscr{K}_{\mathbf{C}}(X)$ and each $v \in \mathscr{K}_{\mathbf{C}}(Y)$, we have

$$\iint f(x)g(y)u(x)v(y)\, d\lambda(x)\, d\mu(y) = \left(\int f(x)u(x)\, d\lambda(x)\right)\left(\int g(y)v(y)\, d\mu(y)\right)$$

by virtue of (13.21.14); hence the formula (13.21.16.1) follows from (13.21.1).

(13.21.17) *Let λ, μ be complex measures on* X, Y, *respectively. Then* $|\lambda \otimes \mu| = |\lambda| \otimes |\mu|$.

We may write $\lambda = f \cdot |\lambda|$ and $\mu = g \cdot |\mu|$, where $|f| = |g| = 1$ (13.16.3). Hence $|f \otimes g| = 1$, and the result therefore follows from (13.21.16) and (13.13.4).

(13.21.18) *Let λ be a complex measure on* X *and μ a complex measure on* Y.

(i) *If λ is concentrated on* $A \subset X$ *and μ is concentrated on* $B \subset Y$ (13.18), *then $\lambda \otimes \mu$ is concentrated on* $A \times B$.

(ii) $\mathrm{Supp}(\lambda \otimes \mu) = \mathrm{Supp}(\lambda) \times \mathrm{Supp}(\mu)$.

(iii) *With the product convention of* (13.11) *we have*

$$\|\lambda \otimes \mu\| = \|\lambda\| \cdot \|\mu\|. \tag{13.21.18.1}$$

In particular, if λ and μ are bounded, then so is $\lambda \otimes \mu$.

By virtue of (13.21.17) we may restrict ourselves to the case in which λ and μ are *positive*. Then (i) follows from the fact that $X \times Y - A \times B$ is the union of the $(\lambda \otimes \mu)$-negligible sets $(X - A) \times Y$ and $X \times (Y - B)$ (13.21.12). It follows from (i) that $\mathrm{Supp}(\lambda \otimes \mu) \subset \mathrm{Supp}(\lambda) \times \mathrm{Supp}(\mu)$.

On the other hand, if $x \in \mathrm{Supp}(\lambda)$ and $y \in \mathrm{Supp}(\mu)$, then for each compact neighborhood V of x in X and each compact neighborhood W of y in Y, we have $\lambda(V) > 0$ and $\mu(W) > 0$, whence by (13.21.15) $(\lambda \otimes \mu)(V \times W) = \lambda(V)\mu(W) > 0$. This establishes (ii), by virtue of the definition of neighborhoods in $X \times Y$. Finally, (iii) follows from (13.21.11) applied to $f = \varphi_X$ and $g = \varphi_Y$, except when one of the factors on the right-hand side is 0 and the other is $+\infty$; but in this case we have $\lambda \otimes \mu = 0$ by (13.21.12), and therefore the formula (13.21.18.1) remains valid in this case.

(13.21.19) There are analogous definitions and results for the *product of any finite number of measures*: if $(X_i)_{1 \leq i \leq n}$ is any finite sequence of locally compact

spaces, and if μ_i is a (complex) measure on X_i for each i, then the unique measure ν on $X = \prod_{i=1}^{n} X_i$ which satisfies the relation

$$\int f_1(x_1) f_2(x_2) \cdots f_n(x_n)\, d\nu(x_1, \ldots, x_n) = \prod_{i=1}^{n} \int f_i(x_i)\, d\mu_i(x_i)$$

is called the *product* of the measures μ_i $(1 \leqq i \leqq n)$, and is denoted by $\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n$ or $\bigotimes_{i=1}^{n} \mu_i$. The existence and uniqueness of ν are proved by induction on n, putting $\nu = (\mu_1 \otimes \cdots \otimes \mu_{n-1}) \otimes \mu_n$, and observing that if ν' also satisfies the required conditions, then we must have

$$\nu'(h \otimes f_n) = ((\mu_1 \otimes \cdots \otimes \mu_{n-1})(h))\mu_n(f_n) = \nu(h \otimes f_n)$$

for all $h \in \mathscr{K}\left(\prod_{i=1}^{n-1} X_i\right)$, by the inductive hypothesis. This characterization of the product of measures shows immediately that the product is *associative*; in particular, we have also

$$\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n = \mu_1 \otimes (\mu_2 \otimes \cdots \otimes \mu_n).$$

We write

$$\iint \cdots \int f\, d\mu_1\, d\mu_2 \cdots d\mu_n \quad \text{or} \quad \iint \cdots \int f(x_1, \ldots, x_n)\, d\mu_1(x_1) \cdots d\mu_n(x_n)$$

instead of $\int f\, d\nu$, and analogous notations for upper and lower integrals. We shall leave to the reader the task of formulating and proving the results corresponding to those established above for the case $n = 2$; the proofs require nothing but simple arguments by induction on n.

The product of the Lebesgue measures on the n factors of $\mathbf{R}^n$ is called *Lebesgue measure* on $\mathbf{R}^n$.

PROBLEMS

1. Let X be a locally compact space, μ a positive measure on X, and f a real-valued function $\geqq 0$ defined on X. In the product space $X \times \mathbf{R}$, let D_f denote the set of points (x, t) such that $0 \leqq t \leqq f(x)$. Also let λ denote Lebesgue measure on $\mathbf{R}$.
   (a) For f to be μ-measurable it is necessary and sufficient that D_f should be measurable with respect to the product measure $\nu = \mu \otimes \lambda$. (To show that the condition is necessary, prove that if f is μ-measurable then D_f is the union of a ν-negligible set and a denumerable family of sets of the form $A \times I$, where A is μ-measurable and I is an interval in $\mathbf{R}_+$. To show that the condition is sufficient, prove that if it is satisfied there exists a dense subset H of $\mathbf{R}$ such that $f^{-1}([\alpha, +\infty])$ is μ-measurable for all $\alpha \in H$, by using (13.21.10).)

(b) Show that for f to be μ-integrable it is necessary and sufficient that D_f should be ν-integrable, and that we have then $\nu(D_f) = \int f\,d\mu$. Moreover, if g is the decreasing real-valued function on $\mathbf{R}_+$ defined by $g(t) = \mu^*(f^{-1}[t, +\infty])$, then $\int f\,d\mu = \int_0^{+\infty} g(t)\,dt$.

(c) Suppose that X is an interval $[0, a]$ (where $a > 0$) and take μ to be Lebesgue measure on X. If f is a *decreasing* μ-integrable function, show that

$$\int_A f(t)\,dt \leqq \int_0^{\mu(A)} f(t)\,dt$$

for every μ-measurable subset A of X. (Use (b).)

2. (a) With the notation of Problem 1, let Γ_f be the graph of f, i.e., the set of all points $(x, f(x))$ in $X \times \mathbf{R}$. If f is μ-measurable, show that Γ_f is ν-negligible (use (13.21.13) and (13.21.10)).

(b) Let $0 < \alpha < 1$ and let $Q(\alpha)$ be the subset of $\mathbf{R}^2$ which is the complement of the union of $]-1, 1[\times]-\alpha, \alpha[$ and $]-\alpha, \alpha[\times]-1, 1[$ in the square $Q = [-1, 1] \times [-1, 1]$. The set $Q(\alpha)$ is the union of its four connected components

$$Q_1(\alpha) = [-1, -\alpha] \times [-1, -\alpha], \qquad Q_2(\alpha) = [-1, -\alpha] \times [\alpha, 1],$$
$$Q_3(\alpha) = [\alpha, 1] \times [\alpha, 1], \qquad Q_4(\alpha) = [\alpha, 1] \times [-1, -\alpha].$$

For $i = 1, 2, 3, 4$ let $h_{i,\alpha}$ denote the similitude mapping Q to $Q_i(\alpha)$ for which

$$h_{1,\alpha}(-1, -1) = (-1, -1), \qquad h_{1,\alpha}(1, -1) = (-1, -\alpha),$$
$$h_{2,\alpha}(-1, -1) = (-1, \alpha), \qquad h_{2,\alpha}(1, -1) = (-\alpha, \alpha),$$
$$h_{3,\alpha}(-1, -1) = (\alpha, \alpha), \qquad h_{3,\alpha}(1, -1) = (1, \alpha),$$
$$h_{4,\alpha}(-1, -1) = (1, -\alpha), \qquad h_{4,\alpha}(1, -1) = (1, -1).$$

Also, in the interval $[0, 7]$ in $\mathbf{R}$, put $I_k = [k, k+1]$ for $0 \leqq k \leqq 6$ and let u_k be the increasing similitude which maps $[0, 7]$ to I_k.

Let f_α be the continuous mapping of $[0, 7]$ into Q which is affine-linear on each of the intervals I_k and is such that the images of $0, 1, 2, \ldots, 7$ are respectively the points $(-1, -1), (-1, -\alpha), (-1, \alpha), (-\alpha, \alpha), (\alpha, \alpha), (1, \alpha), (1, -\alpha), (1, -1)$.

We shall now define a sequence (g_n) of continuous mappings of $[0, 7]$ into Q as follows. Let $(\alpha_n)_{n \geqq 0}$ be a decreasing sequence of numbers belonging to the interval $]0, 1[$. Define $g_0 = f_{\alpha_0}$. For $n \geqq 1$, suppose that g_{n-1} has been defined, and consider all sequences $s = (i_1, \ldots, i_n)$ in which each i_k is one of the integers $0, 1, \ldots, 6$. Put $v_s = u_{i_1} \circ \cdots \circ u_{i_n}$. Then it is sufficient to define $g_n(v_s(t))$ for $0 \leqq t \leqq 7$ and for each of the 7^n sequences s. If at least one of the i_l is odd, we put $g_n(v_s(t)) = g_{n-1}(v_s(t))$. If on the other hand $i_l = 2j_l$ for $1 \leqq l \leqq n$ (with $0 \leqq j_l \leqq 3$), we put $g_n(v_s(t)) = w_s(f_{\alpha_n}(t))$, where $w_s = h_{j_1+1,\alpha_1} \circ h_{j_2+1,\alpha_2} \circ \cdots \circ h_{j_n+1,\alpha_n}$. Show that the sequence (g_n) converges uniformly to an *injective continuous* mapping $g : [0, 7] \to \mathbf{R}^2$, and that the *simple arc* $g([0, 7])$ (Section 4, Appendix to Chapter IX) is *nonnegligible* with respect to Lebesgue measure on $\mathbf{R}^2$, if the sequence (α_n) is suitably chosen.

3. (a) Give an example of two compact spaces X and Y, positive measures λ and μ on X and Y, respectively, and a $\lambda \otimes \mu$-measurable function f such that the two integrals $\int d\mu(y) \int f(x, y)\,d\lambda(x)$ and $\int d\lambda(x) \int f(x, y)\,d\mu(y)$ are both defined and are unequal (cf. Section 5.2. Problem 5).

(b) For each integer $n > 0$, let $A'_n = [2^{-n}, 3 \cdot 2^{-n-1}[$ and $A = [3 \cdot 2^{-n-1}, 2^{-n+1}[$. Let $B'_n = A'_n \times A'_n$, $B''_n = A''_n \times A''_n$, $C'_n = A'_n \times A''_n$, $C''_n = A''_n \times A'_n$ in $\mathbf{R}^2$. Define $f: \mathbf{R}^2 \to \mathbf{R}$ as follows: $f(x, y) = 4^{n+1}$ for $(x, y) \in B'_n \cup B''_n$; $f(x, y) = -4^{n+1}$ for $(x, y) \in C'_n \cup C''_n$, for each integer $n > 0$; $f(x, y) = 0$ otherwise. Show that f is measurable and that the two integrals $\int dy \int f(x, y)\, dx$ and $\int dx \int f(x, y)\, dy$ are defined and equal, but that f is not integrable with respect to Lebesgue measure on $\mathbf{R}^2$.

4. Let X, Y be two locally compact spaces, λ a positive measure on X and μ a positive measure on Y. Let f be a mapping of $X \times Y$ into a metrizable space G, such that: (1) for each $x \in X$, the mapping $f(x, \cdot)$ is μ-measurable; (2) for each $y \in Y$, the mapping $f(\cdot, y)$ is continuous. Show that under these hypotheses f is $(\lambda \otimes \mu)$-measurable. (Reduce to the case where X and Y are compact; by using Egoroff's theorem and the fact that X is metrizable, show that f is almost everywhere (with respect to $\lambda \otimes \mu$) the limit of a sequence of $(\lambda \otimes \mu)$-measurable functions.)

5. Let X, Y be two locally compact spaces, λ a positive measure on X and μ a positive measure on Y. Let f be a real-valued function $\geqq 0$ on $X \times Y$ which is bounded on every compact subset of $X \times Y$ and such that: (1) for almost all $x \in X$ the function $f(x, \cdot)$ is μ-measurable; (2) for each function $h \in \mathscr{K}(Y)$, the function $x \mapsto \int f(x, y)h(y)\, d\mu(y)$, which is defined almost everywhere, is λ-measurable. Show that under these conditions there exists a $(\lambda \otimes \mu)$-measurable function g such that for each $x \in X$ we have $f(x, y) = g(x, y)$ except at the points of a μ-negligible set A_x (depending on x). (Show that, for each function $u \in \mathscr{K}(X \times Y)$, the function $f(x, \cdot)u(x, \cdot)$ is μ-integrable for almost all $x \in X$, and that the function $x \mapsto \int f(x, y)u(x, y)\, d\mu(y)$, defined almost everywhere, is λ-integrable; for this purpose approximate u by functions of the form $v(x)w(y)$. Then remark that the linear form $u \mapsto \int d\lambda(x) \int f(x, y)u(x, y)\, d\mu(y)$ is a positive measure on $X \times Y$ with base $\lambda \otimes \mu$, and apply the Lebesgue–Nikodym theorem. Finally use the fact that Y is a denumerable union of relatively compact open sets U_n and that there exists in $\mathscr{K}(Y)$ a denumerable set of functions D such that every function belonging to $\mathscr{K}(Y)$ is the uniform limit of functions belonging to D and having their supports contained in some U_n.)
(b) Show that the conditions of (a) are satisfied if: (1) for almost all $y \in Y$, the function $f(\cdot, y)$ is λ-measurable; (2) for almost all $x \in X$, the function $f(x, \cdot)$ is continuous almost everywhere with respect to μ. (Use Problem 7(c) of Section 13.9.)
(c) Take $X = Y = [0, 1]$ and take λ and μ to be Lebesgue measure. Assuming the continuum hypothesis, let $x < y$ be a well-ordering on X for which there exists no greatest element and such that, for each $x \in X$, the set of all $z \in X$ such that $z < x$ is denumerable. Show that the characteristic function f of the set of pairs (x, y) for which $x < y$ satisfies the conditions of (a), but that $\int f(x, y)\, d\lambda(x) = 0$ for all $y \in Y$ and $\int f(x, y)\, d\mu(y) = 1$ for all $x \in X$.

6. Let u, v be two increasing real-valued functions on $\mathbf{R}$, each continuous on the right, and such that $u(x) = v(x) = 0$ for $x < 0$. Let w be the increasing function on $\mathbf{R}$, continuous on the right, defined by $w(t) = u(t)v(t)$ for $t \geqq 0$ and $w(t) = 0$ for $t < 0$. Let λ, μ, ν be the Stieltjes measures associated with u, v, w, respectively (Section 13.18, Problem 6).

To each real-valued function f on $\mathbf{R}$ we associate the function f_0 on $\mathbf{R}^2$ defined by $f_0(x, y) = f(x)$ if $y < x$, $f_0(x, y) = f(y)$ if $y \geqq x$. Show that f is ν-integrable if and only if f_0 is $(\lambda \otimes \mu)$-integrable, and that in this case $\int f\, d\nu = \iint f_0\, d\lambda\, d\mu$. (Prove the result first for characteristic functions of intervals.) Hence deduce the formula

$$\int f(x)\, dw(x) = \int f(x)v(x-)\, du(x) + \int f(x)u(x+)\, dv(x).$$

In particular, if u and v are *continuous* on $\mathbf{R}$, we have the formula of *integration by parts*:

$$\int_a^b u(x)\, dv(x) = u(b)v(b) - u(a)v(a) - \int_a^b v(x)\, du(x).$$

Consider the case where u and v are constant on each interval $[n, n+1[$ for each integer $n \geqq 0$ ("Abel's partial summation formula").

7. Let p be a finite real number $\geqq 1$, let X and Y be locally compact spaces, λ a positive measure on X and μ a positive measure on Y. Let $f \geqq 0$ be a function on $X \times Y$ such that f and f^p are $(\lambda \otimes \mu)$-integrable. Prove that

$$\left(\int^* \left(\int f\, d\mu\right)^p d\lambda\right)^{1/p} \leqq \int^* \left(\int f^p\, d\lambda\right)^{1/p} d\mu.$$

(For each $x \in X$, apply Hölder's inequality (13.11, Problem 12(a)) to the function $y \mapsto f(x, y)$, in the form $f(x, y) = g(x, y)\left(\int f^p(x, y)\, d\lambda(x)\right)^{1/pq}$, where q is the exponent conjugate to p.)

8. Let X_i $(1 \leqq i \leqq n)$ be locally compact spaces and μ_i a positive measure on X_i for $1 \leqq i \leqq n$. For each i, let E_i denote the product $\prod_{j \neq i} X_j$. Let f_i be a function $\geqq 0$ which is measurable with respect to $\mu = \bigotimes_{i=1}^{n} \mu_i$ on $X = \prod_{i=1}^{n} X_i$, and *does not depend on* x_i. If f_k^{n-1} is integrable with respect to the measure $\bigotimes_{j \neq k} \mu_j$, for $1 \leqq k \leqq n$, show that the function $\prod_{i=1}^{n} f_i$ is μ-integrable and that

$$\iint \cdots \int f_1 f_2 \cdots f_n\, d\mu_1\, d\mu_2 \cdots d\mu_n \leq \left(\prod_{k=1}^{n} J_k\right)^{1/(n-1)}$$

where $J_k = \int \cdots \int f_k^{n-1}\, d\mu_1 \cdots d\mu_{k-1}\, d\mu_{k+1} \cdots d\mu_n$ for $1 \leqq k \leqq n$. (By induction on n, using the Lebesgue–Fubini theorem and Hölder's inequality.)

Deduce that if A is a μ-measurable subset of X, and A_i its projection on E_i, and if A_i is integrable and of measure m_i with respect to the measure $\bigotimes_{j \neq i} \mu_j$ on E_i, then A is μ-integrable and $\mu(A) \leqq (m_1 \cdots m_n)^{1/(n-1)}$.

Generalize to the case in which, instead of considering the products of the X_i $n-1$ at a time, we consider the $\binom{n}{p}$ products of the X_i p at a time, and integrate over X a product of $\binom{n}{p}$ functions $\geqq 0$, each of which depends on only p of the variables $x_1, \ldots, x_n$.

9. Let $(X_n)_{n\geq 0}$ be an infinite sequence of compact spaces, and let μ_n be a positive me on X_n with total mass 1, for each n.

(a) Show that on $X = \prod_{n=0}^{\infty} X_n$ there exists a unique positive measure μ such tha each integer n and each finite sequence $(f_i)_{0\leq i\leq n}$ of functions $f_i \in \mathscr{C}_{\mathbf{R}}(X_i)$, we $\mu(f) = \prod_{i=0}^{n} \mu_i(f_i)$, where $f(x) = \prod_{i=0}^{n} f_i(\mathrm{pr}_i x)$. (Observe that the continuous func of the form $\prod_{i=0}^{n} f_i(\mathrm{pr}_i x)$, for all choices of n and of $f_i \in \mathscr{C}_{\mathbf{R}}(X_i)$, form a total set i Banach space $\mathscr{C}_{\mathbf{R}}(X)$.) The measure μ is called the *product* of the family (μ_n) a written $\mu = \bigotimes_{n=0}^{n} \mu_n$.

(b) For each n, let A_n be a μ_n-measurable subset of X_n. Show that the pro $A = \prod_{n=0}^{\infty} A_n$ is μ-measurable, and that $\mu(A) = \prod_{n=0}^{\infty} \mu_n(A_n)$.

(c) Let $f \geqq 0$ be a μ-integrable function on X. For each subset L of N, put $L' = N$ and identify X with the product $X_L \times X_{L'}$, where $X_L = \prod_{n\in L} X_n$ and $X_{L'} = \prod_{n\in L'} X$ $x \in X$, put $x_L = \mathrm{pr}_L x$ and $x_{L'} = \mathrm{pr}_{L'} x$, and identify x with $(x_L, x_{L'})$. Finally, le be the product measure $\bigotimes_{n\in L'} \mu_n$ on $X_{L'}$, and put $f_L(x) = \int f(x_L, x_{L'})\, d\mu_{L'}(x_{L'})$.

Now let (L_n) be an increasing sequence of subsets of N, and put $g = \sup_n f_{L_n}$ $h = \sup_n f_{L_n'}$. For each $c > 0$, let A_c be the set of points $x \in X$ at which $g(x)$ and B_c the set of points $x \in X$ at which $h(x) > c$. Show that $c \cdot \mu(A_c) \leqq \int f\, d\mu$ $c \cdot \mu(B_c) \leqq \int f\, d\mu$. (Remark that A_c is the set of points $x \in X$ for which at leas of the $f_{L_n}(x)$ is $> c$, and express A_c as a denumerable union of mutually disjoint G_n such that $c \cdot \mu(G_n) \leqq \int_{G_n} f\, d\mu$.)

(d) Suppose that (L_n) is an increasing sequence of finite subsets of N whose uni N. Show that the f_{L_n} tends to f almost everywhere and that f_{L_n} tends to the cons $\int f\, d\mu$ almost everywhere. (For each $\varepsilon > 0$, consider a continuous function g, dep ing on only finitely many variables, and such that $\int |f - g|\, d\mu \leqq \varepsilon$, and appl to the function $|f - g|$.)

10. Let D be the discrete space $\{0, 1\}$ and X the product space $D^{\mathbf{N}}$. On each factor X_n (of X let μ_n be the measure for which $\mu_n(0) = \mu_n(1) = \frac{1}{2}$, and let $\mu = \bigotimes_{n\in \mathbf{N}} \mu_n$ be product measure on X (Problem 9).

(a) For each point $x = (x_n)_{n\geq 0}$ in X with $x_n = 0$ or $x_n = 1$, put $\pi(x) = \sum_{n=0}^{\infty} x_n 2^-$ Show that π is a continuous mapping of X onto the unit interval $I = [0, 1]$ in R, that the image $\pi(\mu)$ of μ is Lebesgue measure on I. For each $t \in I$, the set $\pi^$ consists of a single point, unless t is of the form $k \cdot 2^{-n}$ with k integral and $0 < k <$ Hence the mapping $f \mapsto f \circ \pi$ induces, on passing to the quotients, an isom isomorphism of $L^p_{\mathbf{C}}(I, \lambda)$ onto $L^p_{\mathbf{C}}(X, \mu)$ for $1 \leqq p \leqq +\infty$.

(b) For each $t \in I$ and each $n \geqq 1$, put $r_n(t) = 1 - 2\mathrm{pr}_{n-1}(\pi^{-1}(t))$ if t is not of form $k \cdot 2^{-m}$ with k integral and $0 \leqq k \leqq 2^m$, and $r_n(t) = 0$ otherwise. The func r_n is called the nth *Rademacher function*. These functions form a nontotal orthonor

system in $\mathscr{L}^2_{\mathbf{C}}(\mathrm{I}, \lambda)$ (show that they are all orthogonal to the functions $\cos 2k\pi t$ for k integral and $\geqq 0$). Show that $r_n(t) = \operatorname{sgn}(\sin 2^n \pi t)$.

(c) For each $x = (x_n)_{n \geqq 0}$ in X, put $u(x) = (x_{n+1})_{n \geqq 0}$. Show that the measure μ is invariant with respect to u and that u is *ergodic* with respect to μ (use (a) and Problem 5(c) of Section 13.12). Deduce that

$$\lim_{n \to \infty} \frac{1}{n} (r_1(t) + \cdots + r_n(t)) = 0$$

almost everywhere with respect to Lebesgue measure ("Borel–Cantelli theorem"). (Observe that $r_n(1 - t) = -r_n(t)$ for $0 \leqq t \leqq \frac{1}{2}$.)

(d) For each finite strictly increasing sequence $(n_l)_{1 \leqq l \leqq k}$ of integers $\geqq 1$, show that

$$\int_0^1 r_{n_1}(t) r_{n_2}(t) \cdots r_{n_k}(t)\, dt = 0.$$

(e) For each finite sequence $(a_k)_{1 \leqq k \leqq n}$ of complex numbers, and each real number $p > 0$, show that

$$\int_0^1 \left| \sum_{k=1}^n a_k r_k(t) \right|^p dt \leqq \left(\frac{p}{2} + 1 \right)^{p/2} \left(\sum_{k=1}^n |a_k|^2 \right)^{p/2}$$

("Khintchine's inequality"). (First consider the case where $p = 2h$, with h an integer $\geqq 1$, by using (d). For $2h - 2 < p \leqq 2h$, use problem 12(e) of Section 13.11.)

(f) With the notation of (e), show that

$$\left(\sum_{k=1}^n |a_k|^2 \right)^{1/2} \leqq 27 \int_0^1 \left| \sum_{k=1}^n a_k r_k(t) \right| dt$$

(use (e) for $p = 4$, and Hölder's inequality for $p = \frac{3}{2}$ and $q = 3$).

11. (a) The hypotheses are those of Section 13.17, Problem 2, and in addition the functions f_n are assumed to be *real*. Let $x \mapsto j(x)$ be a μ-measurable mapping of X into the set $\mathrm{I} = \{1, 2, \ldots, n\}$, and let $k \mapsto w(k)$ be an increasing function on I with values > 0. Then the mapping $(s, u) \mapsto \mathrm{K}_{j(s)}(s, u)/w(j(s))$ of $\mathrm{X} \times \mathrm{X}$ into $\mathbf{R}$ is $\mu \otimes \mu$-measurable. For each μ-measurable subset A of X, show that

$$\int_{\mathrm{X}} d\mu(u) \left(\int_{\mathrm{A}} \frac{\mathrm{K}_{j(s)}(s, u)}{w(j(s))}\, d\mu(s) \right)^2 \leqq \int_{\mathrm{A} \times \mathrm{A}} \frac{|\mathrm{K}_{h(s,t)}(s, t)|}{w^2(h(s, t))}\, d\mu(s)\, d\mu(t)$$

where $h(s, t) = \inf(j(s), j(t))$. (Express the square of the integral on the left as a double integral, so as to reduce the problem to the evaluation of a triple integral over $\mathrm{A} \times \mathrm{A} \times \mathrm{X}$, and use the fact that the f_n are orthogonal.) Show next that

$$\int_{\mathrm{A} \times \mathrm{A}} \frac{|\mathrm{K}_{h(s,t)}(s, t)|}{w^2(h(s, t))}\, d\mu(s)\, d\mu(t) \leqq 2 \int_{\mathrm{A}} d\mu(t) \int_{\mathrm{A}} \frac{|\mathrm{K}_{j(t)}(s, t)|}{w^2(j(s))}\, d\mu(s)$$

(split up $\mathrm{A} \times \mathrm{A}$ into two measurable sets, defined respectively by $j(s) < j(t)$ and $j(s) \geqq j(t)$ for $(s, t) \in \mathrm{A} \times \mathrm{A}$ and use the fact that $\mathrm{K}_n(s, t) = \mathrm{K}_n(t, s)$).

(b) Suppose that there exists an increasing sequence $n \mapsto w(n)$ of numbers > 0 and a μ-measurable subset A of X such that $|\mathrm{H}_n(s)| \leqq c \cdot w(n)$ for all $s \in \mathrm{A}$ and all $n \geqq 1$, where $c > 0$ is a constant. Show that, for each function $g \in \mathscr{L}^2_{\mathbf{R}}(\mathrm{X}, \mu)$, the sequence $(s_n(g)(t))/w(n)$ is bounded for almost all $t \in \mathrm{A}$. (Consider the increasing sequence of μ-integrable functions $v_n(t) = \sup_{1 \leqq k \leqq n} (s_k(g)(t))/w(k)$, and show that the sequence of

integrals $J_n = \int_A v_n(t)\, d\mu(t)$ is bounded above. For this purpose, note that we may write $v_n(t) = (s_{j(t)}(g)(t))/w(j(t))$ for a suitably chosen μ-measurable mapping j of X into I, and majorize J_n^2 by using the Cauchy–Schwarz inequality and (a) above.)

(c) Suppose that the hypotheses of (b) are satisfied and also that $\lim_{n\to\infty} w(n) = +\infty$. If the a_n are real and if the series $\sum_{n=1}^{\infty} a_n^2\, w^2(n)$ converges, show that the series $\sum_{n=1}^{\infty} a_n f_n(t)$ converges almost everywhere in A. (Start by using Problem 8(b) of Section 13.11. Then, in order to majorize $|s_n(t) - s_{n_k}(t)|$ for $n_k \leqq n < n_{k+1}$, determine an increasing sequence (c_n) of numbers >0 such that $\lim_{n\to\infty} c_n = +\infty$ and $\sum_{n=1}^{\infty} a_n^2\, w^2(n) c_n^2 < +\infty$ (Section 5.3, Problem 6), so that the numbers $b_n = a_n w(n) c_n$ are of the form $(g\,|\,f_n)$ for some $g \in \mathscr{L}^2_{\mathbf{R}}(X, \mu)$. Then use the fact that, by virtue of (b), the partial sums $\left|\sum_{k=1}^{n} b_k f_k(t)\right|$ are bounded for almost all $t \in A$, and apply Abel's partial summation formula.)

(d) Deduce from (c) that if $|H_n(s)| \leqq c$ for all $s \in A$ and all n, and if the series $\sum_{n=1}^{\infty} a_n^2$ converges, then the series $\sum_{n=1}^{\infty} a_n f_n(t)$ converges almost everywhere in A. (Use Problem 6 of Section 5.3 again.)

12. (a) Let (c_n) be a sequence of real numbers such that $\sum_{n=1}^{\infty} c_n^2 < +\infty$. Show that the series $\sum_{n=1}^{\infty} c_n r_n(t)$ (the notation is that of Problem 10) converges almost everywhere in I with respect to Lebesgue measure λ. (Express the Rademacher functions as linear combinations of the functions of the Haar orthonrmal system (Section 8.7, Problem 7), and note that Problem 11d) is applicable to the latter orthonormal system (cf. Section 13.17, Problem 2).)

(b) Given any sequence (a_n) of real numbers, a real $\varepsilon > 0$ and a measurable set $A \subset I$, show that there exists an integer n_0 such that

$$\left| \sum_{n \leqq i < j \leqq m} a_i a_j \int_A r_i(t) r_j(t)\, dt \right| \leqq \varepsilon \sum_{k=n}^{m} a_k^2$$

whenever $m > n \geqq n_0$. (Use the Cauchy–Schwarz inequality, the fact that the functions $r_i(t)r_j(t)$ with $i<j$ form an orthonormal system (Problem 10(d)) and Bessel's inequality applied to this system and to the function φ_A.

(c) Let (a_{mn}) be a double sequence of nonnegative numbers such that for each m the set of integers $n \geqq 1$ for which $a_{mn} \neq 0$ is finite, and such that $\lim_{m\to\infty} a_{mn} = 1$ for each n. Also let (c_n) be a sequence of complex numbers such that, if we put

$$S_m(t) = \sum_{n=1}^{\infty} a_{mn} c_n r_n(t),$$

the sequence $(S_m(t))$ is convergent in an integrable set $A \subset I$ of positive measure. Show that $\sum_{n=1}^{\infty} |c_n|^2 < +\infty$. (By virtue of Egoroff's theorem, there exists an integer n_0 and a set $B \subset A$ of positive measure, such that for all $m \geqq n_0$ and $q > p \geqq n_0$, we

have $\left|\sum_{k=p}^{q} a_{mk} c_k r_k(t)\right| \leqq 1$ for all $t \in B$, and therefore

$$\int_B \left|\sum_{k=p}^{q} a_{mk} c_k r_k(t)\right|^2 dt \leqq \lambda(B).$$

Minorize this integral with the help of (b).) (Rademacher–Kolmogoroff theorem.)

13. Let X be a locally compact space, μ a positive measure on X, and $(u_n)_{n \geqslant 0}$ a sequence of μ-integrable complex functions such that, as H runs through the set of finite subsets of N, the set of numbers $\int \left|\sum_{n \in H} u_n(x)\right| d\mu(x)$ is bounded. Show that in these conditions the series $\sum_{n=0}^{\infty} |u_n(x)|^2$ is convergent almost everywhere in X. (Observe that, for all $t \in I = [0, 1]$, the set of numbers $\int \left|\sum_{k=1}^{n} u_k(x) r_k(t)\right| d\mu(x)$ is bounded above by a number independent of n and t; then use Problem 10(f) and the Lebesgue–Fubini theorem.)

14. Let X be a compact space, μ a positive measure on X, and (f_n) an orthonormal sequence in $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$ which is *uniformly bounded* in X.

(a) Let (b_n) be a sequence of real numbers such that $\sum_{n=1}^{\infty} b_n^2 = +\infty$. Show that it is not possible that $\sum_{n=1}^{\infty} |b_n(g \mid f_n)| < +\infty$ for *all* continuous functions g on X. (Argue by contradiction: the functions $u_n(x) = b_n f_n(x)$ would satisfy the conditions of Problem 13, by virtue of the Banach–Steinhaus theorem applied to the linear forms $g \mapsto \sum_{n \in H} b_n(g \mid f_n)$ on $\mathscr{C}(X)$. Hence $\sum_{n=1}^{\infty} b_n^2 |f_n(x)|^2 < +\infty$ almost everywhere in X. Using Egoroff's theorem, the fact that the functions f_n are uniformly bounded and $\int |f_n(x)|^2 \, d\mu(x) = 1$, obtain a contradiction of the hypothesis $\sum_{n=1}^{\infty} b_n^2 = +\infty$.)

(b) Let q be a real number such that $1 < q < 2$. Show that it is not possible that $\sum_{n=1}^{\infty} |(g \mid f_n)|^q < +\infty$ for *all* continuous functions g on X. (If $p = q/(q-1)$, show that if the assertion were false there would exist a sequence (b_n) of real numbers > 0 such that $\sum_n b_n^2 = +\infty$, $\sum_n b_n^p < +\infty$ and $\sum_n |b_n(g \mid f_n)| < +\infty$ for all continuous functions g.)

(c) Deduce from (b) that there exists a continuous function g on X such that, for all q satisfying $1 < q < 2$, we have $\sum_n |(g \mid f_n)|^q = +\infty$. (Use the principle of condensation of singularities (Section 12.16, Problem 14).)

15. On the space $\mathbf{R}^n$, let λ be Lebesgue measure and let $\|x\|$ be a norm for which the unit ball $\|x\| \leqq 1$ has measure 1. Let $(x_k)_{k \geqslant 1}$ be an infinite sequence of distinct points in an integrable bounded set B such that $\lambda(B) = 1$. For each integer $m > 1$, let d_m denote the smallest of the numbers $\|x_i - x_j\|$ with $1 \leqq i < j \leqq m$. Show that $\liminf_{m \to \infty} m \, d_m \leqq c_n^{-1}$, where

$$c_n = 1 + n \int_0^1 \frac{(1-t)^n}{\quad} dt$$

(cf. Section 12.7, Problem 6). (Argue by contradiction, and assume that for some $\varepsilon > 0$ there exists m_0 such that $md_m^n > h_n^n$ for all $m \geqq m_0$, where $h_n^n = c_n^{-1} + \varepsilon$. For $1 \leqq i \leqq m$, let B_i be the ball with center x_i and radius $\frac{1}{2}h_n m^{-1/n}$, and for $m \leqq i \leqq 2^n m$ let B_i be the ball with center x_i and radius $\frac{1}{2}h_n(2i^{-1/n} - m^{-1/n})$. Show that the $2^n m$ balls B_i are pairwise disjoint, and calculate the measure of their union by the Euler–MacLaurin summation formula.)

16. Let X, Y be two locally compact spaces and A a universally measurable subset of $X \times Y$.
(a) For each $x \in X$, show that the section $A(x)$ of A at x is a universally measurable subset of Y. Also, for each measure $\mu \geqq 0$ on Y, the function $x \mapsto \mu^*(A(x))$ is universally measurable (use the Lebesgue–Fubini theorem).
(b) If Y is compact and if A is closed in $X \times Y$, then the function $x \mapsto \mu^*(A(x))$ is upper semicontinuous.
(c) Let μ be a positive measure on Y such that the section $A^{-1}(y)$ is denumerable for almost all $y \in Y$ (with respect to μ). Show that the set N of points $x \in X$ for which $\mu^*(A(x)) > 0$ contains no nondenumerable compact subset. (By using Section 3.9, Problem 4, reduce to the case where this compact set contains no isolated point, and show that such a set is the support of a diffuse measure $\neq 0$, by using Section 13.18, Problem 6(b) and Section 4.2, Problem 3(b).)

17. Let μ be a bounded positive measure on X, with total mass 1. If $f \in \mathscr{L}^1_{\mathbf{C}}$ is such that $\int \log|1 + \zeta f| \, d\mu \leqq 0$ for every complex number ζ, then f is μ-negligible. (Use the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1 - \zeta e^{it}| \, dt = \log^+|\zeta|$$

(which is valid for all $\zeta \in \mathbf{C}$) together with the Lebesgue–Fubini theorem, to evaluate the integral $\int \log^+|Rf| \, d\mu$, where $R > 0$, and deduce that this integral must be zero.)

18. Let I denote the discrete space $\{0, 1, \ldots, m-1\}$. Let X denote the compact product space $I^{\mathbf{Z}}$, and μ a positive measure on I with total mass 1 (so that μ is defined by the finite sequence of masses $p_j = \mu(\{j\})$ such that $p_j \geqq 0$ and $\sum_{j=0}^{m-1} p_j = 1$). On each factor X_n of X let μ_n denote the measure μ, and let ν be the product measure $\bigotimes_{n \in \mathbf{Z}} \mu_n$ on X (Problem 9).
(a) For each $x = (x)_{n \in \mathbf{Z}}$ in X, put $u(x) = (x_{n+1})_{n \in \mathbf{Z}}$. Show that u is a homeomorphism of X onto X and that the measure ν is invariant under u. The triple (X, ν, u) is called the *Bernoulli scheme* $B(p_0, \ldots, p_{m-1})$.
(b) Consider in particular the Bernoulli scheme $B(\frac{1}{2}, \frac{1}{2})$. For each $x = (x_n)_{n \in \mathbf{Z}} \in X$, where $x_n = 0$ or 1 for all $n \in \mathbf{Z}$, let $f(x)$ denote the canonical image in $\mathbf{T}^2$ of the point $(y, z) \in \mathbf{R}^2$, where $y = \sum_{n=0}^{\infty} x_{-n} 2^{-n-1}$ and $z = \sum_{n=1}^{\infty} x_n 2^{-n}$. Let β be the normalized Haar measure (14.3) on the compact group $\mathbf{T}^2$. Show that f is continuous and that $f(\nu) = \beta$; also that the set of points $t \in \mathbf{T}^2$ such that $f^{-1}(t)$ does not consist of a single point is β-negligible.

Let φ be the restriction of the canonical mapping $\pi : \mathbf{R}^2 \to \mathbf{T}^2$ to the set K of points (x, y) such that $0 \leqq x < 1$ and $0 \leqq y < 1$, so that φ is a continuous bijection of K onto $\mathbf{T}^2$. For $t \in \mathbf{T}^2$ and $(x, y) = \varphi^{-1}(t)$, put

$$v(t) = \begin{cases} \pi(2x, \tfrac{1}{2}y) & \text{if } 0 \leqq x < \tfrac{1}{2}, \\ \pi(2x, \tfrac{1}{2}(y+1)) & \text{if } \tfrac{1}{2} \leqq x < 1. \end{cases}$$

Show that the mapping $v : \mathbf{T}^2 \to \mathbf{T}^2$ ("baker's transformation") is continuous almost everywhere and that $v \circ f = f \circ u$ (where u is the mapping defined in (a)).

We shall see later that u and v are ergodic (relative to ν and β, respectively) (Section 15.11, Problem 16).

(c) Show that for the Bernoulli scheme $B(p_0, \ldots, p_{k-1})$, the entropy $h(u)$ (Section 13.9, Problem 28) is given by

$$h(u) = -(p_0 \log p_0 + \cdots + p_{k-1} \log p_{k-1}).$$

(Use the Kolmogoroff–Sinai theorem, by starting with the partition $\alpha = (A_j)_{0 \leqq j \leqq k-1}$, where A_j is the set of $(x_n)_{n \in \mathbf{Z}}$ such that $x_0 = j$. Use Problem 9(a) and the definition of u, and remark that the set

$$A_{j_0} \cap u^{-1}(A_{j_1}) \cap \cdots \cap u^{-n+1}(A_{j_{n-1}})$$

is the set of $(x_n)_{n \in \mathbf{Z}}$ such that $x_0 = j_0, x_1 = j_1, \ldots, x_{n-1} = j_{n-1}$. The problem is then to calculate the sum

$$\sum_{m_0 + \cdots + m_{k-1} = n} \frac{n!}{m_0! \cdots m_{k-1}!} p_0^{m_0} \cdots p_{k-1}^{m_{k-1}} (m_0 \log p_0 + \cdots + m_{k-1} \log p_{k-1}).$$

For this purpose, observe that

$$\sum_{m_0 + \cdots + m_{k-1} = n} \frac{n!}{m_0! \cdots m_{k-1}!} m_0 p_0^{m_0 - 1} p_1^{m_1} \cdots p_{k-1}^{m_{k-1}} = \frac{d}{dp_0}(p_0 + p_1 + \cdots + p_{k-1})^n.)$$

Deduce from this calculation that the bijections u of the Bernoulli schemes $B(\frac{1}{2}, \frac{1}{2})$ and $B(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ are not conjugate (Section 13.12, Problem 11).

19. Let X be a locally compact space, let μ be a positive measure on X, and let f, g be two nonnegative μ-measurable functions on X. For each $\alpha > 0$, let A_α be the set of $x \in X$ such that $f(x) > \alpha$. Suppose that:

(i) $\mu(A_\alpha) < +\infty$ for all $\alpha > 0$;

(ii) $g \in \mathscr{L}^p_{\mathbf{R}}(X, \mu)$ for some $p \in]1, +\infty[$;

(iii) for each $\alpha > 0$, $\mu(A_\alpha) \leqq \dfrac{1}{\alpha} \displaystyle\int_{A_\alpha} g \, d\mu$.

Show that these conditions imply that $f \in \mathscr{L}^p_{\mathbf{R}}(X, \mu)$ and that

$$N_p(f) \leqq \frac{p}{p-1} N_p(g)$$

(Wiener's inequality). (Consider first the case where f is bounded and of compact support, and integrate from 0 to ∞ the inequality

$$t^{p-1} \int \varphi_{A_t} \, d\mu \leqq t^{p-2} \int g \varphi_{A_t} \, d\mu$$

(cf. Problem 1). For the general case, consider the functions $\sup(n, f\varphi_{K_n})$, where (K_n) is an increasing sequence of compact sets whose union is X.)

20. Let U be a continuous endomorphism of the space $L^1_{\mathbf{R}}(X, \mu)$ satisfying the hypotheses of Section 13.11, Problem 17, so that U extends to an endomorphism of each of the spaces $L^p_{\mathbf{R}}(X, \mu)$, $1 \leqq p \leqq +\infty$ (*loc. cit.*). For each function $f \in \mathscr{L}^p_{\mathbf{R}}$, put

$$R_n(f) = \frac{1}{n}(f + U \cdot f + \cdots + U^{n-1} \cdot f),$$

$$R_n^*(f) = \sup_{1 \leqslant k \leqslant n} |R_k(f)|,$$

$$R^*(f) = \sup_n R_n^*(f).$$

(a) Show that if $1 < p < +\infty$ and $f \in \mathscr{L}^p_{\mathbf{R}}$, then $R^*(f) \in \mathscr{L}^p_{\mathbf{R}}$ and

$$N_p(R^*(f)) \leqq \frac{p}{p-1} N_p(f).$$

(Use Problem 19 above, and Section 13.11, Problem 17(c).)

(b) For each function $f \in \mathscr{L}^2_{\mathbf{R}}$, let $P \cdot f$ be the limit in $L^2_{\mathbf{R}}$ of the sequence $((R_n(f))^{\sim})$ (Section 12.15, Problem 12(c)). We have $UP = PU = P$. Show that the sequence $(R_n(f))$ converges almost everywhere to $P \cdot f$ (Dunford–Schwartz ergodic theorem). (Put

$$S(f) = \limsup_{n \to \infty} R_n(f),$$

$$I(f) = \liminf_{n \to \infty} R_n(f).$$

Observe that $I(f) \leqq S(f) \leqq R^*(f)$, and that $S(R_m(f) - P \cdot f) = S(f) - P \cdot f$ and $I(R_m(f) - P \cdot f) = I(f) - P \cdot f$ for all $m \geqq 1$, and use (a) with $R_m(f) - P \cdot f$ in place of f.)

(c) Show that $N_1(P \cdot f) \leqq N_1(f)$ for all $f \in \mathscr{L}^1_{\mathbf{R}} \cap \mathscr{L}^\infty_{\mathbf{R}}$, and hence that P extends to a contraction on the space $L^1_{\mathbf{R}}$. Deduce that the sequence $(R_n(f))$ converges almost everywhere to f, for each function $f \in \mathscr{L}^1_{\mathbf{R}}$. (Put

$$L(f) = \limsup_{n \to \infty} |R_n(f) - P \cdot f|,$$

observe that $L(f) \leqq R^*(f - g) + P \cdot |f - g|$ for all $g \in \mathscr{L}^1_{\mathbf{R}} \cap \mathscr{L}^\infty_{\mathbf{R}}$, and make use of Section 13.11, Problem 7(d).)

21. The notation is that of Section 13.17, Problem 7. For each $\alpha > 0$ and each function $g \in \mathscr{L}^1_{\mathrm{loc},C}(Y, \nu)$, let $A_\alpha(g)$ denote the set of points $y \in Y$ such that $|g(y)| > \alpha$. If $p \in [1, +\infty[$, the endomorphism U is said to be of *weak type* (p, p) if there exists a constant $C > 0$ such that, for each μ-integrable step function f and each $\alpha > 0$, we have

$$\nu(A_\alpha(U \cdot f)) \leqq \frac{C}{\alpha^p} \int |f|^p \, d\mu.$$

Let p, q be such that $1 \leqq p < q < +\infty$. Show that, if U is of weak type (p, p) and also of weak type (q, q), then U is of type (r, r) (Section 13.17, Problem 7) for all r such that $p < r < q$ (Marcinkiewicz's interpolation theorem). (For each $f \in \mathscr{L}^1_{\mathbf{C}}(X, \mu)$ and $\alpha > 0$, let f'_α be the function which is equal to $f(x)$ if $|f(x)| > \alpha$, and zero otherwise, and put $f''_\alpha = f - f'_\alpha$. Show that, for each $t > 0$,

$$t^{r-1}\mu(A_{2t}(U \cdot f)) \leqq C' t^{r-p-1} \int |f'_t|^p \, d\mu + C'' t^{r-q-1} \int |f''_t|^q \, d\mu.$$

Integrate from 0 to ∞ and use Problem 1.)

CHAPTER XIV

INTEGRATION IN LOCALLY COMPACT GROUPS

Haar measure and convolution on arbitrary locally compact groups have become indispensable tools for the modern analyst, as they were in classical analysis on the real line and in finite-dimensional Euclidean spaces. Together with convolution of distributions, which generalizes convolution of measures and which we shall introduce in Chapter XVII, they are the fundamental notions in harmonic analysis (Chapter XXII) and the theory of linear representations of compact groups (Chapter XXI).

We have again followed, though in less detail, the exposition of N. Bourbaki [22]. In fact, since in this treatise practically the only locally compact groups we shall consider will be *Lie groups* (Chapters XVI, XIX, and XXI), for which there is a much simpler proof of the existence of a Haar measure, we could have restricted ourselves entirely to Lie groups. Nevertheless, it seemed worthwhile to bring out the fact that the theory of integration on a locally compact group is entirely independent of any differential structure; and the totally discontinuous locally compact groups have ceased to be mere curiosities since the advent of p-adic and "adelic" groups in the theory of numbers [36].

Throughout this chapter, the phrase "locally compact group" will mean "separable metrizable locally compact group."

1. EXISTENCE AND UNIQUENESS OF HAAR MEASURE

Let G be a (separable, metrizable) locally compact group. If f is any mapping of G into a set E and if s is any element of G, we define the *left* and *right translations* $\gamma(s)f$ and $\delta(s)f$ of f by s, which are mappings of G into E, by

the formulas

$$(\gamma(s)f)(x) = f(s^{-1}x), \qquad (\delta(s)f)(x) = f(xs). \tag{14.1.1}$$

It follows immediately from this definition that we have

$$\gamma(st)f = \gamma(s)(\gamma(t)f), \qquad \delta(st)f = \delta(s)(\delta(t)f) \tag{14.1.1.1}$$

for all s, $t \in G$.

If μ is a (complex) measure on G, we denote by $\gamma(s)\mu$ and $\delta(s)\mu$ the measures on G which are the images of μ under the homeomorphisms $x \mapsto sx$ and $x \mapsto xs^{-1}$, respectively (13.1.6), so that

$$\langle f, \gamma(s)\mu\rangle = \langle \gamma(s^{-1})f, \mu\rangle, \qquad \langle f, \delta(s)\mu\rangle = \langle \delta(s^{-1})f, \mu\rangle \tag{14.1.2}$$

for all functions $f \in \mathscr{K}_{\mathbf{c}}(G)$. From this definition it follows that

$$\gamma(st)\mu = \gamma(s)(\gamma(t)\mu), \qquad \delta(st)\mu = \delta(s)(\delta(t)\mu) \tag{14.1.2.1}$$

for all s, $t \in G$.

The measure μ is said to be *left* (resp. *right*) *invariant* if

$$\gamma(s)\mu = \mu \quad (\text{resp. } \delta(s)\mu = \mu) \tag{14.1.2.2}$$

for all $s \in G$.

If a measure $\mu \neq 0$ on G is left invariant, then $\mathrm{Supp}(\mu) = G$, because $\mathrm{Supp}(\gamma(s)\mu) = s \cdot \mathrm{Supp}(\mu)$ for all $s \in G$ by virtue of (13.19.4), and $\mathrm{Supp}(\mu) \neq \varnothing$. Similarly for right-invariant measures.

Let μ be a left-invariant measure on G and let f be a μ-integrable mapping of G into $\overline{\mathbf{R}}$ or $\mathbf{C}$. Then for each $s \in G$ the function $x \mapsto f(s^{-1}x)$ is also μ-integrable, and we have

$$\int f(s^{-1}x)\, d\mu(x) = \int f(x)\, d\mu(x) \tag{14.1.2.3}$$

by (13.7.10). In particular, if A is a μ-integrable set, then so is sA, and

$$\mu(s\mathrm{A}) = \mu(\mathrm{A}). \tag{14.1.2.4}$$

If f is any mapping of G into a set E, we write

$$\check{f}(x) = f(x^{-1}) \qquad \text{for all} \quad x \in G. \tag{14.1.3}$$

If μ is any measure on G, we denote by $\check{\mu}$ the image of μ under the mapping $x \mapsto x^{-1}$ of G onto G, so that we have

$$\langle f, \check{\mu} \rangle = \langle \check{f}, \mu \rangle \tag{14.1.4}$$

for all $f \in \mathscr{K}_{\mathbf{C}}(G)$. It follows immediately from the definitions that we have $(\gamma(s)f)^{\check{}} = \delta(s)\check{f}$, each side being the function $x \mapsto f(s^{-1}x^{-1})$; and therefore, for any measure μ on G, we have $(\gamma(s)\mu)^{\check{}} = \delta(s)\check{\mu}$. Hence if μ is a left-invariant measure, $\check{\mu}$ is a right-invariant measure, and vice versa.

(14.1.5) *Let* G *be a locally compact group. Then there exists a nonzero left-invariant positive measure* μ *on* G, *and every other left-invariant measure on* G *is of the form* $a\mu$, *where* $a \in \mathbf{C}$.

(1) *Existence.* Let $\mathscr{K}_+^*$ denote the set of functions $g \in \mathscr{K}_{\mathbf{R}}(G)$ which are $\geqq 0$ and not the zero function. For each $f \in \mathscr{K}_{\mathbf{R}}(G)$ and each $g \in \mathscr{K}_+^*$, there exist positive real numbers $c_1, \ldots, c_r$ and elements $s_1, \ldots, s_r$ in G such that

$$f \leqq \sum_{i=1}^{r} c_i \gamma(s_i) g \tag{14.1.5.1}$$

(i.e., such that $f(x) \leqq \sum_{i=1}^{r} c_i g(s_i^{-1}x)$ for all $x \in G$). For there exists a non-empty open subset U in G such that $a = \inf_{x \in U} g(x) > 0$; since $\mathrm{Supp}(f)$ is compact, there exist a finite number of points $s_i \in G$ $(1 \leqq i \leqq r)$ such that the $s_i U$ cover $\mathrm{Supp}(f)$, and then (14.1.5.1) is satisfied by taking $c_i = \|f\|/a$ for all i. We shall denote by $(f : g)$ the *greatest lower bound* of the numbers $\sum_{i=1}^{r} c_i$, for all systems $(c_1, \ldots, c_r, s_1, \ldots, s_r)$ satisfying (14.1.5.1). The symbol $(f : g)$ has the following properties:

(i) $(\gamma(s)f : g) = (f : g)$ for all $f \in \mathscr{K}_{\mathbf{R}}(G)$, $g \in \mathscr{K}_+^*$, $s \in G$;

(ii) $(af : g) = a(f : g)$ for all $f \in \mathscr{K}_{\mathbf{R}}(G)$, $g \in \mathscr{K}_+^*$, $a \geqq 0$;

(iii) $(f_1 + f_2 : g) \leqq (f_1 : g) + (f_2 : g)$ for all $f_1, f_2 \in \mathscr{K}_{\mathbf{R}}(G)$, $g \in \mathscr{K}_+^*$;

(iv) $(f : g) \geqq \sup_{x \in G} f(x) / \sup_{x \in G} g(x)$ for all $f \in \mathscr{K}_{\mathbf{R}}(G)$, $g \in \mathscr{K}_+^*$;

(v) $(f : h) \leqq (f : g)(g : h)$ for all $f \in \mathscr{K}_{\mathbf{R}}(G)$ and g, h in $\mathscr{K}_+^*$;

(vi) $0 < \dfrac{1}{(f_0 : f)} \leqq \dfrac{(f : g)}{(f_0 : g)} \leqq (f : f_0)$ for all f, f_0, g in $\mathscr{K}_+^*$.

Properties (i), (ii), and (iii) are immediate consequences of the definitions. Given (14.1.5.1), there exists (3.17.10) $s \in G$ such that

$$\sup_{x \in G} f(x) = f(s) \leqq \sum_{i=1}^{r} c_i g(s_i^{-1} s) \leqq \left(\sum_{i=1}^{r} c_i \right) \sup_{x \in G} g(x),$$

and (iv) follows. For (v), observe that if we have $f \leqq \sum_i a_i \gamma(s_i) g$ and $g \leqq \sum_j b_j \gamma(t_j) h$, then $f \leqq \sum_{i,j} a_i b_j \gamma(s_i t_j) h$, and therefore

$$(f : h) \leqq \sum_{i,j} a_i b_j = \Big(\sum_i a_i \Big) \Big(\sum_j b_j \Big).$$

Since we may take $\sum_i a_i$ (resp. $\sum_j b_j$) arbitrarily close to $(f : g)$ (resp. $(g : h)$), we have (v). Finally, (vi) follows from (v) applied to f_0, f, g and to f, f_0, g.

By hypothesis, there exists a denumerable fundamental system (V_n) of neighborhoods of the neutral element e in G. For each n, let g_n be a function belonging to $\mathscr{K}_+^*$ such that $\mathrm{Supp}(g_n) \subset V_n$ (4.5.2). Let f_0 be a function in $\mathscr{K}_+^*$, fixed once and for all, and put

(14.1.5.2) $$I_n(f) = (f : g_n)/(f_0 : g_n)$$

for all $f \in \mathscr{K}_+^*$; let $I_n(0) = 0$. From (ii) and (iii) above it follows immediately that the mappings $f \mapsto I_n(|f|)$ are *seminorms* on $\mathscr{K}_{\mathbf{R}}(G)$, and from (i) that $I_n(\gamma(s) f) = I_n(f)$ for all $s \in G$.

Next, there exists a sequence of relatively compact open sets U_p which cover G and are such that $\bar{U}_p \subset U_{p+1}$ (3.18.3). The space $\mathscr{C}(U_p)$ is separable (7.4.4) and therefore so is $\mathscr{K}(G; \bar{U}_p) \cap \mathscr{K}_+^*$ (3.10.9). Hence there exists a dense sequence $(f_{mp})_{m \geqq 1}$ in the latter space. Since the sequence of values $I_n(f_{mp})$ $(n \geqq 1)$ lies in the compact interval with endpoints $1/(f_0 : f_{mp})$ and $(f_{mp} : f_0)$ by (vi), it follows from (12.5.9) that if we replace the sequence (g_n) by a suitably chosen subsequence, we may assume that the sequence $(I_n(f_{mp}))_{n \geqq 1}$ tends to a limit > 0 *for all* m, p. Also it is clear that if f, f' in $\mathscr{K}_+^*$ are such that $f \leqq f'$, then $I_n(f) \leqq I_n(f')$. Let h_p be a continuous mapping of G into [0, 1] with compact support, taking the value 1 on $\bar{U}_p$ ((3.18.2) and (4.5.2)). Then, for each function $f \in \mathscr{K}(G; \bar{U}_p) \cap \mathscr{K}_+^*$, we have $I_n(f) \leqq \|f\| I_n(h_p)$ for all n, and therefore

$$|I_n(f) - I_n(f')| \leqq I_n(|f - f'|) \leqq I_n(h_p) \|f - f'\|$$

for all $f, f' \in \mathscr{K}(G, \bar{U}_p) \cap \mathscr{K}_+^*$ and all n, because $f \mapsto I_n(|f|)$ is a seminorm. In other words, the set of restrictions of the I_n to each of the subspaces $\mathscr{K}(G, \bar{U}_p) \cap \mathscr{K}_+^*$ is *equicontinuous*. By (7.5.5) it follows that $\lim_{n \to \infty} I_n(f) = I(f)$ exists for all $f \in \mathscr{K}_+^*$, and takes values $\neq 0$.

We shall now show that

(14.1.5.3) $$\mathrm{I}(f+f') = \mathrm{I}(f) + \mathrm{I}(f')$$

for all f, f' in $\mathscr{K}_+^*$. By (iii), it is enough to prove that $\mathrm{I}(f) + \mathrm{I}(f') \leqq \mathrm{I}($ Let $\varepsilon > 0$ be a real number, and let h be a function $\geqq 0$ belonging to such that $h(x) \geqq 1$ on the union of the (compact) supports of f and f' (and (4.5.2)). Then it is enough to prove that there exists a compact ne hood V of e in G, such that, for all $g \in \mathscr{K}_+^*$ with $\mathrm{Supp}(g) \subset \mathrm{V}$, we h

(14.1.5.4) $$(f : g) + (f' : g) \leqq (f+f' : g) + \varepsilon(h : g).$$

To prove this, put $u = f + f' + \frac{1}{2}\varepsilon h$, and let v (resp. v') be the f which coincides with f/u (resp. f'/u) on $\mathrm{Supp}(f+f')$ and is zero on t plement of this set. Since at every frontier point x of $\mathrm{Supp}(f+f')$ $f(x) + f'(x) = 0$, and therefore $f(x) = f'(x) = 0$, it follows that v belong to $\mathscr{K}_{\mathbf{R}}(\mathrm{G})$ and are $\geqq 0$. The functions v and v' are therefore un continuous with respect to a left-invariant distance defining the topolo ((12.9.1) and (3.16.5)), and therefore, for each $\eta > 0$, there exists a c neighborhood V of e such that $|v(s) - v(t)| \leqq \eta$ and $|v'(s) - v'(t)| \leqq \eta$ pairs (s, t) such that $s^{-1}t \in \mathrm{V}$. Now let $g \in \mathscr{K}_+^*$ be such that Supp(For each $s \in \mathrm{G}$ we have $v \cdot \gamma(s)g \leqq (v(s) + \eta) \cdot \gamma(s)g$. (This is obviously points where $\gamma(s)g$ vanishes, hence at points outside $s\mathrm{V}$; and if $t \in s\mathrm{V}$ $v(t) \leqq v(s) + \eta$.) Similarly, we have $v' \cdot \gamma(s)g \leqq (v'(s) + \eta)\gamma(s)g$. This b let c_i $(1 \leqq i \leqq n)$ be real numbers $\geqq 0$, and s_i $(1 \leqq i \leqq n)$ elements of that $u \leqq \sum_{i=1}^{n} c_i \gamma(s_i)g$. Then we have

$$f = vu \leqq \sum_{i=1}^{n} c_i\, v \cdot \gamma(s_i)g \leqq \sum_{i=1}^{n} c_i(v(s_i) + \eta)\gamma(s_i)g$$

and a similar set of inequalities with f replaced by f'. Consequently

$$(f : g) + (f' : g) \leqq \sum_{i=1}^{n} c_i(v(s_i) + v'(s_i) + 2\eta) \leqq (1 + 2\eta)\sum_{i=1}^{n} c_i,$$

because $v + v' \leqq 1$. From the definition of u, and properties (ii), (iii), we obtain

$$\begin{aligned}(f : g) + (f' : g) &\leqq (1 + 2\eta)(u : g) \leqq (1 + 2\eta)((f+f' : g) + \tfrac{1}{2}\varepsilon(h : g)) \\ &\leqq (f+f' : g) + \tfrac{1}{2}\varepsilon(h : g) + 2\eta((f+f') : h)(h : g) + \varepsilon\end{aligned}$$

so that, by taking η such that

$$\eta(2((f+f') : h) + \varepsilon) \leqq \tfrac{1}{2}\varepsilon,$$

we get (14.1.5.4)

Now extend I to the whole of $\mathscr{K}_{\mathbf{R}}(G)$ by putting $I(0) = 0$ and, for each $f = f_1 - f_2$ with f_1, f_2 in $\mathscr{K}_+^*$, $I(f) = I(f_1) - I(f_2)$. From (14.1.5.3) it follows immediately that $I(f)$ depends only on f and not on the choice of expression $f = f_1 - f_2$. Clearly (14.1.5.3) remains valid for all f, f' in $\mathscr{K}_{\mathbf{R}}(G)$. Furthermore, the original definition of I shows immediately that $I(\lambda f) = \lambda I(f)$ for all $f \in \mathscr{K}_+^*$ and all real numbers $\lambda > 0$; and this relation obviously extends to the case where $f \in \mathscr{K}_{\mathbf{R}}(G)$ and $\lambda \in \mathbf{R}$. We may therefore conclude from (13.3.1) that I is a *positive measure* on G; also I is not zero and, by construction, $I(\gamma(s)f) = I(f)$ for all $f \in \mathscr{K}_{\mathbf{R}}(G)$. In other words, we have constructed a *nonzero left-invariant positive measure* on G.

(2) *Uniqueness.* Let μ (resp. ν) be a nonzero left (resp. right) invariant measure on G. Then $\check{\nu}$ is a left-invariant measure. We shall show that μ and $\check{\nu}$ are proportional, and this will complete the proof of (14.1.5). Let $f \in \mathscr{K}_{\mathbf{C}}(G)$ be such that $\mu(f) \neq 0$, and consider the function D_f on G defined by

$$D_f(s) = \mu(f)^{-1} \int f(t^{-1}s)\, d\nu(t).$$

We shall show that D_f is *continuous* on G. In fact this will be a consequence of the following more general result:

(14.1.5.5) *Let* G *be a locally compact group,* H *a closed subgroup,* α *a measure on* H, *and* $f: G \to \mathbf{C}$ *a continuous mapping. Suppose that either* $\mathrm{Supp}(f)$ *is compact or* $\mathrm{Supp}(\alpha)$ *is compact. Then the mappings*

$$s \mapsto \int f(st)\, d\alpha(t) \qquad \text{and} \qquad s \mapsto \int f(ts)\, d\alpha(t)$$

are continuous on G.

Consider, for example, the first of these integrals. Let $s_0 \in G$ and let V_0 be a compact neighborhood of s_0. Given $\varepsilon > 0$, we have to find a neighborhood $V \subset V_0$ of s_0 such that for all $s \in V$ we have $\left|\int (f(st) - f(s_0 t))\, d\alpha(t)\right| \leqq \varepsilon$. If $K = \mathrm{Supp}(f)$ is compact and $L = V_0^{-1}K$, then

$$\int (f(st) - f(s_0 t))\, d\alpha(t) = \int_L (f(st) - f(s_0 t))\, d\alpha(t);$$

since f is uniformly continuous (with respect to a right-invariant distance on G) (3.16.5), there exists a neighborhood W of e in G such that the relation $s \in Ws_0$ implies $|f(st) - f(s_0 t)| \leqq \varepsilon/|\alpha|(L)$ for all $t \in G$, and we may take $V = V_0 \cap Ws_0$. If on the other hand $S = \mathrm{Supp}(\alpha)$ is compact, we have

$$\int (f(st) - f(s_0 t))\, d\alpha(t) = \int_S (f(st) - f(s_0 t))\, d\alpha(t),$$

and if $t \in S$ and $s \in V_0$, then $st \in V_0 S$, which is compact. The restriction $f | V_0 S$ is uniformly continuous and therefore we may take W so that the relation $s \in W s_0$ implies $|f(st) - f(s_0 t)| \leqq \varepsilon / |\alpha|(S)$ for all $t \in S$; as before, we take $V = V_0 \cap W s_0$.

Now let g be any function belonging to $\mathscr{K}_c(G)$. Then the function $(s, t) \mapsto f(s)g(ts)$ is continuous on $G \times G$ and has compact support. By (13.21.7) and the left-invariance of μ, we have

$$\begin{aligned}\mu(f)\nu(g) = \left(\int f(s)\, d\mu(s)\right)\left(\int g(t)\, d\nu(t)\right) &= \int d\mu(s) \int f(s)g(ts)\, d\nu(t) \\ &= \int d\nu(t) \int f(s)g(ts)\, d\mu(s) \\ &= \int d\nu(t) \int f(t^{-1}s)g(s)\, d\mu(s) \\ &= \mu(g \cdot \mu(f)D_f) = \mu(f)\mu(D_f \cdot g),\end{aligned}$$

and since $\mu(f) \neq 0$ by hypothesis, it follows that

$$\nu(g) = \mu(D_f \cdot g).$$

This shows first of all that D_f does not depend on f, for if f' is another function in $\mathscr{K}_c(G)$ such that $\mu(f') \neq 0$, then it follows from above that $D_f \cdot \mu = D_{f'} \cdot \mu$, and hence (13.15.3) that D_f and $D_{f'}$ are equal almost everywhere with respect to μ. But because $\mu \neq 0$ and is invariant, its support is the whole of G; also we have seen that D_f and $D_{f'}$ are continuous on G; hence the set of $s \in G$ at which $D_f(s) \neq D_{f'}(s)$ is open and negligible, therefore empty. In other words, $D_f = D_{f'} = D$, say. We have therefore, by the definition of the function D,

$$\mu(f)D(e) = \check{\nu}(f)$$

for every $f \in \mathscr{K}_c(G)$ such that $\mu(f) \neq 0$. This formula, being true in the complement of a hyperplane in $\mathscr{K}_c(G)$, is true in the whole of $\mathscr{K}_c(G)$, since both sides are linear forms in f. Since $\nu \neq 0$ we have $D(e) \neq 0$; therefore μ and $\check{\nu}$ are proportional, and the proof of (14.1.5) is complete. Q.E.D.

Any left (resp. right) invariant *positive* nonzero measure on G is called a *left* (resp. *right*) *Haar measure* on G. From (14.1.5), any two left (resp. right) Haar measures on G are proportional.

PROBLEMS

1. Let G be a locally compact group, A a dense subset of G, μ a left Haar measure on G, and H a μ-measurable subset of G with the following property: for each $s \in A$, the sets $sH \cap \complement H$ and $H \cap \complement sH$ are μ-negligible. Show that either H or its complement is negligible (prove that the measure $\varphi_H \cdot \mu$ is left-invariant).

2. Let G be a locally compact group, μ a left Haar measure on G, and A, B subsets of G.
(a) Suppose that one of the following two conditions is satisfied: (α) A is μ-integrable; (β) $\mu^*(A) < +\infty$, and B is μ-measurable. Show that in either case the function $f(s) = \mu^*(sA \cap B)$ is uniformly continuous on G with respect to a right-invariant distance on G. (For any two subsets M, N of G, put

$$\rho(M, N) = \mu^*((M \cap \complement N) \cup (N \cap \complement M)).$$

Consider first the case where A is compact, and show that for each $\varepsilon > 0$ there exists a neighborhood U of e in G such that $\rho(sA \cap B, stA \cap B) \leqq \varepsilon$ for all $s \in G$ and all $t \in U$. Then apply Problem 5 of Section 13.9. If B is μ-measurable and $\mu^*(A) < +\infty$, observe that there exists a decreasing sequence (A_n) of μ-integrable subsets of G containing A and such that $\inf(\mu(A_n)) = \mu(A)$, and show that

$$\mu^*(sA \cap B) = \inf(\mu^*(sA_n \cap B))$$

(Section 13.9, Problem 2(a)). Use the fact that $\mu(A_n - A_{n+1})$ tends to 0 with $1/n$.)
(b) If A is μ-integrable and $\mu^*(B) < +\infty$, then the function f is also uniformly continuous with respect to a left-invariant distance on G. If also A^{-1} is μ-integrable, then $\int_G f(s)\, ds = \mu(A^{-1})\mu^*(B)$. (Reduce to the case where B is μ-integrable, and observe that in that case $\mu(sA \cap B) = \mu(A \cap s^{-1}B)$, and that $\varphi_{sA \cap B} = \varphi_{sA}\varphi_B$ and $\varphi_{sA}(t) = \varphi_{tA-1}(s)$.)
(c) Deduce from (a) that in the two cases considered there, the interiors of AB and BA are nonempty if A and B are not μ-negligible.
(d) In the group $G = \mathbf{SL}_2(\mathbf{R})$, give an example of a compact set A and a μ-measurable set B such that the function $f(s) = \mu(sA \cap B)$ is not uniformly continuous with respect to a left-invariant measure on G. (Observe that there exists a sequence (t_n) of elements of G tending to e and a sequence (s_n) of elements of G such that the sequence $s_n^{-1} t_n s_n$ tends to the point at infinity.)

3. Let G be a locally compact group, μ a left Haar measure on G, and A an integrable subset of G such that $\mu(A) > 0$. Show that the set H(A) of elements $s \in G$ such that $\mu(A) = \mu(A \cap sA)$ is a compact group. (Use Problem 2 to show that H(A) is closed in G. To show that H(A) is compact, consider a compact subset B of A such that $\mu(B) > \frac{1}{2}\mu(A)$, and prove that $H(A) \subset BB^{-1}$.)

4. Let G be a *commutative* locally compact group, written additively. Let μ be a Haar measure on G and let A, B be two integrable subsets of G.
(a) For each $s \in G$ let

$$A' = \sigma_s(A, B) = A \cup (B + s), \qquad B' = \tau_s(A, B) = (A - s) \cap B.$$

Show that $\mu(A') + \mu(B') = \mu(A) + \mu(B)$ and that $A' + B' \subset A + B$. (Note that $A + \phi = \phi$ for all subsets A of G.)

(b) Suppose that $0 \in A \cap B$. A pair (A', B') of integrable subsets of G is said to be *derived from* (A, B) if there exists a sequence $(s_k)_{1 \leqslant k \leqslant n}$ of elements of G and two sequences $(A_k)_{0 \leqslant k \leqslant n}$ and $(B_k)_{0 \leqslant k \leqslant n}$ of subsets of G such that

$$A_0 = A, \qquad B_0 = B, \qquad A_k = \sigma_{s_k}(A_{k-1}, B_{k-1}), \qquad B_k = \tau_{s_k}(A_{k-1}, B_{k-1})$$

for $1 \leqq k \leqq n$, and $s_k \in A_{k-1}$ for $1 \leqq k \leqq n$, and $A' = A_n$, $B' = B_n$. Show that there exists a sequence (E_n, F_n) of pairs of subsets of G such that (i) $E_0 = A$ and $F_0 = B$, (ii) (E_{n+1}, F_{n+1}) is derived from (E_n, F_n), (iii) $\mu((E_n - s) \cap F_n) \geqq \mu(F_{n+1}) - 2^{-n}$ for all n and all $s \in E_n$. Let $E_\infty = \bigcup_n E_n$, $F_\infty = \bigcap_n F_n$. Show that for each $s \in E_\infty$ we have $\mu((E_\infty - s) \cap F_\infty) = \mu(F_\infty)$.

(c) Suppose that $\mu(F_\infty) > 0$. Show that the function

$$f(s) = \mu((E_\infty - s) \cap F_\infty)$$

takes only the values 0 and $\mu(F_\infty)$. Let C be the set of elements $s \in G$ such that $f(s) = \mu(F_\infty)$. Show that C is *open* and *closed*, that $\mu(C) = \mu(E_\infty)$ and that C is the closure of E_∞ (use Problems 2(a) and 2(b)). Let D be the set of all $s \in F_\infty$ such that the intersection of F_∞ with every neighborhood of s has measure >0. Show that $\mu(D) = \mu(F_\infty)$ and that $E_\infty + D \subset C$. Deduce that D is contained in the subgroup H(C) defined in Problem 3, and that H(C) is a compact open subgroup of G. Show also that $C + H(C) = C$, that $\mu(C) \geqq \mu(A) + \mu(B) - \mu(H(C))$ and that $C \subset A + B$ (consider the measure of $E_\infty \cap (c - F_\infty)$ for each $c \in C$).

(d) Let A, B be two integrable subsets of G. Deduce from (c) that *either* $\mu_*(A + B) \geqq \mu(A) + \mu(B)$, *or* there exists a compact open subgroup H in G such that $A + B$ contains a coset of H, and that in this case $\mu_*(A + B) \geqq \mu(A) + \mu(B) - \mu(H)$. Consider the case when G is connected.

5. (a) Let A (resp. B) be the set of real numbers $x = x_0 + \sum_{i=1}^{\infty} x_i 2^{-i}$ where x_0 is an integer, each x_i $(i \geqq 1)$ is 0 or 1, and $x_i = 0$ for all even $i > 0$ (resp. $x_i = 0$ for all odd $i > 0$). Show that each of A, B has zero Lebesgue measure but that $A + B = \mathbf{R}$.

(b) Deduce from (a) that there exists a Hamel basis H of $\mathbf{R}$ (over $\mathbf{Q}$) contained in $A \cup B$ and therefore of measure zero. The set P_1 of numbers of the form rh, where $r \in \mathbf{Q}$ and $h \in H$, is also of measure 0.

(c) Let P_n denote the set of real numbers which have at most n nonzero coordinates relative to the basis H. Show that if P_n is negligible and P_{n+1} Lebesgue-measurable, then P_{n+1} is negligible. (Let $h_0 \in H$, and show that the set S of numbers $x \in P_{n+1}$ in which the coefficient of h_0 is nonzero, is negligible. Using Problem 2(c), show that if P_{n+1} were not negligible there would exist two points x', x'' in $P_{n+1} \cap \complement S$ such that $(x' - x'')/h_0$ were rational, and hence obtain a contradiction.)

(d) Deduce from (b) and (c) that there exist two negligible sets C, D in $\mathbf{R}$ such that $C + D$ is not measurable (with respect to Lebesgue measure).

6. Let G be a group acting (on the left) on a set X. A subset P (resp. C) of X is said to be a G-*packing* (resp. a G-*covering*) if for each $s \neq e$ in G we have $s \cdot P \cap P = \varnothing$ (resp. if $X = \bigcup_{s \in G} s \cdot C$). A subset P which is both a G-packing and a G-covering is called a G-*tessellation*.

(a) Suppose that X is separable, metrizable and locally compact, that G is at most denumerable and acts continuously on X (with respect to the discrete topology on G)

and that there exists a positive nonzero G-invariant measure μ on X. Let P (resp. C) be a G-packing (resp. a G-covering) such that P and C are μ-integrable. Show that $\mu(C) \geqq \mu(P)$. (Remark that $\mu(C) \geqq \sum_{s \in G} \mu(C \cap s \cdot P) = \sum_{s \in G} \mu(s^{-1} \cdot C \cap P)$.)

(b) Suppose that there exists a G-invariant distance d on X defining the topology of X. Let $\Delta(G)$ denote the greatest lower bound of the numbers $\mu(C)$, where C runs through all integrable G-coverings of X. Let r be a real number >0 such that there exists a point $a \in X$ for which $\mu(B(a; r)) > \Delta(G)$. Show that there exists $s \neq e$ in G such that $d(a, s \cdot a) < 2r$.

(c) Suppose that X is a locally compact group, μ a left Haar measure on X, and G a denumerable subgroup acting on X by left translations. Show that if A is an integrable subset of X such that $\mu(A) > \Delta(G)$, then there exists $s \in G \cap AA^{-1}$ such that $s \neq e$.

(d) With the same hypotheses as in (a), let F be a μ-integrable G-tessellation and let G_0 be a subgroup of finite index h in G. If $s_1, \ldots, s_h$ are a system of representatives of the right cosets of G_0 in G, show that $F_0 = \bigcup_{1 \leqslant i \leqslant h} s_i \cdot F$ is a G_0-tessellation.

(e) Same hypotheses as in (a). Let $f \geqq 0$ be a μ-integrable function on X. Show that there exist two points a, b in X such that

$$\mu(C) \sum_{s \in G} f(s \cdot a) \geqq \int_X f(x)\, d\mu(x) \qquad \text{and} \qquad \mu(P) \sum_{s \in G} f(s \cdot b) \leqq \int_X f(x)\, d\mu(x).$$

(Observe that if $g \geqq 0$ is an integrable function and E an integrable subset of X, then there exists $c \in E$ such that $\int_E g(x)\, d\mu(x) \leqq g(c)\mu(E)$, and $c' \in E$ such that

$$\int_E g(x)\, d\mu(x) \geqq g(c')\mu(E).)$$

2. PARTICULAR CASES AND EXAMPLES

(14.2.1) On the additive group $\mathbf{R}$, Lebesgue measure (13.1.4) is a Haar measure (left and right, since $\mathbf{R}$ is commutative). This follows from the formula for change of variable (8.7.4) applied to the function $\varphi(\xi) = \xi + \alpha$, which gives $\int_{-\infty}^{+\infty} f(t + \alpha)\, dt = \int_{-\infty}^{+\infty} f(t)\, dt$ for all $f \in \mathscr{K}_{\mathbf{C}}(\mathbf{R})$ and all $\alpha \in \mathbf{R}$.

(14.2.2) Now consider the multiplicative group $\mathbf{R}_+^*$ of real numbers >0. This is a locally compact commutative group ((13.18.4), (4.1.2), and (4.1.4)) For each function $f \in \mathscr{K}_{\mathbf{C}}(\mathbf{R}_+^*)$, there exists a compact interval $[a, b]$ with $0 < a < b$, containing the support of f. Hence for each interval $[c, d]$ in $\mathbf{R}_+^*$ containing Supp(f), the integral $\int_c^d (f(t)\, dt)/t$ is defined and its value, which we denote by $\int_0^{+\infty} (f(t)\, dt)/t$, is independent of the choice of interval $[c, d]$ containing Supp(f). We assert that $f \mapsto \int_0^{+\infty} (f(t)\, dt)/t$ is a Haar measure on $\mathbf{R}_+^*$. From the formula for change of variable (8.7.4), it follows that, for each $s > 0$,

$$\int_0^{+\infty} \frac{f(t)\,dt}{t} = \int_0^{+\infty} \frac{f(st)}{st}\, s\,dt = \int_0^{+\infty} \frac{f(st)\,dt}{t},$$

which proves our assertion.

(14.2.3) *Let* G *be a locally compact group with neutral element* e, *and let* μ *be a (left or right) Haar measure on* G. *Then* G *is discrete if and only if* $\mu(\{e\}) > 0$; *and* G *is compact if and only if* $\mu^*(G) < +\infty$ (i.e., if and only if μ is a *bounded* measure (13.20)).

It is clear that if G is compact then μ is bounded. If G is discrete, $\{e\}$ is an open neighborhood of e, and hence $\mu(\{e\}) > 0$ because the support of μ is the whole of G. Conversely, let V be a compact neighborhood of e. If $\mu(\{e\}) > 0$, we have $\mu(\{s\}) = \mu(\{e\})$ for all $s \in G$, because μ is invariant; hence the number of points of V is *finite* and $\leqq \mu(V)/\mu(\{e\})$. Since G is Hausdorff it follows that G is discrete.

Suppose now that μ is bounded and (say) left-invariant. Consider the set $\mathfrak{E}$ of finite subsets $\{s_1, s_2, \ldots, s_n\}$ in G such that $s_i V \cap s_j V = \varnothing$ whenever $i \neq j$. We have

$$n\mu(V) = \mu(s_1 V \cup s_2 V \cup \cdots \cup s_n V) \leqq \mu(G),$$

hence $n \leqq \mu(G)/\mu(V)$. Hence $\mathfrak{E}$ contains a subset $\{s_1, \ldots, s_n\}$ having the largest possible number of elements. For each $s \in G$ it then follows that sV must meet at least one of the $s_i V$, that is to say, $s \in s_i VV^{-1}$. Hence G is the union of the sets $s_i VV^{-1}$, which are compact (12.10.4), and so G is compact.

(14.2.4) *Let* G *be a locally compact group,* V *an open subset of* G, *and* μ *a nonzero positive measure on* V, *having the following property: if* U *is an open subset of* V *and if* $s \in G$ *is such that* $sU \subset V$, *then the image of the measure* μ_U, *induced by* μ *on* U (13.1.8), *under the homeomorphism* $x \mapsto sx$ (13.1.6), *is the measure* μ_{sU} *induced by* μ *on* sU. *Then there exists a unique left Haar measure* α *on* G *which induces* μ *on* V.

For each $s \in G$, let μ_s be the image of μ under the homeomorphism $x \mapsto sx$ of V onto sV. The restriction of μ_s to $V \cap sV$ is the image of $\mu_{s^{-1}V \cap V}$ under the restriction of $x \mapsto sx$ to $s^{-1}V \cap V$. By hypothesis, this image is $\mu_{V \cap sV}$. By translation it follows that for all s, t in G the measures μ_s and μ_t have the same restriction to $sV \cap tV$. By virtue of (13.1.9), there exists a positive measure α on G which induces μ_s on sV for all $s \in G$. Clearly α is left-invariant, and is therefore the unique left Haar measure on G which induces μ on V.

We shall use this *local* definition of a Haar measure in Chapter XIX to construct a left Haar measure on a *Lie group*. Here we note the following consequence of (14.2.4) :

(14.2.5) *Let* G *be a locally compact group,* H *a discrete normal subgroup of* G, *and* $\pi : G \to G/H$ *the canonical homomorphism. Also let* V *be an open neighborhood of the neutral element of* G *such that the restriction of* π *to* V *is a homeomorphism of* V *onto the neighborhood* $\pi(V)$ *of the neutral element of* G/H (12.11.2). *Let* λ *be a left Haar measure on* G. *If* μ *is the image under* $\pi \mid V$ *of the restriction* λ_V *of* λ *to* V, *then* μ *is the restriction to* $\pi(V)$ *of a left Haar measure on* G/H.

For every open set in $\pi(V)$ is of the form $\pi(U)$, where $U \subset V$ is open, and the relation $\pi(s)\pi(U) \subset \pi(V)$ is equivalent to $sU \subset V$; hence it follows immediately from the definitions that μ satisfies the condition of (14.2.4).

Example

(14.2.6) The mapping $\varphi : t \mapsto e^{2\pi i t}$ is a strict morphism (12.12.7) of $\mathbf{R}$ onto the compact group $\mathbf{U}$ of complex numbers of absolute value 1, by virtue of (9.5.2) and (9.5.7). The kernel of φ is the discrete subgroup $\mathbf{Z}$ consisting of the integers, and $\mathbf{U}$ may therefore be canonically identified with the quotient group $\mathbf{R}/\mathbf{Z} = \mathbf{T}$ (also called the 1-*dimensional torus* or the *additive group of real numbers modulo* 1). Apply (14.2.5) to the case where $V =]-\frac{1}{2}, \frac{1}{2}[$; bearing in mind that a Haar measure μ on $\mathbf{U}$ must be diffuse (14.2.3) and that the complement of $\varphi(V)$ in $\mathbf{U}$ consists of a single point, we see that a function f on $\mathbf{U}$ is μ-integrable if and only if the function $t \mapsto f(e^{2\pi i t})$ is Lebesgue-integrable on $]-\frac{1}{2}, \frac{1}{2}[$, and that we then have $\int f\, d\mu = \int_{-1/2}^{1/2} f(e^{2\pi i t})\, dt$.

(14.2.7) *Let* G_1, G_2 *be two locally compact groups, and* μ_1 (resp. μ_2) *a left Haar measure on* G_1 (resp. G_2). *Then* $\mu_1 \otimes \mu_2$ *is a left Haar measure on* $G_1 \times G_2$.

For each function $f \in \mathscr{K}_{\mathbf{c}}(G_1 \times G_2)$ and each $(s_1, s_2) \in G_1 \times G_2$ we have

$$\begin{aligned}
&\iint f(s_1 x_1, s_2 x_2)\, d\mu_1(x_1)\, d\mu_2(x_2) \\
&\quad = \int d\mu_1(x_1) \int f(s_1 x_1, s_2 x_2)\, d\mu_2(x_2) \\
&\quad = \int d\mu_1(x_1) \int f(s_1 x_1, x_2)\, d\mu_2(x_2)
\end{aligned}$$

$$= \int d\mu_2(x_2) \int f(s_1 x_1, x_2)\, d\mu_1(x_1)$$

$$= \int d\mu_2(x_2) \int f(x_1, x_2)\, d\mu_1(x_1)$$

$$= \iint f(x_1, x_2)\, d\mu_1(x_1)\, d\mu_2(x_2)$$

by virtue of (13.21.2). Hence the result.

In particular, *Lebesgue measure* on $\mathbf{R}^n$ (13.21.19) is a Haar measure on the additive group $\mathbf{R}^n$.

PROBLEMS

1. Let G be a locally compact group, μ a left Haar measure on G, A a subset of G, and B a relatively compact μ-integrable subset of G such that $\mu(B) > 0$. Show that, if $\mu_*(AB) < +\infty$, then A is relatively compact (imitate the proof of (14.2.3)).

2. (a) Let G be a discrete subgroup of rank n in the additive group $\mathbf{R}^n$, acting on $\mathbf{R}^n$ by translations. Show that, if λ is Lebesgue measure, the number $\Delta(G)$ (Section 14.1, Problem 6(b)) is equal to the absolute value of the determinant (with respect to the canonical basis of $\mathbf{R}^n$) of a $\mathbf{Z}$-basis of G (use Problem 6(a) of Section 14.1). Deduce that if A is a closed symmetric convex set in $\mathbf{R}^n$ with nonempty interior (Section 12.14, Problem 11) such that $\lambda(A) \geqq 2^n \Delta(G)$, then $A \cap G$ contains a point other than 0 (*Minkowski's theorem*).

   (b) Let $u_i : (x_j) \mapsto \sum_{j=1}^{n} c_{ij} x_j$ $(1 \leqq i \leqq m)$ be linear forms on $\mathbf{R}^n$ with *integer* coefficients c_{ij}, and suppose that $m < n$. Let p be an integer > 1, and let A be a symmetric convex set in $\mathbf{R}^n$ with nonempty interior. Show that, for each $r > 0$ satisfying $\lambda(A) r^n \geqq 2^n p^m$, there exists a point $x \neq 0$ in rA with integer coefficients, such that $u_i(x) \equiv 0 \pmod{p}$ for $1 \leqq i \leqq m$. (Apply Minkowski's theorem to the subgroup G_0 of $\mathbf{Z}^n$ consisting of all $z \in \mathbf{Z}^n$ such that $u_i(z) \equiv 0 \pmod{p}$ for $1 \leqq i \leqq m$, and use Problem 6(d) of Section 14.1.) In particular, show that if c_1, c_2 are any two integers, there exist integers x_1, x_2, not both zero, such that $|x_1| \leqq \sqrt{p}$, $|x_2| \leqq \sqrt{p}$ and $c_1 x_1 + c_2 x_2 \equiv 0 \pmod{p}$ (*Thue's theorem*).

   (c) Let a, b be two integers. Use (b) to show that there exist integers x_1, x_2, x_3, x_4, not all zero, such that

$$a x_1 + b x_2 \equiv x_3 \pmod{p}, \qquad b x_1 - a x_2 \equiv x_4 \pmod{p}$$

   and

$$y = x_1^2 + x_2^2 + x_3^2 + x_4^2 \leqq \sqrt{2}\, p.$$

   Show that if p is prime one can find two integers a, b such that $a^2 + b^2 + 1 \equiv 0 \pmod{p}$ (assuming p is odd, observe that when z takes the $\frac{1}{2}(p+1)$ integer values

$0, 1, 2, \ldots, \frac{1}{2}(p-1)$, the residues mod p of the numbers z^2 are all distinct). Deduce that in this case $y = p$. Hence, using the multiplicative property of norms of quaternions, deduce that every positive integer is the sum of at most four squares (*Lagrange's theorem*).

3. (a) Let λ be Lebesgue measure on $\mathbf{R}$. Let f be a real-valued function $\geqq 0$ on $\mathbf{R}$ which is λ-integrable, bounded and of compact support. Put $\gamma = \sup_{t \in \mathbf{R}} f(t)$. For each $w \in \mathbf{R}$, let $U_f(w)$ denote the set of $t \in \mathbf{R}$ such that $f(t) \geqq w$, and put $\nu_f(w) = \lambda^*(U_f(w))$. Show that, for all $\alpha > 1$,

$$\int_{-\infty}^{+\infty} f^\alpha(t)\, dt = \int_0^\gamma \nu_f(w) \alpha w^{\alpha-1}\, dw.$$

(b) Let g be another function satisfying the same conditions as f, and put $\delta = \sup_{t \in \mathbf{R}} g(t)$. Let h be the function on $\mathbf{R}^2$ defined by $h(u, v) = f(u) + g(v)$ if $f(u)g(v) \neq 0$, and $h(u, v) = 0$ otherwise; also let $k(t) = \sup_{u+v=t} h(u, v)$, so that k is positive, λ-integrable, bounded and compactly supported. Show that, for all $\alpha > 1$,

$$\int_{-\infty}^{+\infty} k^\alpha(t)\, dt \geqq (\gamma + \delta)^\alpha \left(\frac{1}{\gamma^\alpha} \int_{-\infty}^{+\infty} f^\alpha(t)\, dt + \frac{1}{\delta^\alpha} \int_{-\infty}^{+\infty} g^\alpha(t)\, dt \right).$$

(Observe that if $0 < w \leqq 1$ we have $U_k(\gamma w + \delta w) \supset U_f(\gamma w) + U_f(\delta w)$, and use (a) above and Problem 4(d) of Section 14.1).

(c) Let λ_n be Lebesgue measure on $\mathbf{R}^n$, and let A, B be two λ_n-integrable subsets of $\mathbf{R}^n$. Show that

$$((\lambda_n)_*(A + B))^{1/n} \geqq (\lambda_n(A))^{1/n} + (\lambda_n(B))^{1/n}$$

(*Brunn–Minkowski inequality*). (Reduce to the case where A and B are compact. Then use induction on n, Problem 4(d) of Section 14.1, the theorem of Lebesgue–Fubini, the inequality established in (b) above, and Hölder's inequality.)

4. Let p be a prime number. The normalized Haar measure μ on the compact group $\mathbf{Z}_p$ of p-adic integers (Section 12.9, Problem 4) is such that the measure of any closed ball of radius p^{-k} is equal to p^{-k}. (Show that $\mathbf{Z}_p$ is the union of p^k closed balls of radius p^{-k}, no two of which intersect.)

3. THE MODULUS FUNCTION ON A GROUP; THE MODULUS OF AN AUTOMORPHISM

Let G be a locally compact group, μ a left Haar measure on G. For all $s, t \in G$ we have

$$\gamma(t)(\delta(s)\mu) = \delta(s)(\gamma(t)\mu) = \delta(s)\mu$$

directly from the definitions (14.1.2), and therefore $\delta(s)\mu$ is also a left-invariant positive measure on G. Hence there is a unique real number $\Delta_G(s) > 0$ (also written $\Delta(s)$) such that $\delta(s)\mu = \Delta_G(s)\mu$ (14.1.5), and this number is clearly, by

virtue of (14.1.5) again, independent of the choice of left Haar measure μ. The mapping $s \mapsto \Delta_G(s)$ is called the *modulus function* on G. If f is any μ-integrable function, the function $x \mapsto f(xs)$ is therefore also μ-integrable, and we have

$$\text{(14.3.1.1)} \qquad \int f(xs)\, d\mu(x) = \Delta(s^{-1}) \int f(x)\, d\mu(x).$$

In particular, if A is any μ-integrable subset of G, then As is also μ-integrable, and

$$\text{(14.3.1.2)} \qquad \mu(\mathrm{A}s) = \Delta(s)\mu(\mathrm{A}).$$

(14.3.2) *The mapping $s \mapsto \Delta_G(s)$ is a continuous homomorphism of* G *into the multiplicative group* $\mathbf{R}_+^*$.

This follows immediately from the formula (14.3.1.1) and the lemma (14.1.5.5).

The group G is said to be *unimodular* if $\Delta_G(s) = 1$ for all $s \in G$. In this case there is no distinction to be made between left and right Haar measures, and we call them simply *Haar measures* on G.

(14.3.3) *If there exists in* G *a compact neighborhood* V *of the neutral element e which is invariant under all inner automorphisms of* G, *then* G *is unimodular. This is the case, in particular, when* G *is compact, or discrete, or commutative.*

For each $s \in G$ we have $\mu(V) = \mu(s^{-1}Vs) = \Delta_G(s)\mu(V)$ by (14.3.1.2) and (14.1.2.4); hence the result, since $\mu(V) \neq 0$ (14.1.2).

(14.3.4) *If μ is any left Haar measure on* G, *then $\check{\mu} = \Delta^{-1} \cdot \mu$. If f is any μ-integrable function on* G, *then the function $x \mapsto f(x^{-1})\Delta(x)^{-1}$ is μ-integrable, and we have*

$$\text{(14.3.4.1)} \qquad \int f(x^{-1})\Delta(x)^{-1}\, d\mu(x) = \int f(x)\, d\mu(x).$$

For each $s \in G$ we have

$$\delta(s)(\Delta^{-1} \cdot \mu) = (\delta(s)\Delta^{-1}) \cdot (\delta(s)\mu) = (\Delta(s)^{-1}\Delta^{-1}) \cdot (\Delta(s)\mu) = \Delta^{-1} \cdot \mu,$$

which shows that $\Delta^{-1} \cdot \mu$ is a right Haar measure on G. Since $\check{\mu}$ is also a right Haar measure, there exists a constant $a > 0$ such that $\check{\mu} = a\Delta^{-1} \cdot \mu$. It follows that $\mu = a(\Delta^{-1} \cdot \mu)^{\check{}} = a\Delta \cdot \check{\mu} = a^2\mu$ (13.14.5), whence $a^2 = 1$ and therefore

$a = 1$ (because $a > 0$). This proves the first assertion. The second then follows (13.14.3).

We deduce that if f is a locally μ-integrable function, then

(14.3.4.2) $$(f \cdot \mu)^{\check{}} = (\Delta^{-1}\check{f}) \cdot \mu.$$

(14.3.5) In particular, if G is *unimodular*, then for each μ-integrable function the functions $\gamma(s)f$, $\delta(s)f$ and $\check{f}$ are μ-integrable, and we have

(14.3.5.1) $$\int f(sx)\, d\mu(x) = \int f(xs)\, d\mu(x) = \int f(x^{-1})\, d\mu(x) = \int f(x)\, d\mu(x).$$

In particular, if A is any μ-integrable set in G, then

(14.3.5.2) $$\mu(s\mathrm{A}) = \mu(\mathrm{A}s) = \mu(\mathrm{A}^{-1}) = \mu(\mathrm{A})$$

for all $s \in \mathrm{G}$.

When G is *infinite and compact* (resp. *infinite and discrete*) and therefore unimodular (14.3.3), the *normalized Haar measure* on G is the unique Haar measure μ on G for which $\mu(\mathrm{G}) = 1$ (resp. $\mu(\{e\}) = 1$).

(14.3.6) Now let u be an *automorphism* of the topological group G. It is clear that the image $u^{-1}(\mu)$ of a left Haar measure μ on G (13.1.6) is another left Haar measure; hence (14.1.5) there exists a number $a > 0$, independent of the choice of μ, such that $u^{-1}(\mu) = a\mu$. This number a is called the *modulus* of the automorphism u, and is denoted by $\mathrm{mod}_{\mathrm{G}}(u)$ or $\mathrm{mod}(u)$. For every μ-integrable function f we have therefore

(14.3.6.1) $$\int f(u^{-1}(x))\, d\mu(x) = (\mathrm{mod}\ u) \int f(x)\, d\mu(x)$$

and in particular, for any μ-integrable set A,

(14.3.6.2) $$\mu(u(\mathrm{A})) = (\mathrm{mod}\ u)\mu(\mathrm{A}).$$

In particular, for each $s \in \mathrm{G}$, let i_s be the inner automorphism $x \mapsto s^{-1}xs$. Then we have $i_s^{-1} = \delta(s)\gamma(s)$, and therefore

$$i_s^{-1}(\mu) = \delta(s)\mu = \Delta(s)\mu,$$

which proves that

(14.3.7) $$\mathrm{mod}(i_s) = \Delta(s).$$

If G is either *compact* or *discrete*, then we have $\mathrm{mod}(u) = 1$ for *every* automorphism u of G. For it is clear that $u(G) = G$ and $u(\{e\}) = \{e\}$, and we may apply (14.3.6.2) with $A = G$ and $A = \{e\}$, respectively.

We deduce also from (14.3.6.2) that if u and v are two automorphisms of G, then

$$\mathrm{mod}(u \circ v) = \mathrm{mod}(u) \cdot \mathrm{mod}(v). \tag{14.3.8}$$

(14.3.9) *If u is any automorphism of the vector space $\mathbf{R}^n$, then*

$$\mathrm{mod}(u) = |\det u|.$$

Let $U = (\alpha_{ij})$ be the matrix of the automorphism u with respect to the canonical basis of $\mathbf{R}^n$. Let E_{ij} denote the $n \times n$ matrix which has the element in the (i,j) place equal to 1 and all other elements equal to 0. If $i \neq j$ and $\lambda \in \mathbf{R}$, put

$$B_{ij}(\lambda) = I_n + \lambda E_{ij}.$$

Then we have the following lemma:

(14.3.9.1) *Every invertible $n \times n$ matrix U is a product of matrices of the form $B_{ij}(\lambda)$ and a matrix of the form $I_n + (a-1)E_{nn}$.*

Consider invertible matrices of the form

$$X = \begin{pmatrix} 1 & 0 & \cdots & 0 & \xi_{1,n-h} & \cdots & \xi_{1n} \\ 0 & 1 & \cdots & 0 & \xi_{2,n-h} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \xi_{n-h-1,n-h} & \cdots & \xi_{n-h-1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \xi_{n,n-h} & \cdots & \xi_{nn} \end{pmatrix}$$

where $h \geqq 0$; if $h = n-1$, then X is an arbitrary invertible matrix. The proof is by induction on h. The matrix $B_{ij}(\lambda)X$ is obtained by adding λ times the jth row to the ith row of X. If $h = 0$ we must have $\xi_{nn} \neq 0$; hence if we multiply on the left successively by the matrices $B_{in}(-\xi_{nn}^{-1}\xi_{in})$ for $1 \leqq i \leqq n-1$, we shall obtain the matrix $I_n + (\xi_{nn} - 1)E_{nn}$, which proves the lemma in the case $h = 0$. Now suppose that the lemma has been proved for $h = 0, 1, \ldots, k-1 < n$, and consider the case $h = k$. From the expansion for $\det(X)$ we see Ann 7.4 that there must exist an element $\xi_{i,n-k} \neq 0$ for some i such that $n-k \leqq i \leqq n$. Premultiplying X by $B_{ji}((1-\xi_{j,n-k})\xi_{i,n-k}^{-1})$ for some index $j \neq i$ such that $n-k \leqq j \leqq n$, we see that we may assume that $\xi_{j,n-k} = 1$. Multiplying successively by the matrices $B_{rj}(-\xi_{r,n-k})$ for $r \neq j$, we end up

with a matrix for which $\xi_{r,n-k} = 0$ if $r \neq j$, and $\xi_{j,n-k} = 1$. Finally, multiplying this matrix by $B_{j,n-k}(-1)B_{n-k,j}(1)$, we obtain a matrix of the same form but with $j = n - k$. This completes the induction step, and hence also the proof of the lemma.

Bearing in mind (14.3.8) we see that in order to prove (14.3.9) it is enough to consider the cases where u takes one of the forms

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, ax_n), \tag{14.3.9.2}$$

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_j + \lambda x_i, \ldots, x_n). \tag{14.3.9.3}$$

Now by (13.21.2), if f is any function belonging to $\mathscr{K}(\mathbf{R}^n)$, we have

$$\int \cdots \int f(x_1, \ldots, x_{n-1}, ax_n)\, dx_1 \cdots dx_n$$
$$= \int \cdots \int dx_1 \cdots dx_{n-1} \int f(x_1, \ldots, ax_n)\, dx_n$$

and the formula for change of variable (8.7.4) gives

$$\int_{-\infty}^{+\infty} (x_1, \ldots, x_{n-1}, ax_n)\, dx_n = |a^{-1}| \int_{-\infty}^{+\infty} f(x_1, \ldots, x_n)\, dx_n,$$

so that (14.3.9) is true when u is of the form (14.3.9.2). Likewise, we may write

$$\int \cdots \int f(x_1, \ldots, x_j + \lambda x_i, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n$$
$$= \int \cdots \int dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n \int f(x_1, \ldots, x_j + \lambda x_i, \ldots, x_n)\, dx_j,$$

and by virtue of the fact that Lebesgue measure is translation-invariant,

$$\int_{-\infty}^{+\infty} f(x_1, \ldots, x_j + \lambda x_i, \ldots, x_n)\, dx_j = \int_{-\infty}^{+\infty} f(x_1, \ldots, x_j, \ldots, x_n)\, dx_j.$$

This proves (14.3.9) when u is of the form (14.3.9.3).

(14.3.10) *Application to the calculation of integrals*: I. *Parallelotope and simplex.*

In $\mathbf{R}^n$, let P be the "parallelotope" constructed on n linearly independent vectors $x_1, \ldots, x_n$, i.e., the set of all vectors $\sum_{i=1}^{n} \xi_i x_i$ with $0 \leqq \xi_i \leqq 1$ for each

index i. If μ is Lebesgue measure on $\mathbf{R}^n$, and if $x_i = (\alpha_{1i}, \ldots, \alpha_{ni})$, then we have

$$\mu(\mathrm{P}) = |\det(\alpha_{ij})|. \tag{14.3.10.1}$$

For let u be the automorphism of the vector space $\mathbf{R}^n$ given by $u(e_i) = x_i$ for $1 \leqq i \leqq n$ (where $e_1, \ldots, e_n$ are the vectors of the canonical basis of $\mathbf{R}^n$); we have $\mathrm{P} = u(\mathrm{K})$, where K is the parallelotope on the vectors $e_1, \ldots, e_n$. Hence from (14.3.6.2) and (14.3.9) we have $\mu(\mathrm{P}) = |\det u|\, \mu(\mathrm{K})$; but it follows immediately from (13.21.15) that $\mu(\mathrm{K}) = 1$. Hence the formula (14.3.10.1).

Now let S be the "simplex" on the vectors $x_1, \ldots, x_n$, i.e., the set of all $x = \sum_{i=1}^{n} \xi_i x_i$ such that $\xi_i \geqq 0$ for all i, and $\xi_1 + \cdots + \xi_n \leqq 1$. Then

$$\mu(\mathrm{S}) = \frac{1}{n!} |\det(\alpha_{ij})|. \tag{14.3.10.2}$$

Here again, by using the same linear transformation u, we are reduced to proving the formula (14.3.10.2) when $x_i = e_i$ for $1 \leqq i \leqq n$. Let S_n denote the corresponding simplex, write μ_n in place of μ, and let $a_n = \mu_n(\mathrm{S}_n)$. Identifying $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$, consider for each $\lambda \in \mathbf{R}$ the section $\mathrm{S}_n(\lambda)$ in $\mathbf{R}^{n-1}$. This set is empty if $\lambda < 0$ or $\lambda > 1$. If $0 \leqq \lambda \leqq 1$, then $\mathrm{S}_n(\lambda)$ is the set of points $(\xi_1, \ldots, \xi_{n-1}) \in \mathbf{R}^{n-1}$ such that

$$\xi_1 \geqq 0, \ldots, \xi_{n-1} \geqq 0, \quad \xi_1 + \cdots + \xi_{n-1} \leqq 1 - \lambda$$

and is therefore the image of $\mathrm{S}_n(0)$ under the homothety with ratio $1 - \lambda$. Applying (14.3.6.2) and (14.3.9) to this homothety in $\mathbf{R}^{n-1}$, we obtain

$$\mu_{n-1}(\mathrm{S}_n(\lambda)) = (1 - \lambda)^{n-1} a_{n-1}.$$

Apply (13.21.8) to the compact set S_n, and we get

$$a_n = \int_0^1 (1 - \lambda)^{n-1} a_{n-1}\, d\lambda = \frac{1}{n}\, a_{n-1}.$$

Since clearly $a_1 = 1$, we have $a_n = 1/n!$.

(14.3.11) *Application to the calculation of integrals*: II. *Closed ball.*

With the same notation as in (14.3.10), we shall now calculate the measure $\mathrm{V}_n = \mu_n(\mathbf{B}_n)$ of the Euclidean unit ball $\mathbf{B}_n$, i.e., the set of all $(x_1, \ldots, x_n) \in \mathbf{R}^n$

such that $\sum_{i=1}^{n} x_i^2 \leqq 1$. The same method that was used for the calculation of $\mu_n(S_n)$ gives

$$V_n = \int_{-1}^{1} \mu_{n-1}(\mathbf{B}_n(\lambda))\, d\lambda.$$

Now $\mathbf{B}_n(\lambda)$ is the set of all $(x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ such that $\sum_{i=1}^{n-1} x_i^2 \leqq 1 - \lambda^2$, and hence is obtained from $\mathbf{B}_{n-1}$ by the homethety with ratio $(1-\lambda^2)^{1/2}$, and therefore

$$\mu_{n-1}(\mathbf{B}_n(\lambda)) = (1 - \lambda^2)^{(n-1)/2} V_{n-1}.$$

Making the change of variable $\lambda = \sin\theta$, we get

$$\text{(14.3.11.1)} \qquad V_n = V_{n-1} \int_{-\pi/2}^{\pi/2} \cos^n\theta\, d\theta = 2V_{n-1} \int_0^{\pi/2} \cos^n\theta\, d\theta.$$

Put $c_n = \int_0^{\pi/2} \cos^n\theta\, d\theta$, and apply the formula of integration by parts (8.7.5); then if $n \geqq 2$ we obtain

$$c_n = (n-1) \int_0^{\pi/2} \cos^{n-2}\theta \sin^2\theta\, d\theta = (n-1)(c_{n-2} - c_n),$$

so that $nc_n = (n-1)c_{n-2}$. Since $c_0 = \frac{1}{2}\pi$ and $c_1 = 1$, we have finally

$$c_{2n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{\pi}{2}, \qquad c_{2n-1} = \frac{2 \cdot 4 \cdot 6 \cdots (2n-2)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}.$$

Since $V_1 = 2$ and $V_2 = \pi$ we obtain from (14.3.11.1)

$$\text{(14.3.11.2)} \qquad \begin{cases} V_{2n} = \dfrac{\pi^n}{n!}, \\ V_{2n-1} = \dfrac{2^n \cdot \pi^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}. \end{cases}$$

In terms of the gamma function,* these formulas can be written in the form

$$\text{(14.3.11.3)} \qquad V_n = \pi^{n/2}/\Gamma(\tfrac{1}{2}n + 1).$$

* See for example my book, *Calcul infinitésimal* (Hermann, Paris, 1968).

PROBLEMS

1. Let G be a locally compact group containing a compact open subgroup H. Fo automorphism u of G, show that $\mathrm{mod}(u)$ is a *rational* number. (Observe that $u(H$ is a subgroup of finite index in both H and $u(H)$.) Show that the set of elements such that $\Delta_G(s) = 1$ is an open subgroup of G, containing H.

2. Let G be a compact group, μ a Haar measure on G, and u a (continuous) end phism of G such that $u(G)$ is *open* in G and the kernel $G_u = u^{-1}(e)$ is a *finite* sub of G.
   (a) Show that there exists a real number $h(u) > 0$ and an open neighborhc of e in G such that, for every open set $V \subset U$, the set $u(V)$ is open in $\mu(u(V)) = h(u)\mu(V)$ (use (14.2.5)).
   (b) Show that $h(u) = \mathrm{Card}(G/u(G))/\mathrm{Card}(G_u)$. (Calculate $\mu(u(G))$ in two di ways, using (a) above and (14.4.2).)

3. Let $\mathbf{Q}^*$ be the set of nonzero rational numbers, endowed with the discrete top On the locally compact space $G = \mathbf{R} \times \mathbf{Q}^*$ a law of composition is defined formula $(x, r)(x', r') = (rx' + x, rr')$. Show that with respect to this law of compo G is a locally compact group which is locally isomorphic to the additive gro but not unimodular.

4. (a) Let G be a locally compact group and let χ be a continuous homomorphi G into the multiplicative group $\mathbf{C}^*$. Show that if a complex measure ν on G i that $\gamma(s)\nu = \chi(s)\nu$ for all $s \in G$, then $\nu = a\chi \cdot \mu$ where μ is a left Haar measure and a is a complex constant.
   (b) A complex measure ν on G is said to be *left quasi-invariant* if $\gamma(s)\nu$ is *equ* to ν (13.15.6) for each $s \in G$. Show that ν is left quasi-invariant if and only if measure *equivalent* (13.15.6) to a left Haar measure μ on G. (Reduce to the case use the criterion (b′) of (13.15.5), consider the double integral

$$\iint f(x)\varphi_A(xy)\, d\mu(x)\, d\nu(y),$$

   where A is compact, $f \in \mathscr{K}(G)$, and $f \geqq 0$, and use the theorem of Lebesgue–Fu

5. (a) Let G be a locally compact group and let X, Y be two closed subgroups such that $X \cap Y = \{e\}$ and such that the set $\Omega = XY$ (i.e., the set of all xy, $x \in X$ and $y \in Y$) contains a neighborhood of e in G. Show that Ω is open in that the mapping $(x, y) \mapsto xy^{-1}$ of $X \times Y$ onto Ω is a homeomorphism (cf. (12.1 by considering $X \times Y$ as acting on Ω by the rule $(x, y) \cdot z = xzy^{-1}$).
   (b) Let μ_G, μ_X, μ_Y be left Haar measures on G, X, Y, respectively, and let μ restriction of μ_G to Ω. Show that, up to a constant factor, μ is the ima $\mu_X \otimes (\chi^{-1} \cdot \mu_Y)$ under the homeomorphism $(x, y) \mapsto xy^{-1}$ of $X \times Y$ onto Ω, wh denotes the restriction of Δ_G to Y. Deduce that a real-valued function f defined is μ-integrable if and only if the function $(x, y) \mapsto f(xy)\Delta_G(y)\Delta_Y(y)^{-1}$ is $(\mu_X \otimes$ integrable, and that we then have

$$\int_\Omega f(z)\, d\mu(z) = a \iint_{X \times Y} f(xy)\Delta_G(y)\Delta_Y(y)^{-1}\, d\mu_X(x)\, d\mu_Y(y),$$

   where a is a constant independent of f.

(c) Suppose that Y is a *normal* subgroup of G. Then the measure μ is (up to a constant factor) the image of $\mu_X \otimes \mu_Y$ under the homeomorphism $(x, y) \mapsto xy$ of $X \times Y$ onto Ω, and for each $x \in X$ and $y \in Y$ we have $\Delta_G(xy) = \Delta_X(x)\,\Delta_Y(y) \bmod(i_x)$, where i_x is the automorphism $v \mapsto x^{-1}vx$ of Y (cf. (14.4.6)).
(d) Consider the locally compact space $G = \mathbf{R} \times \mathbf{R}^*$, with the law of composition $(x, y)(x', y') = (yx' + x, yy')$. Show that this locally compact group is not unimodular (use (c)).
(e) Let G be a locally compact group. On the locally compact space $E = \mathbf{R} \times G$, a law of composition is defined by the formula $(\xi, x)(\xi', x') = (\xi + \Delta_G(x)\xi', xx')$. Show that, with respect to this law of composition, E is a *unimodular* locally compact group (use (b)). The group G (which is not necessarily unimodular) is isomorphic to a subgroup and to a quotient group of E.

6. Let p be a prime number. On the group $\mathbf{Z}_p$ (Section 12.9, Problem 4) we define a ring structure as follows: if $z = (z_n)$ and $z' = (z'_n)$, then $zz' = (z_n z'_n)$ (the $\mathbf{Z}/p^n\mathbf{Z}$ being quotient rings of $\mathbf{Z}$). The canonical injection of $\mathbf{Z}$ into $\mathbf{Z}_p$ (*loc. cit.*) is a ring homomorphism, which identifies $\mathbf{Z}$ with a dense subring of $\mathbf{Z}_p$. Show that $\mathbf{Z}_p$ is an integral domain. Its field of fractions is called the *field of p-adic numbers* and is denoted by $\mathbf{Q}_p$. The fundamental system of neighborhoods of 0 in $\mathbf{Z}_p$ formed by the balls with center 0 is also a fundamental system of neighborhoods of 0 in $\mathbf{Q}_p$ for a topology compatible with the additive group structure of $\mathbf{Q}_p$. With respect to this topology, $\mathbf{Q}_p$ is separable, metrizable and locally compact (Section 12.8, Problem 1). The field $\mathbf{Q}$ of rational numbers is dense in $\mathbf{Q}_p$, and the p-adic distance on $\mathbf{Q}$ (3.2.6) extends to a distance d on $\mathbf{Q}_p$ defining the topology of $\mathbf{Q}_p$. For $z \in \mathbf{Q}_p$ we write $|z|_p = d(0, z)$. For each $s \in \mathbf{Q}_p$, show that the modulus of the homothety $z \mapsto sz$ of $\mathbf{Q}_p$ is $|s|_p$. Deduce that, for each automorphism u of the vector space $\mathbf{Q}_p^n$ over $\mathbf{Q}_p$, we have $\bmod(u) = |\det u|_p$.

7. Let λ_n denote Lebesgue measure in the space $\mathbf{R}^n$ endowed with the Euclidean scalar product $(x \mid y) = \sum_{j=1}^{n} \xi_j \eta_j$. For every universally measurable bounded set A, let $\sigma_p(A)$ be the set defined as follows. Identify $\mathbf{R}^n$ with the product $\mathbf{R}^p \times \mathbf{R}^{n-p}$, and for each point $x' \in \mathrm{pr}_1(A)$ let $B_{n-p}(x')$ be the closed Euclidean ball in $\mathbf{R}^{n-p}$ with center 0 such that the measure $\lambda_{n-p}(B_{n-p}(x'))$ is equal to the measure $\lambda_{n-p}(A(x'))$ of the section of A at x'. Then $\sigma_p(A)$ is defined to be the union of the sets $\{x'\} \times B_{n-p}(x')$ as x' runs through $\mathrm{pr}_1(A)$. Hence $\mathrm{pr}_1(\sigma_p(A)) = \mathrm{pr}_1(A)$.
(a) Show that if A is compact, then so is $\sigma_p(A)$. (Use Problem 16(b) of Section 13.21.) Deduce that if A is universally measurable, then $\sigma_p(A)$ is λ_n-measurable and that $\lambda_n(\sigma_p(A)) = \lambda_n(A)$.
(b) Show that if A and B are universally measurable bounded sets in $\mathbf{R}^n$, then

$$\lambda_n(\sigma_p(A) \cap \sigma_p(B)) = \lambda_n(A \cap B) + \lambda_n(\sigma_p(A \cap \complement B) \cap \sigma_p(B \cap \complement A)).$$

(c) Show that the mapping $A \mapsto \sigma_p(A)$ is not continuous with respect to the topology on the set of nonempty bounded closed subsets of $\mathbf{R}^n$ defined by the distance of Section 3.16, Problem 3. (Observe that in this topology every compact set can be approximated arbitrarily closely by a finite set, which is of measure zero.)
(d) If A, B are compact subsets of $\mathbf{R}^n$, show that

$$\sigma_{n-1}(A) + \sigma_{n-1}(B) \subset \sigma_{n-1}(A + B)$$

(cf. Section 14.2, Problem 4(d)). In particular, we have

$$\sigma_{n-1}(V_r(A)) \supset V_r(\sigma_{n-1}(A))$$

for all $r > 0$, in the notation of Section (3.6). (Observe that $V_r(A) = A + B'(0; r$

8. Let $H \subset \mathbf{R}^n$ be a hyperplane passing through the origin 0, and let T be a rota transforming H into $\mathbf{R}^{n-1}$. For each universally measurable bounded subset A o put $\sigma_H(A) = T^{-1} \cdot \sigma_{n-1}(T \cdot A)$ (the "Steiner symmetrization" of A with respect to (a) Suppose that A is closed and contained in a closed ball B with center 0. Show if W is an open subset of the sphere S which is the boundary of B and if $W \cap A =$ then $\sigma_H(A)$ does not intersect W nor the image of W under the symmetry with res to H.

(b) Deduce from (a) that, under the same hypotheses, there exist finitely n hyperplanes $H_1, \ldots, H_r$ passing through 0, such that the set $\sigma_{H_r} \sigma_{H_{r-1}} \cdots \sigma_H$ is contained in B and does not meet S unless $A = B$ (use the compactness o (c) Suppose that A is compact. If B_0 is the closed ball with center 0 such $\lambda_n(A) = \lambda_n(B_0)$, show that there exists a sequence (H_m) of hyperplanes passing thro 0, such that the sequence of compact sets $A_m = \sigma_{H_m} \sigma_{H_{m-1}} \cdots \sigma_{H_1}(A)$ tends to E the topology defined in Section 3.16, Problem 3. (Use the result of this problem showing first that the sequence (A_m) can be assumed to have a limit A′ such $A' \subset B'(0; R)$, where R is the greatest lower bound of the radii of closed balls center 0 containing a transform of A under the composition of a finite numbe Steiner symmetrizations σ_H with respect to hyperplanes H passing through 0. T argue by contradiction and use (b) above to show that $B'(0; R) = B_0$.)

9. (a) Let A be a nonempty compact subset of the space $\mathbf{R}^n$ endowed with the Euclic scalar product. For each unit vector $u \in \mathbf{R}^n$, put $h(A; u) = \sup_{x \in A}(x \mid u)$, and $b(A; u$ $h(A; u) + h(A; -u)$ (the *width* of A in the direction u). The least upper bound of numbers $b(A; u)$ as u varies on the unit sphere S_{n-1} is the diameter $\delta(A)$ of A (Sec 6.3, problem 2). For any pair of compact sets A, B and any real number $\alpha > 0$ have $h(A + B; u) = h(A; u) + h(B; u)$ and $h(\alpha A; u) = \alpha h(A; u)$, from which follows that $\delta(A + B) \leqq \delta(A) + \delta(B)$.

(b) Let s be a finite sequence $((\alpha_j, U_j))_{1 \leqslant j \leqslant m}$ of pairs in which the α_j are real numb $\geqq 0$ such that $\sum_j \alpha_j = 1$, and the U_j are rotations about 0 (i.e., elements of $\mathbf{SO}(n,$ The *rotational mean* of A corresponding to the finite sequence s is defined to be compact set $\rho_s(A) = \sum_j \alpha_j U_j(A)$. We have $\lambda_n(\rho_s(A)) \geqq \lambda_n(A)$ (Section 14.1, Prob 4(d)). For each $u \in S_{n-1}$, show that

$$h(\rho_s(A); u) = \sum_j \alpha_j h(A; U_j^{-1}(u)).$$

(c) Let A be a nonempty compact set contained in a closed ball B with cente and let S be the frontier of B (a sphere). Show that if there exists a nonempty o subset W of S which does not meet A, then there exists a rotational mean $\rho_s(A)$ s that $\rho_s(A) \subset B$ and $S \cap \rho_s(A) = \varnothing$ (same method as Problem 7; use the compactr of S).

(d) Let R be the greatest lower bound of radii of closed balls with center 0 contain a transform of A under the composition of a finite number of rotational means $B_0 = B'(0; R)$, show that there exists a sequence of sets $A_p = \rho_{s_p} \rho_{s_{p-1}} \cdots \rho_{s_1}$ tending to B_0, relative to the distance defined in Section 3.16, Problem 3. (Argue

contradiction, by first showing that the sequence (A_p) may be assumed to have a limit $A' \subset B_0$, and then using (c) above to prove that A' contains the frontier S_0 of B_0; finally remark that $\frac{1}{2}S_0 + \frac{1}{2}S_0 = B_0$.)

(e) If A is any compact subset of $\mathbf{R}^n$, show that

$$\lambda_n(A) \leqq 2^{-n} V_n(\delta(A))^n$$

where the diameter $\delta(A)$ is relative to the Euclidean distance, and V_n is the Lebesgue measure of the ball $\mathbf{B}_n$ ("Bieberbach's inequality"). (Use (d) and the inequality $\delta(\rho_s(A)) \leqq \delta(A)$.)

10. With the notation of Problem 7, for any compact subset A of $\mathbf{R}^n$ the *upper* (resp. *lower*) *Minkowski area* of A is defined to be the number

$$\alpha^+(A) = \limsup_{r \to 0} (\lambda_n(V_r(A)) - \lambda_n(A))/r$$

(resp.

$$\alpha^-(A) = \liminf_{r \to 0} (\lambda_n(V_r(A)) - \lambda_n(A))/r)$$

where r tends to 0 through positive values.

(a) Give an example of a compact set A such that $\alpha^+(A) = 1$ and $\alpha^-(A) = 0$. (Take $n = 2$, and consider a union of finite sets of the form $c_k + A_k$, where the sequence (c_k) tends to 0, and for each k the set A_k is a product $I_k \times J_k$, where I_k, J_k are finite sequences of the form $(pa_k)_{0 \leqslant p \leqslant r_k}$, $(qb_k)_{0 \leqslant q \leqslant s_k}$. Choose the sequences (c_k), (a_k), (b_k), (r_k), (s_k) appropriately.)

(b) Show that

$$\alpha^-(A) \geqq n V_n^{1/n} (\lambda_n(A))^{(n-1)/n}$$

(the *isoperimetric inequality*). (Use the Brunn–Minkowski inequality.)

(c) Let A be a compact convex set (Section 8.5, Problem 8). Show that, for each compact convex set B, the function of ξ

$$-(\lambda_n(A + \xi B))^{1/n}$$

is convex for $0 \leqq \xi < +\infty$, by using the Brunn–Minkowski inequality and the relations $(\alpha + \beta)A = \alpha A + \beta A$, $(\alpha + \beta)B = \alpha B + \beta B$ for all real numbers α, β. Deduce that $\alpha^+(A) = \alpha^-(A)$ (use Problem 8 of Section 8.5). The number $\alpha(A) = \alpha^+(A) = \alpha^-(A)$ is called the *Minkoswki area* of A.

(d) If the compact convex set A is contained in $\mathbf{R}^{n-1}$, then $\alpha(A) = \lambda_{n-1}(A)$.

(e) If 0 is an interior point of a compact convex set $A \subset \mathbf{R}^n$, then

$$n r_i^{-1} \lambda_n(A) \geqq \alpha(A) \geqq n r_e^{-1} \lambda_n(A),$$

where r_i is the maximum of the radii of closed balls with center 0 and contained in A, and r_e is the minimum of the radii of closed balls with 0 which contain A. (Use the inequalities

$$\left(\frac{r_e + r}{r_e}\right) A \subseteq V_r(A) \subseteq \left(\frac{r_i + r}{r_i}\right) A.)$$

11. Let G be a *compact* group, h a left invariant distance on G which defines the topology of G (12.9.1) and μ a Haar measure on G. Show that

$$h'(x, y) = \int h(xs, ys)\, d\mu(s)$$

is a bi-invariant distance on G, equivalent to h.

4. HAAR MEASURE ON A QUOTIENT GROUP

Let G be a locally compact group, G′ a closed normal subgroup of G, and $G'' = G/G'$ the quotient group, which is locally compact ((12.11.3) and (12.10.9)). Let α', α'' be left Haar measures on G′, G″, respectively, and let $\pi : G \to G''$ be the canonical homomorphism.

(14.4.1) *For each* $f \in \mathscr{K}(G)$ *and each* $x \in G$, *the function*

$$g(x) = \int_{G'} f(x\xi)\, d\alpha'(\xi)$$

is continuous on G, *and* $g(x\xi) = g(x)$ *for all* $\xi \in G'$, *so that we may write* $g(x) = h(\pi(x))$, *where* h *is continuous on* G″ (12.10.6). *The support of* h *is compact. The positive linear form*

$$f \mapsto \int_{G''} h(x'')\, d\alpha''(x'')$$

is a left Haar measure on G.

The continuity of g follows from (14.1.5.5), and we have

$$g(x\zeta) = \int_{G'} f(x\zeta\xi)\, d\alpha'(\xi) = g(x)$$

because α' is left-invariant. If $S = \mathrm{Supp}(f)$, then $h(x'') \neq 0$ implies that $x'' \in \pi(S)$, and $\pi(S)$ is compact in G″ (3.17.9). Finally, if s is any element of G, put $f_1 = \gamma(s^{-1})f$, and let g_1, h_1 be the corresponding functions. Then it is clear that $g_1(x) = g(sx)$, and therefore $h_1(x'') = h(\pi(s)x'')$. Hence the last part of the proposition follows from the left-invariance of α'' (the fact that the linear form $f \mapsto \int_{G''} h(x'')\, d\alpha''(x'')$ is not zero follows from the fact that the supports of α' and α'' are equal to G′ and G″, respectively).

If we denote by α the left Haar measure on G defined by (14.4.1), then by abuse of notation we write

$$\int_G f(x)\, d\alpha(x) = \int_{G''} d\alpha''(\dot{x}) \int_{G'} f(x\xi)\, d\alpha'(\xi), \tag{14.4.2}$$

where $\dot{x} = \pi(x)$, and $f \in \mathscr{K}(G)$.

This notational analogy with the product measure (13.21.2) is continued in the following propositions:

(14.4.3) (i) *If f is any mapping of* G *into* $\overline{\mathbf{R}}$, *the number* $\int^* f(x\xi)\, d\alpha'(\xi)$ *depends only on* $\dot{x} = \pi(x)$. *If* $f \in \mathscr{I}(G)$, *the function* $\dot{x} \mapsto \int^* f(x\xi)\, d\alpha'(\xi)$ *belongs to* $\mathscr{I}(G'')$, *and we have*

$$\int^* f(x)\, d\alpha(x) = \int^* d\alpha''(\dot{x}) \int^* f(x\xi)\, d\alpha'(\xi). \tag{14.4.3.1}$$

(ii) *If f is any mapping of* G *into* $\overline{\mathbf{R}}$, *then*

$$\int^* f(x)\, d\alpha(x) \geqq \int^* d\alpha''(\dot{x}) \int^* f(x\xi)\, d\alpha'(\xi), \tag{14.4.3.2}$$

$$\int_* f(x)\, d\alpha(x) \leqq \int_* d\alpha''(\dot{x}) \int_* f(x\xi)\, d\alpha'(\xi). \tag{14.4.3.3}$$

(iii) *Let* N *be an α-negligible set in* G, *and let* M *be the set of all $x \in G$ such that $x^{-1}(N \cap xG')$ is not α'-negligible in G'. Then $\pi(M)$ is α''-negligible.*

The proofs follow step by step those of (13.21.3), (13.21.4), and (13.21.5), using (14.4.1) and the left invariance of α'.

(14.4.4) *Let u be an α-measurable mapping of* G *into a topological space* E, *and let* N *be the set of all $x \in G$ such that the partial mapping $\xi \mapsto u(x\xi)$ is not α'-measurable. Then $\pi(N)$ is α''-negligible.*

The proof follows the lines of the proof of (13.21.6) using (14.4.3) in place of (13.21.5).

(14.4.5) *Let f be an α-integrable mapping of* G *into* $\overline{\mathbf{R}}$, *and let* N *be the set of all $x \in G$ such that the partial mapping $\xi \mapsto f(x\xi)$ of G' into $\overline{\mathbf{R}}$ is not α'-integrable. Then $\pi(N)$ is α''-negligible; for each $\dot{x} \notin \pi(N)$, the number $\int f(x\xi)\, d\alpha'(\xi)$ is the same for all points $x \in \dot{x}$; the function $\dot{x} \mapsto \int f(x\xi)\, d\alpha'(\xi)$, defined almost everywhere on G'', is α''-integrable; and the formula* (14.4.2) *is valid.*

Here again, we have only to follow step by step the proof of the theorem of Lebesgue–Fubini (13.21.7), using (14.4.3) and (14.4.4).

(14.4.6) *Let u be an automorphism of the topological group* G *such that $u(G') = G'$. Let u' be the restriction of u to G', and let u'' be the automorphism of G'' induced by u* (so that $u''(\pi(x)) = \pi(u(x))$ for all $x \in G$). *Then* $\mathrm{mod}_G(u) = \mathrm{mod}_{G'}(u') \cdot \mathrm{mod}_{G''}(u'')$.

Let f be any function belonging to $\mathscr{K}(G)$, and put

$$h(x'') = \int f(x\xi)\, d\alpha'(\xi) \qquad \text{for} \quad \pi(x) = x''.$$

By virtue of (14.4.2) we have

$$\int f(u^{-1}(x))\, d\alpha(x) = \int d\alpha''(\dot{x}) \int f(u^{-1}(x)u^{-1}(\xi))\, d\alpha'(\xi),$$

and since $u^{-1}(\xi) = u'^{-1}(\xi)$ by definition,

$$\int f(u^{-1}(x)u^{-1}(\xi))\, d\alpha'(\xi) = (\mathrm{mod}(u')) \cdot \int f(u^{-1}(x)\xi)\, d\alpha'(\xi)$$
$$= h(u''^{-1}(\dot{x})).$$

Since $\int h(u''^{-1}(x''))\, d\alpha''(x'') = (\mathrm{mod}(u'')) \int h(x'')\, d\alpha''(x'')$, we obtain the result.

(14.4.7) *For each $x \in G$ we have*

$$\Delta_G(x) = \Delta_{G''}(\pi(x)) \bmod_{G'}(i'_x)$$

where i'_x denotes the automorphism $s' \mapsto x^{-1}s'x$ of G'.

Apply (14.4.6) with $u = i_x$, and use (14.3.7). In particular, if x' have $\Delta_G(x') = \Delta_{G'}(x')$.

PROBLEMS

1. Let G be a locally compact group. A measure μ on G is said to be *rela* (resp. *right-*) *invariant* if for each $s \in G$ there exists a complex number $\chi(s)$ that $\gamma(s)\mu = \chi(s)\mu$ (resp. $\delta(s)\mu = \chi(s)\mu$). Show that χ is a continuous homo of G into the multiplicative group $\mathbf{C}^*$, and that μ is of the form $a\chi^{-1} \cdot \beta$, w left (resp. right) Haar measure on G and a is a complex number $\neq 0$ (Se Problem 4(a)). Consider the converse of this result, and deduce that every left-invariant measure on G is also relatively right-invariant.

2. Let G be a locally compact group, H a closed subgroup of G, α a left Haar m G, β a left Haar measure on H, π the canonical mapping of G onto G/H.
   (a) For each function $f \in \mathscr{K}(G)$, show that there exists a unique function $f^\flat$ such that $f^\flat(\pi(x)) = \int_H f(x\xi)\, d\beta(\xi)$. Conversely, for each $h \in \mathscr{K}(G/H)$, the function $f \in \mathscr{K}(G)$ such that $f^\flat = h$. (If C is the support of h, consider a comp K of G such that $\pi(K) = C$ (12.10.9) and a function $g \in \mathscr{K}(G)$ such that $g($ all $x \in K$, and observe that $(g \cdot (h \circ \pi))^\flat = g^\flat \cdot h$.
   (b) A measure $\mu \neq 0$ on G/H is said to be *relatively invariant* under G i

$s \in G$ there exists a complex number $\chi(s) \neq 0$ such that, for every function $h \in \mathscr{K}(G/H)$, we have

$$\int h(s^{-1} \cdot z)\, d\mu(z) = \chi(s) \int h(z)\, d\mu(z).$$

The measure μ is said to be *invariant* under G if $\chi(s) = 1$ for all $s \in G$. In this case the linear form $\nu : f \mapsto \int f^{\flat}(z)\, d\mu(z)$ is a relatively left-invariant measure on G (Problem 1). Deduce that χ is a continuous homomorphism of G into $\mathbf{C}^*$ and that for each $\xi \in H$ we have $\chi(\xi) = \Delta_G(\xi)/\Delta_H(\xi)$. (For $\xi \in H$, consider the function $f_1(x) = f(x\xi)$ on G, and calculate $\nu(f_1)$ in two ways.) Conversely, if there exists a continuous homomorphism $\chi : G \to \mathbf{C}^*$ which extends the homomorphism $\xi \mapsto \Delta_G(\xi)/\Delta_H(\xi)$ of H into $\mathbf{C}^*$, then there exists a measure $\neq 0$ on G/H which is relatively invariant under G. (Prove that if $f \in \mathscr{K}(G)$ is such that $f^{\flat} = 0$, then we must have $\int \chi(x^{-1}) f(x)\, d\alpha(x) = 0$; to this end, observe that for each function $g \in \mathscr{K}(G)$ we have

$$\int_H \Delta_H(\xi^{-1})\, d\beta(\xi) \int_G g(x)\chi^{-1}(x) f(x\xi^{-1})\, d\alpha(x) = 0,$$

and use (a) above and the theorem of Lebesgue–Fubini.)

(c) In particular, if H is unimodular, there exist nonzero measures on G/H which are relatively invariant under G. If there exists such a measure μ on G/H which is positive and bounded, show that μ is G-invariant and that G is unimodular. (If G′ is the kernel of Δ_G, show that $H \subset G'$ and that the canonical image of μ on G/G′ is a Haar measure on this group, and is bounded; deduce that G/G′ is compact, and finally that $G' = G$.)

3. Consider the general linear group $\mathbf{GL}(2, \mathbf{R})$, which may be identified with the set of all matrices

$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

with determinant $\neq 0$. The topology induced from that of $\mathbf{R}^4$ is compatible with the group structure (12.8.1) and makes $\mathbf{GL}(2, \mathbf{R})$ a locally compact group. Show that the measure μ induced on $\mathbf{GL}(2, \mathbf{R})$ by Lebesgue measure on $\mathbf{R}^4$ is relatively invariant (Problem 1), and deduce that $|\det(X)|^{-2} \cdot \mu$ is a left and right Haar measure on $\mathbf{GL}(2, \mathbf{R})$.

Let $\mathbf{T}(2, \mathbf{R})^*$ be the subgroup of $\mathbf{GL}(2, \mathbf{R})$ consisting of all matrices of the form

$$Z = \begin{pmatrix} x & y \\ 0 & t \end{pmatrix}.$$

Show that a left Haar measure on this group is given by the formula

$$f \mapsto \iiint f(x, y, t)|x^{-2}t^{-1}|\, dx\, dy\, dt$$

and that the modulus function on this group is the function $Z \mapsto |x^{-1}t|$. Deduce that there exists no relatively $\mathbf{GL}(2, \mathbf{R})$-invariant measure on the homogeneous space $\mathbf{GL}(2, \mathbf{R})/\mathbf{T}(2, \mathbf{R})^*$ (which can be identified with the projective line $\mathbf{P}_1(\mathbf{R})$). (Use the fact that the commutator subgroup of $\mathbf{GL}(2, \mathbf{R})$ is $\mathbf{SL}(2, \mathbf{R})$.)

4. Let G be a locally compact group and H a closed subgroup of G such that there exists a positive *bounded* G-invariant measure $\mu \neq 0$ on G/H. If U is any relatively compact open neighborhood of e in G, show that there exists an integer $m \geqq 1$ such that, for

each $s \in G$, there exists an integer n such that $1 \leqq n \leqq m$ and $s^n \in UHU^{-1}$. (If $\pi : G \to G/H$ is the canonical mapping, observe that $\mu(\pi(U)) > 0$ and that the measures of the sets $s^j \cdot \pi(U)$ are all equal.)

5. Let G be a locally compact group and μ a right Haar measure on G.

(a) In order that a positive measure $\alpha \neq 0$ on G, with support S, should be such that S is a closed subgroup of G and the measure induced on S by α a right Haar measure on S, it is necessary and sufficient that $\delta(s)\alpha = \alpha$ for all $s \in S$. The set of elements $t \in G$ such that $\delta(t)\alpha = \alpha$ is then equal to S.

(b) Let Γ denote the set of positive measures α on G satisfying the condition of (a) above. Show that Γ is vaguely closed in $M_+(G) - \{0\}$. If $f \in \mathscr{K}_+(G)$ is such that $f(e) > 0$, show that the set Γ_f of measures $\alpha \in \Gamma$ such that $\int f(s)\, d\alpha(s) = 1$ is vaguely compact (prove that $\sup_{\alpha \in \Gamma_f} \alpha(K) < +\infty$ for every compact subset K of G). Deduce that the mapping $\alpha \mapsto (\alpha(f), \alpha/\alpha(f))$ is a homeomorphism of Γ onto the product space $\mathbf{R}_+^* \times \Gamma_f$.

(c) For each $\alpha \in \Gamma$, let $H_\alpha = \mathrm{Supp}(\alpha)$: it is a closed subgroup of G. Let d be a right-invariant distance on G such that $d(x, y) \leqq 1$ for all x, y in G (12.9.1). Let (K_n) be an increasing sequence of compact sets, covering G and such that K_n is contained in the interior of K_{n+1} for each n (3.18.3). If h is the Hausdorff distance on G corresponding to d (Section 3.16, Problem 3), then for any two nonempty closed sets M, N in G we define $h_n(M, N) = 1$ if either $M \cap K_n$ or $N \cap K_n$ is empty, and $h_n(M, N) = h(M \cap K_n, N \cap K_n)$ otherwise. Endow the set $\mathfrak{F}(G)$ of nonempty closed subsets of G with the topology defined by the pseudo-distances h_n. Show that the mapping $\alpha \mapsto H_\alpha$ is a homeomorphism of Γ_f (endowed with the vague topology) onto the set Σ of closed subgroups of G, endowed with the topology induced by that of $\mathfrak{F}(G)$.

6. With the notation of Problem 5, consider the subset Γ^0 of Γ consisting of measures $\alpha \in \Gamma$ such that H_α is *unimodular*. This is also the set of measures $\alpha \in \Gamma$ such that $\alpha(f) = \alpha(\check{f})$ for all $f \in \mathscr{K}(G)$, and is therefore vaguely closed in Γ. For each $\alpha \in \Gamma^0$, put $Q_\alpha = G/H_\alpha$; then there exists a relatively G-invariant measure μ_α on Q_α such that

$$\int_G f(x)\, d\mu(x) = \int_{Q_\alpha} d\mu_\alpha(\dot{x}) \int_{H_\alpha} f(xs)\, d\alpha(s)$$

for every $f \in \mathscr{K}(G)$ (where $\dot{x}$ is the coset xH_α) (Problem 2).

(a) If $\alpha \in \Gamma^0$ and $f \in \mathscr{K}(G)$, put $f_\alpha(x) = \int_{H_\alpha} f(xs)\, d\alpha(s)$. Show that the mapping $\alpha \mapsto \|f_\alpha\|$ is vaguely continuous, and deduce that the mapping $\alpha \mapsto \|\mu_\alpha\|$ is lower semi-continuous with respect to the vague topology.

(b) Let $g \geqq 0$ be a real-valued μ-integrable function, and let $\Gamma^0(g)$ be the set of measures $\alpha \in \Gamma^0$ such that $\int^* g(xs)\, d\alpha(s) \geqq 1$ for all $x \in G$. Show that the mapping $\alpha \mapsto \|\mu_\alpha\|$ of $\Gamma^0(g)$ into $\bar{\mathbf{R}}$ is vaguely continuous. (It is enough to show that the mapping is upper semicontinuous. Let $h \in \mathscr{K}_+(G)$ be such that $\int |g(x) - h(x)|\, d\mu(x) \leqq \varepsilon$, and let $K = \mathrm{Supp}(h)$. If $\pi_\beta : G \to Q_\beta$ is the canonical mapping, show that $\mu_\beta(Q_\beta - \pi_\beta(K)) \leqq \varepsilon$ for all $\beta \in \Gamma^0(g)$. On the other hand, consider a function $f \in \mathscr{K}_+(G)$ such that $\int_G f(xs)\, d\alpha(s) \leqq 1$ for all $s \in G$ and $\int_G f(xs)\, d\alpha(s) = 1$ for all $s \in K$ (use Problem 2(a)). If U_ε is the set of measures $\beta \in \Gamma^0(g)$ such that $\int_G f(xs\, d\beta(s) > 1 - \varepsilon$ for all $x \in K$, show that $(1 - \varepsilon)\mu_\beta(\pi_\beta(K)) \leqq \|\mu_\alpha\|$.)

(c) Let Γ_c^0 be the set of measures $\alpha \in \Gamma^0$ such that G/H_α is *compact*. Show that $\Gamma^0(g) \subset \Gamma_c^0$ for all $g \in \mathscr{K}_+(G)$. Let $\alpha \in \Gamma_c^0$, and let $g \in \mathscr{K}_+(G)$ be such that $\int g(xs)\, d\alpha(s) = 2$ for all $x \in G$ (Problem 2(a)). For each compact $L \subset G$, the set W of measures $\beta \in \Gamma^0$ such that $\int g(xs)\, d\beta(s) \geqq 1$ for all $x \in L$ is a neighborhood of α in Γ^0 with respect to the vague topology. Show that if G is generated by a compact neighborhood U of e, and if we take $L = UK$ above, then $W \subset \Gamma^0(g)$. Deduce that the restriction to Γ^0 of the mapping $\alpha \mapsto \|\mu_\alpha\|$ is vaguely continuous in this case.
(d) Let $\Gamma_d \subset \Gamma^0$ be the set of measures α such that H_α is *discrete*, and let N be the subset of Γ_d consisting of measures α such that $\alpha(\{e\}) = 1$. For each relatively compact open neighborhood U of e in G, let N_U be the set of all $\alpha \in N$ such that $H_\alpha \cap U = \{e\}$. Show that N_U is compact (observe that the relation $\alpha \in N_U$ is equivalent to $\alpha(\{e\}) \geqq 1$ and $\alpha(U) \leqq 1$). As U runs through the set of relatively compact open neighborhoods of e in G, the interiors of the sets N_U cover N. A subset M of N is relatively compact in N if and only if there exists a relatively compact open neighborhood U of e in G such that $M \subset N_U$.

7. With the notation of Problems 5 and 6, suppose that there exists a neighborhood of e in G which contains no *finite* subgroup of G other than $\{e\}$. Show that the mapping $\alpha \mapsto \alpha(\{e\})$ of Γ_d into $\mathbf{R}_+^*$ is vaguely continuous. (Show that there exists a neighborhood V of e in G and a neighborhood W of α in Γ_d such that the relation $\beta \in W$ implies $(V^2 - V) \cap H_\beta = \varnothing$: argue by contradiction.)

8. With the notation of Problems 5 and 6, suppose that G is *commutative* and generated by a compact neighborhood of e. Let N_c denote the subset of N consisting of measures α such that $Q_\alpha = G/H_\alpha$ is compact, so that $N_c = N \cap \Gamma_c^0$; the set N_c is open in N (Problem 6(c)). A subset A of N_c is relatively compact in N_c if and only if it satisfies the following two conditions: (1) there exists an open neighborhood U of e in G such that $H_\alpha \cap U = \{e\}$ for all $\alpha \in A$; (2) there exists a constant k such that $\mu_\alpha(G/H_\alpha) \leqq k$ for all $\alpha \in A$. (Use Problem 6 and the fact that μ_α is a Haar measure on G/H_α.)

9. Translate the results of Problems 6 to 8 into statements about subspaces of the space Σ of closed subgroups of G (Problem 5(c)): in particular, the subspace Σ^0 of unimodular closed subgroups, the subspace Σ_c^0 of subgroups $H \in \Sigma^0$ such that G/H is compact, the subspace **D** of discrete subgroups, and the subspace $\mathbf{D}_c = \mathbf{D} \cap \Sigma_c^0$. In particular, obtain *Mahler's criterion*: if for each discrete subgroup $H \in \mathbf{D}_c$ we denote by $v(H)$ the total mass of G/H relative to the measure μ_α corresponding to the Haar measure α on H such that $\alpha(\{e\}) = 1$, then a subset **A** of $\mathbf{D}_c$ is relatively compact in $\mathbf{D}_c$ if and only if (1) there exists a neighborhood U of e in G such that $H \cap U = \{e\}$ for all $H \in \mathbf{A}$; and (2) there exists a constant k such that $v(H) \leqq k$ for all $H \in \mathbf{A}$. Consider the case $G = \mathbf{R}^n$.

5. CONVOLUTION OF MEASURES ON A LOCALLY COMPACT GROUP

Let $\mu_1, \ldots, \mu_n$ be a finite number of (complex) measures on a locally compact group G. The sequence $(\mu_1, \ldots, \mu_n)$ is said to be *convolvable* if, for each function $f \in \mathscr{K}_{\mathbf{C}}(G)$, the function

$$(x_1, \ldots, x_n) \mapsto f(x_1 x_2 \cdots x_n)$$

is *integrable with respect to the product measure* $\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n$ *on* G^n. It follows immediately from this definition and from (13.21.17) that the sequence $(\mu_i)_{1 \leqq i \leqq n}$ is convolvable if and only if the sequence $(|\mu_i|)_{1 \leqq i \leqq n}$ of absolute values of these measures is convolvable. Moreover, it is clear that

$$f \mapsto \int \cdots \int f(x_1 x_2 \cdots x_n)\, d|\mu_1|(x_1) \cdots d|\mu_n|(x_n)$$

is then a *positive linear form* on $\mathscr{K}_{\mathbf{R}}(G)$ and therefore a *positive measure* on G (13.3.1); and for all $f \in \mathscr{K}_{\mathbf{R}}(G)$ we have

$$\text{(14.5.1)} \qquad \left| \int \cdots \int f(x_1 x_2 \cdots x_n)\, d\mu_1(x_1) \cdots d\mu_n(x_n) \right| \leqq \int \cdots \int |f(x_1 x_2 \cdots x_n)|\, d|\mu_1|(x_1) \cdots d|\mu_n|(x_n)$$

by virtue of (13.16.5) and (13.21.17). It follows directly (13.1.1) that $f \mapsto \int \cdots \int f(x_1 x_2 \cdots x_n)\, d\mu_1(x_1) \cdots d\mu_n(x_n)$ is also a (complex) *measure* on G. This measure is denoted by $\mu_1 * \mu_2 * \cdots * \mu_n$ and is called the *convolution product* or *convolution* of the sequence $(\mu_i)_{1 \leqq i \leqq n}$. The formula (14.5.1) also shows that

$$\text{(14.5.2)} \qquad |\mu_1 * \mu_2 * \cdots * \mu_n| \leqq |\mu_1| * |\mu_2| * \cdots * |\mu_n|.$$

For each function $f \in \mathscr{K}_{\mathbf{C}}(G)$, the function $(x_1, \ldots, x_n) \mapsto f(x_1 x_2 \cdots x_n)$ is continuous and therefore measurable with respect to every measure on G^n; hence, by virtue of (13.21.10), the sequence $(\mu_1, \ldots, \mu_n)$ is convolvable if and only if

$$\text{(14.5.3)} \qquad \int^* d|\mu_{\sigma(1)}|(x_{\sigma(1)}) \int^* d|\mu_{\sigma(2)}|(x_{\sigma(2)}) \cdots \cdots \int^* |f(x_1 x_2 \cdots x_n)|\, d|\mu_{\sigma(n)}|(x_{\sigma(n)}) < +\infty$$

for *some* permutation σ of $\{1, 2, \ldots, n\}$. An equivalent condition is that, for each compact subset K of G, the (closed) set $A \subset G^n$ of points $(x_1, x_2, \ldots, x_n)$ such that $x_1 x_2 \cdots x_n \in K$ is $(\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n)$-*integrable*.

If a sequence (μ, ν) of two measures is convolvable, we say that μ and ν (in this order) are convolvable, or that μ is *convolvable on the left with* ν, or that ν is *convolvable on the right* with μ. If μ and ν are convolvable and μ', ν' are two measures such that $|\mu'| \leqq |\mu|$ and $|\nu'| \leqq |\nu|$, then it is clear from (14.5.3) that μ' and ν' are convolvable.

(14.5.4) *If the sequence* $(\mu_i)_{1 \leqq i \leqq n}$ *is convolvable, and if* $S_i = \mathrm{Supp}(\mu_i)$ (13.19), *then* $\mathrm{Supp}(\mu_1 * \mu_2 * \cdots * \mu_n) \subset \overline{S_1 S_2 \cdots S_n}$, *and the two sides are equal if all the measures* μ_i *are positive.*

Let x be a point not in the closure of $S_1 S_2 \cdots S_n$, and let V be an open neighborhood of x not meeting $S_1 S_2 \cdots S_n$. If f is any continuous function whose support is contained in V, we can write (13.21.18)

$$\int \cdots \int f(x_1 x_2 \cdots x_n)\, d\mu_1(x_1) \cdots d\mu_n(x_n)$$
$$= \int_{S_1} d\mu_1(x_1) \cdots \int_{S_n} f(x_1 x_2 \cdots x_n)\, d\mu_n(x_n).$$

But if $x_i \in S_i$ for $1 \leqq i \leqq n$, we have $f(x_1 x_2 \cdots x_n) = 0$ by hypothesis, and the first assertion of (14.5.4) is proved. If all the μ_i are positive, then so is

$$\mu = \mu_1 * \mu_2 * \cdots * \mu_n.$$

If U is a μ-negligible open set, K a compact subset of U, and $f \in \mathscr{K}_{\mathbf{R}}(G)$ a function with values in [0, 1] which is equal to 1 on K and to 0 on $\complement$U (4.5.2), then we have

$$\int \cdots \int f(x_1 x_2 \cdots x_n)\, d\mu_1(x_1) \cdots d\mu_n(x_n) = 0;$$

therefore the open set of points $(x_1, \ldots, x_n)$ such that $f(x_1 x_2 \cdots x_n) > \frac{1}{2}$ is $(\mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n)$-negligible, and consequently does not intersect the support $S_1 \times S_2 \times \cdots \times S_n$ of this measure (13.21.18). Since the mapping $(x_1, \ldots, x_n) \mapsto x_1 x_2 \cdots x_n$ is continuous, we conclude that K does not intersect $\overline{S_1 S_2 \cdots S_n}$, and therefore that $\overline{S_1 S_2 \cdots S_n}$ is contained in $\mathrm{Supp}(\mu)$. This completes the proof.

6. EXAMPLES AND PARTICULAR CASES OF CONVOLUTION OF MEASURES

(14.6.1) A Dirac measure ε_s (13.1.3) on G is convolvable (on either side) with *every* measure μ on G, and we have

(14.6.1.1) $$\varepsilon_s * \mu = \gamma(s)\mu, \qquad \mu * \varepsilon_s = \delta(s^{-1})\mu,$$

(14.6.1.2) $$\varepsilon_s * \varepsilon_t = \varepsilon_{st}.$$

For example, let us verify the first of the formulas (14.6.1.1). We have to show that, for every function $f \in \mathscr{K}_{\mathbf{c}}(G)$, the continuous function $(x,y) \mapsto f(xy)$ is $(\varepsilon_s \otimes \mu)$-integrable. Now the function $x \mapsto f(xy)$ is ε_s-integrable, and $\int f(xy)\, d\varepsilon_s(x) = f(sy)$. Since the function $y \mapsto f(sy)$ is continuous and has compact support, it is μ-integrable. Our assertion therefore follows from (13.21.10), and then the theorem of Lebesgue–Fubini (13.21.7) gives $\iint f(xy)\, d\varepsilon_s(x)\, d\mu(y) = \int f(sy)\, d\mu(y)$. The formula to be proved follows now from (14.1.1) and (14.1.2).

(14.6.2) *Every finite family* $(\mu_1, \ldots, \mu_n)$ *of bounded measures on* G *is convolvable. The measure* $\mu_1 * \mu_2 * \cdots * \mu_n$ *is bounded, and*

$$\|\mu_1 * \mu_2 * \cdots * \mu_n\| \leqq \|\mu_1\| \cdot \|\mu_2\| \cdots \|\mu_n\|. \tag{14.6.2.1}$$

To prove the first assertion, observe that the measure $\mu_1 \otimes \cdots \otimes \mu_n$ on G^n is bounded (13.21.18). For every function $f \in \mathscr{K}_{\mathbf{c}}(G)$, the function $(x_1, \ldots, x_n) \mapsto f(x_1 x_2 \cdots x_n)$ is continuous and bounded on G^n, and therefore $(\mu_1 \otimes \cdots \otimes \mu_n)$-integrable (13.20.4). To prove (14.6.2.1), it is enough to remark that if $f \in \mathscr{K}_{\mathbf{c}}(G)$ and $\|f\| \leqq 1$, then

$$\left| \int \cdots \int f(x_1 x_2 \cdots x_n)\, d\mu_1(x_1) \cdots d\mu_n(x_n) \right| \leqq \|\mu_1\| \cdot \|\mu_2\| \cdots \|\mu_n\|$$

by virtue of (13.21.18).

(14.6.3) *A left Haar measure* λ *on* G *is convolvable on the right with any bounded measure* μ *on* G, *and* $\mu * \lambda = \mu(1)\lambda$.

We may restrict ourselves to the case where $\mu \geqq 0$ (cf. (14.7.1.2)). For each function $f \in \mathscr{K}_+(G)$ we have

$$\int^* d\mu(x) \int^* f(xy)\, d\lambda(y) = \int^* d\mu(x) \int f(y)\, d\lambda(y) = \lambda(f)\, \|\mu\|$$

and therefore, by virtue of (13.21.9), the function $(x, y) \mapsto f(xy)$ is $(\mu \otimes \lambda)$-integrable and its integral is equal to $\lambda(f)\, \|\mu\|$.

The same calculation shows that λ *is not convolvable with itself* if G is not compact, for then the function 1 is not λ-integrable (14.2.3).

(14.6.4) *Let* $(\mu_1, \ldots, \mu_n)$ *be a finite sequence of measures on* G, *all of which except possibly for one have compact support. Then the sequence* $(\mu_1, \ldots, \mu_n)$ *is convolvable.*

Let $S_i = \mathrm{Supp}(\mu_i)$, and suppose that S_i is compact except possibly for one index j. Let $f \in \mathscr{K}_{\mathbf{C}}(G)$ and let K be the compact support of f. It is enough to show that the set of points $(x_1, \ldots, x_n) \in G^n$ which belong to the support $\prod_{i=1}^{n} S_i$ of $\mu_1 \otimes \cdots \otimes \mu_n$ (13.21.18) and are such that $x_1 x_2 \cdots x_n \in K$, is relatively compact in G^n. Now, the conditions $x_i \in S_i$ for all i, and $x_1 x_2 \cdots x_n \in K$, imply that

$$x_j \in S_{j-1}^{-1} \cdots S_1^{-1} K S_n^{-1} \cdots S_{j+1}^{-1},$$

and this set is compact (12.10.5). Since by hypothesis S_i is compact whenever $i \neq j$, our assertion is proved (3.20.16).

The result of (14.6.2) or (14.6.3) shows that two measures can be convolvable without either of them having a compact support. Later (14.10.7) we shall see examples of *unbounded* measures which are convolvable.

7. ALGEBRAIC PROPERTIES OF CONVOLUTION

(14.7.1) Let λ, μ, ν be three measures on G, and suppose that the pairs (λ, μ) and (λ, ν) are convolvable. Then so is the pair $(\lambda, \mu + \nu)$ and we have

$$\lambda * (\mu + \nu) = \lambda * \mu + \lambda * \nu. \tag{14.7.1.1}$$

For by virtue of the relation $|\mu + \nu| \leqq |\mu| + |\nu|$ we may restrict ourselves to the case where λ, μ and ν are positive, and in this case the result follows immediately from (13.16.1).

Similarly, if (λ, ν) and (μ, ν) are convolvable, then so is $(\lambda + \mu, \nu)$ and we have

$$(\lambda + \mu) * \nu = \lambda * \nu + \mu * \nu. \tag{14.7.1.2}$$

Also it is clear that if the pair (λ, μ) is convolvable, then so is $(a\lambda, b\mu)$ for all scalars a and b, and

$$(a\lambda) * (b\mu) = (ab)\lambda * \mu. \tag{14.7.1.3}$$

(14.7.2) *Let λ, μ, ν be three measures $\neq 0$ on* G.

(i) *If the sequence (λ, μ, ν) is convolvable, then so are the sequences (λ, μ), $(|\lambda| * |\mu|, \nu)$, (μ, ν), $(\lambda, |\mu| * |\nu|)$, and we have*

$$\lambda * \mu * \nu = (\lambda * \mu) * \nu = \lambda * (\mu * \nu). \tag{14.7.2.1}$$

(ii) *If the sequences* (λ, μ) *and* $(|\lambda| * |\mu|, \nu)$ *are convolvable, then so is* (λ, μ, ν). *Likewise if the sequences* (μ, ν) *and* $(\lambda, |\mu| * |\nu|)$ *are convolvable.*

We may restrict ourselves to the case in which λ, μ, and ν are positive. Suppose that the sequence (λ, μ, ν) is convolvable; then for every compact subset K of G the set of triples (x, y, z) such that $xyz \in K$ is $(\lambda \otimes \mu \otimes \nu)$-integrable. Let A be the set of pairs (x, y) such that $xy \in K$. For each compact subset K′ of G, the set $A \times K' \subset G^3$ is contained in the set of triples (x, y, z) such that $xyz \in KK'$, and since KK′ is compact, it follows that $A \times K'$ is $((\lambda \otimes \mu) \otimes \nu)$-integrable. Since $\nu \neq 0$, this implies that A is $(\lambda \otimes \mu)$-integrable (13.21.11) and hence that (λ, μ) is convolvable. Consequently, for any $f \in \mathscr{K}_+(G)$, it follows from the hypothesis and the Lebesgue–Fubini theorem that

$$\int^* d\nu(z) \int^* f(tz)\, d(\lambda * \mu)(t) = \int^* d\nu(z) \iint f(xyz)\, d\lambda(x)\, d\mu(y)$$
$$= \iiint f(xyz)\, d\lambda(x)\, d\mu(y)\, d\nu(z)$$

since the function $t \mapsto f(tz)$ is in $\mathscr{K}_+(G)$, for each $x \in G$. This shows that $\lambda * \mu$ and ν are convolvable. One proves in the same way that (μ, ν) and $(\lambda, \mu * \nu)$ are convolvable. The formula (14.7.2.1) is then a consequence of the Lebesgue–Fubini theorem.

Conversely, suppose that (λ, μ) and $(\lambda * \mu, \nu)$ are convolvable, and let f be a function belonging to $\mathscr{K}_+(G)$. For each $z \in G$, the function $t \mapsto f(tz)$ belongs to $\mathscr{K}_+(G)$, hence is $(\lambda * \mu)$-integrable, and we have

$$\int f(tz)\, d(\lambda * \mu)(t) = \iint f(xyz)\, d\lambda(x)\, d\mu(y).$$

Hence, by Lebesgue–Fubini, it follows that

$$\int^* d\nu(z) \int^* d\mu(y) \int^* f(xyz)\, d\lambda(x)$$
$$= \int^* d\nu(z) \int^* f(tz)\, d(\lambda * \mu)(t) < +\infty,$$

which proves that (λ, μ, ν) is convolvable.

One can give examples of measures λ, μ, ν on G such that the pairs (λ, μ), $(\lambda * \mu, \nu)$, (μ, ν) and $(\lambda, \mu * \nu)$ are convolvable but $(\lambda * \mu) * \nu \neq \lambda * (\mu * \nu)$ (Problem 1).

(14.7.3) *If the sequence* $(\mu_1, \ldots, \mu_n)$ *is convolvable, then so is the sequence* $(\check{\mu}_n, \ldots, \check{\mu}_1)$, *and we have*

$$(\mu_1 * \mu_2 * \cdots * \mu_n)^{\vee} = \check{\mu}_n * \check{\mu}_{n-1} * \cdots * \check{\mu}_1. \tag{14.7.3.1}$$

For (14.1.4 and 13.7.10) we have

$$\int \cdots \int^{*} |f(x_1 x_2 \cdots x_n)| \, d\,|\mu_1|(x_1) \cdots d\,|\mu_n|(x_n) < +\infty$$

if and only if

$$\int \cdots \int^{*} |f(x_n^{-1} x_{n-1}^{-1} \cdots x_1^{-1})| \, d\,|\check{\mu}_n|(x_n) \cdots d\,|\check{\mu}_1|(x_1) < +\infty,$$

and these two integrals are equal.

On the other hand, if the sequence (λ, μ) is convolvable, it does not necessarily follow that (μ, λ) is convolvable (Problem 2). But if G is *commutative* this will be the case, and we shall have $\lambda * \mu = \mu * \lambda$.

In particular, it follows from the preceding results that

(14.7.4) *On the set* $\mathrm{M}^1_{\mathbf{C}}(\mathrm{G})$ *of bounded measures on* G, *the law of composition* $(\lambda, \mu) \mapsto \lambda * \mu$ (*together with the vector space structure*) *defines a* C-*algebra structure*; *the unit element is the Dirac measure* ε_e *at the neutral element* e *of* G. *The set* $\mathrm{M}^c_{\mathbf{C}}(\mathrm{G})$ *of compactly supported measures on* G *is a subalgebra of* $\mathrm{M}^1_{\mathbf{C}}(\mathrm{G})$. *The algebra* $\mathrm{M}^1_{\mathbf{C}}(\mathrm{G})$ *is commutative if and only if* G *is commutative.*

The fact that G is commutative if $\mathrm{M}^1_{\mathbf{C}}(\mathrm{G})$ is commutative follows from the formula (14.6.1.2).

If G is *discrete*, the algebra $\mathrm{M}^c_{\mathbf{C}}(\mathrm{G})$ consists of all linear combinations $\sum_{s \in \mathrm{G}} a_s \varepsilon_s$, where $a_s = 0$ for all but a finite number of points $s \in \mathrm{G}$ (3.16.3), and the formula (14.6.1.2) shows that

$$\left(\sum_{s \in \mathrm{G}} a_s \varepsilon_s\right) * \left(\sum_{s \in \mathrm{G}} b_s \varepsilon_s\right) = \sum_{s \in \mathrm{G}} \left(\sum_{tu = s} a_t b_u\right) \varepsilon_s.$$

This is what is called in algebra the *group algebra* of the group G over the field **C**.

PROBLEMS

1. On the additive group $\mathbf{R}$, let λ be Lebesgue measure, let $\mu = \varphi_{\mathrm{I}} \cdot \lambda$, where I is the i
$[0, +\infty]$, and let $a < b$ be two distinct points of $\mathbf{R}$. Show that the convolution p
$((\varepsilon_a - \varepsilon_b) * \mu) * \lambda$ is defined, but that μ and λ are not convolvable. Show th
convolutions $\mu * ((\varepsilon_a - \varepsilon_b) * \lambda)$ and $(\mu * (\varepsilon_a - \varepsilon_b)) * \lambda$ are both defined and are ur

2. Let G be a locally compact group which is not unimodular.
(a) Show that there exists a bounded positive measure μ on G such that
is not bounded (take μ to be discrete).
(b) Let λ be a left Haar measure on G. By (14.6.3), μ and λ are convolvable.
that λ and μ are not convolvable.

3. Let G be a locally compact group.
(a) Let μ, ν be two *positive* measures on G. If $\mu * \nu = \varepsilon_e$, show that $\mu = a$
$\nu = a^{-1}\varepsilon_{x^{-1}}$ for some $x \in \mathrm{G}$ and $a \neq 0$ (cf. 14.5.4).
(b) Give an example of a positive measure on the group $\mathbf{Z}/2\mathbf{Z}$ whose support
whole group and which has an inverse (with respect to convolution product).

4. (a) In the set $\mathrm{M}^1_{\mathbf{R}}(\mathbf{R})$, consider the two sequences of bounded measures $\mu_n =$
$\nu_n = \varepsilon_{-n}$, both of which tend to 0 in the topology $\mathscr{T}_2$ defined in Section 13.20, Pr
1. Show that the sequence of measures $\mu_n * \nu_n$ does not tend to 0 with respect
topology $\mathscr{T}_1$.
(b) Let G be a locally compact group and let (μ_n), (ν_n) be two sequences
bounded measures on G. Suppose that $\mu_n \to \mu$ in the topology $\mathscr{T}_2$, and that
in the topology $\mathscr{T}_3$ (notation of Section 13.20, Problem 1). Show that the se
$\mu_n * \nu_n$ tends to $\mu * \nu$ in the topology $\mathscr{T}_2$. (Observe that if $f, g \in \mathscr{K}(\mathrm{G})$, the fu
$(x, y) \mapsto g(y)f(xy)$ has compact support and can be uniformly approximated
linear combination of functions $u_i \otimes v_i$, where u_i and v_i are in $\mathscr{K}(\mathrm{G})$.) Give an ex
(with $\mathrm{G} = \mathbf{R}$) where $\mu = \nu = 0$ and $\mu_n * \nu_n$ does not tend to 0 with respect
topology $\mathscr{T}_3$. Show that if $\mu_n \to 0$ with respect to $\mathscr{T}_3$ and if the sequence of
$(\|\nu_n\|)$ is bounded, then $\mu_n * \nu_n \to 0$ with respect to $\mathscr{T}_2$. If $\mu_n \to \mu$ with respect t
and $\nu_n \to \nu$ with respect to $\mathscr{T}_3$, show that $\mu_n * \nu_n \to \mu * \nu$ with respect to $\mathscr{T}_3$ (s
method).
(c) With the same notation, show that if, in the topology $\mathscr{T}_6$, $\mu_n \to \mu$ and
then $\mu_n * \nu_n \to \mu * \nu$ (use Problem 2 of Section 13.20, and Egoroff's theorem).
(d) Take G to be the group $\mathbf{R}^2$. Let a, b be the vectors of the canonical basis of
$\mathbf{R}$; let ρ_n be the measure on $\mathrm{I} = [0, \pi] \subset \mathbf{R}$ whose density with respect to Leb
measure is the function $\sin(2^n x)$; let μ_n be the measure $\rho_n \otimes \varepsilon_0$ on G, and let ν_n
measure $\varepsilon_{b/2^n} - \varepsilon_0$ on G. Show that $\mu_n \to 0$ with respect to $\mathscr{T}_6$ and that $\nu_n \to$
respect to $\mathscr{T}_3$, but that the sequence $(\mu_n * \nu_n)$ does not tend to 0 with respect t

5. Let G be a compact group and μ a positive measure on G such that $\mathrm{Supp}(\mu)$
and $\mu * \mu = \mu$. Show that $\mu(\mathrm{G}) = 1$ and then that μ is a Haar measure on G
$f \in \mathscr{C}_+(\mathrm{G})$, and put $g(x) = \int f(yx)\, d\mu(y)$, which is a continuous function on G.
that $g(x) = \int g(yx)\, d\mu(y)$, and deduce that g is constant, by considering the
points at which it attains its upper bound.)

6. (a) Let μ be a measure on a locally compact group G. Show that $\mu * \nu = \nu * \mu$ for all measures ν such that both $\mu * \nu$ and $\nu * \mu$ are defined, if and only if $\mu * \varepsilon_x = \varepsilon_x * \mu$ for all $x \in G$.
(b) Suppose that G is compact. If μ is any measure on G, show that the measure $\mu^\natural$ defined by

$$\mu^\natural(f) = \int f(xyx^{-1})\, d\mu(y)\, d\beta(x)$$

(where β is a Haar measure on G) satisfies $\mu^\natural * \nu = \nu * \mu^\natural$ for every measure ν on G.

7. Let G be a compact group and β the Haar measure on G for which $\beta(G) = 1$. Let μ be a positive measure on G such that $\mu(G) = 1$ and $\mu \geqq c\beta$, where $0 < c < 1$. Show that $\|\mu^{*n} - \beta\| \leqq 2^n(1-c)^n$, where μ^{*n} denotes the convolution product of n measures equal to μ (use (14.6.3)).

8. Let r be a real number such that $0 < r \leqq \frac{1}{2}$. For each integer $n > 1$, let $\rho_{n,r}$ denote the measure $\frac{1}{2}(\varepsilon_{rn} + \varepsilon_{-rn})$ on $\mathbf{R}$, and let $\mu_{n,r}$ denote $\rho_{1,r} * \rho_{2,r} * \cdots * \rho_{n,r}$.
(a) Show that the sequence $(\mu_{n,r})_{n \geqq 1}$ converges vaguely to a measure μ on $\mathbf{R}$ with support contained in $I = [-1, 1]$ (prove that for every interval $U \subset \mathbf{R}$ the sequence $(\mu_{n,r}(U))$ converges).
(b) Show that, if $r < \frac{1}{2}$, the measure μ is *disjoint* from Lebesgue measure on $\mathbf{R}$, but that $\mu_{1/2}$ is the measure induced on I by Lebesgue measure.
(c) Let $\nu_{1/4}$ be the image of $\mu_{1/4}$ under the homothety $t \mapsto 2t$ on $\mathbf{R}$. Show that $\mu_{1/4} * \nu_{1/4} = \mu_{1/2}$, although $\mu_{1/4}$ and $\nu_{1/4}$ are each disjoint from Lebesgue measure (use Problem 4(b)).

8. CONVOLUTION OF A MEASURE AND A FUNCTION

In the rest of this chapter we shall fix once and for all a *left Haar measure* β on the group G. If f is any mapping of G into $\overline{\mathbf{R}}$ or $\mathbf{C}$, the norm $N_p(f)$ ($p = 1$, 2 or $+\infty$) *is taken with respect to the measure* β.

(14.8.1) *Let μ be a measure on* G *and let f be a (complex) locally β-integrable function on* G (13.13.1). *For the measures μ and $f \cdot \beta$ to be convolvable, it is necessary and sufficient that there should exist a β-negligible set* N *such that the function $s \mapsto f(s^{-1}x)$ is μ-integrable for all $x \notin N$, and the function*

$$x \mapsto \int |f(s^{-1}x)|\, d|\mu|(s)$$

(defined almost everywhere with respect to β) is locally β-integrable. When this condition is satisfied, the function $g(x) = \int f(s^{-1}x)\, d\mu(s)$, defined almost everywhere with respect to β, is locally β-integrable, and $\mu * (f \cdot \beta)$ is equal to $g \cdot \beta$.

We shall first prove the following lemma:

(14.8.1.1) *The image of the measure* $\mu \otimes \beta$ *under the homeom* $(s, x) \mapsto (s, s^{-1}x)$ *of* G^2 *onto* G^2 *is the measure* $\mu \otimes \beta$.

We may assume that $\mu \geqq 0$. Then, for every $F \in \mathscr{K}_+(G^2)$, the f $(s, x) \mapsto F(s, s^{-1}x)$ belongs to $\mathscr{K}_+(G^2)$, and we have

$$\begin{aligned}\iint F(s, s^{-1}x)\, d\mu(s)\, d\beta(x) &= \int d\mu(s) \int F(s, s^{-1}x)\, d\beta(x) \\ &= \int d\mu(s) \int F(s, x)\, d\beta(x) \\ &= \iint F(s, x)\, d\mu(s)\, d\beta(x),\end{aligned}$$

by virtue of the left-invariance of β and the Lebesgue–Fubini theore proves the lemma.

Now suppose that μ and $f \cdot \beta$ are convolvable. Since $|f \cdot \beta| =$ (13.13.4) we may limit ourselves to the case where $\mu \geqq 0$ and $f \geqq 0$. F function $h \in \mathscr{K}_C(G)$, the function $(s, x) \mapsto h(sx)f(x)$ is $(\mu \otimes \beta)$-integr hypothesis (13.21.16 and 3.14.3); by virtue of (14.8.1.1), the same is tru function $(s, x) \mapsto h(x)f(s^{-1}x)$ (13.7.10). If A_h is the set of points $x \in$ that $h(x) \neq 0$, it follows from the theorem of Lebesgue–Fubini tha exists a β-negligible set N_h in A_h such that, for each $x \in A_h \cap \complement N$ function $s \mapsto f(s^{-1}x)$ is μ-integrable. Taking a *sequence* of functio $\mathscr{K}_+(G)$ such that the corresponding sets A_h cover G (4.5.2), we see t function $s \mapsto f(s^{-1}x)$ is μ-integrable except at the points x of a β-negli N. Furthermore, it follows from the Lebesgue–Fubini theorem t function $x \mapsto h(x) \int f(s^{-1}x)\, d\mu(s)$, defined almost everywhere with res β, is β-integrable, and that

$$\iint h(sx)f(x)\, d\mu(s)\, d\beta(x) = \int h(x)\, d\beta(x) \int f(s^{-1}x)\, d\mu(s).$$

This proves that the condition of (14.8.1) is necessary, and that $\mu * (f$ $g \cdot \beta$ (13.13.1).

Conversely, suppose that the condition is satisfied. We have to sho for each function $h \in \mathscr{K}_+(G)$, the function $(s, x) \mapsto h(sx)f(x)$ is (integrable. Now this function is $(\mu \otimes \beta)$-measurable because h is cont (13.21.13), and therefore it is enough to show that

$$\iint^* h(sx)f(x)\, d\mu(s)\, d\beta(x) < +\infty.$$

But the lemma (14.8.1.1) shows that this is equivalent to the relation

$$\iint^{*} h(x)f(s^{-1}x)\,d\mu(s)\,d\beta(x) < +\infty.$$

and therefore (13.21.10) equivalent to

$$\int^{*} h(x)\,d\beta(x) \int^{*} f(s^{-1}x)\,d\mu(s) < +\infty.$$

But this is a consequence of our hypotheses (13.13.1).

When the conditions of (14.8.1) are satisfied, we say that *the measure μ and the function f are convolvable.* Any function which is equal almost everywhere (with respect to β) to the function g of (14.8.1) is called (by abuse of language) the *convolution* of μ and f, and is written $\mu * f$ (or $\mu * {}^{\beta}f$ if the Haar measure β needs to be brought into the notation). Thus we have, almost everywhere with respect to β,

$$(\mu * f)(x) = \int f(s^{-1}x)\,d\mu(s). \tag{14.8.2}$$

If one of the functions equal almost everywhere to g is *continuous* on G, then it is the only function with this property, because the support of β is equal to G; in this case it is *this* function which is denoted by $\mu * f$. We then have, in particular,

$$(\mu * f)(e) = \int f(s^{-1})\,d\mu(s). \tag{14.8.3}$$

Similarly we define f and μ to be convolvable if the measures $f \cdot \beta$ and μ are convolvable. For this it is necessary and sufficient that, for almost all x (with respect to β) the function $s \mapsto f(xs^{-1})\Delta(s^{-1})$ should be μ-integrable and that the function

$$x \mapsto \int |f(xs^{-1})\Delta(s^{-1})|\,d\,|\mu|\,(s)$$

should be locally β-integrable. The measure $(f \cdot \beta) * \mu$ then has a density with respect to β, and we denote any one of these densities by $f * \mu$, so that for almost all x (with respect to β) we have

$$(f * \mu)(x) = \int f(xs^{-1})\Delta(s^{-1})\,d\mu(s). \tag{14.8.4}$$

In particular, for each $x \in G$ and each locally β-integrable func
the convolutions $\varepsilon_x * f$ and $f * \varepsilon_x$ are defined and (with the above conve
are given by the formulas

(14.8.5) $$(\varepsilon_x * f)(y) = f(x^{-1}y), \quad (f * \varepsilon_x)(y) = f(yx^{-1})\Delta(x^{-1}).$$

9. EXAMPLES OF CONVOLUTIONS OF MEASURES AND FUNCTI

Generally speaking, if we are looking for usable sufficient condition
measure μ and a function f on G to be convolvable, *the stronger the con
imposed on one of the factors, the weaker the conditions that need to be in
on the other.* Moreover, the function $\mu * f$ is "at least as regular" a
general. (For a Lie group G, we shall obtain these characteristics o
volution in a more general context in Chapter XVII.)

(14.9.1) *A measure μ with compact support is convolvable with every
β-integrable function f. If in addition f is continuous* (resp. *continuou
compact support*), *then the integral on the right-hand side of* (14.8.2) *is
for all $x \in G$, and the function*

$$x \mapsto (\mu * f)(x) = \int f(s^{-1}x)\, d\mu(s)$$

is continuous (resp. *continuous with compact support*).

The first assertion is a particular case of (14.6.4), and the fac
$s \mapsto f(s^{-1}x)$ is μ-integrable is a consequence of (13.19.3). The continu
$\mu * f$ when f is continuous follows from (14.1.5.5). Finally, that $\mu *$
compact support when f has compact support is a particular case of (1

(14.9.2) *Let μ be a bounded measure on* G.

(i) *For $p = 1, 2$ or $+\infty$, the measure μ is convolvable with every fu
$f \in \mathscr{L}^p_{\mathbf{C}}(G, \beta)$. The function $\mu * f$ (defined almost everywhere by* (14.8.2)
belongs to $\mathscr{L}^p_{\mathbf{C}}(G, \beta)$, and we have

(14.9.2.1) $$N_p(\mu * f) \leqq \|\mu\| \cdot N_p(f).$$

(ii) *If f is continuous and bounded, then the integral on the righ
side of* (14.8.2) *is defined for all $x \in G$, and the function $x \mapsto \int f(s^{-1}x)$ a
continuous and bounded on* G.

(iii) *If $f \in \mathscr{C}^0_{\mathbf{C}}(G)$* (13.20.5), *then $\mu * f \in \mathscr{C}^0_{\mathbf{C}}(G)$.*

(i) It follows from (14.8.1.1) that the function $(s, x) \mapsto f(s^{-1}x)$ is $(\mu \otimes \beta)$-measurable. Moreover, if $p = 1$, the measure $f \cdot \beta$ is bounded (13.14.4), hence μ and f are convolvable (14.6.2) and we have

$$\|\mu * (f \cdot \beta)\| \leqq \|\mu\| \cdot \mathrm{N}_1(f).$$

But $\|\mu * (f \cdot \beta)\| = \mathrm{N}_1(\mu * f)$, which proves (14.9.2.1) in this case. For $p = 2$ we have, by (13.11.2.2), for any function $h \in \mathscr{K}_{\mathbf{C}}(\mathrm{G})$,

$$\left(\iint^* |h(x)f(s^{-1}x)|\, d\,|\mu|(s)\, d\beta(x)\right)^2 \leqq \left(\iint^* |h(x)|^2\, d\beta(x)\, d\,|\mu|(s)\right) \cdot \left(\iint^* |f(s^{-1}x)|^2\, d\beta(x)\, d\,|\mu|(s)\right);$$

also, by (13.21.9) and the left-invariance of β,

$$\iint^* |f(s^{-1}x)|^2\, d\beta(x)\, d\,|\mu|(s) = \int^* d\,|\mu|(s) \int^* |f(s^{-1}x)|^2\, d\beta(x) = \|\mu\|\,(\mathrm{N}_2(f))^2,$$

from which it follows that μ and $f \cdot \beta$ are convolvable. Moreover, by (13.11.2.2),

$$\left(\int^* |f(s^{-1}x)|\, d\,|\mu|(s)\right)^2 \leqq \|\mu\| \cdot \int^* |f(s^{-1}x)|^2\, d\,|\mu|(s),$$

so that the above relations and (13.21.9) imply

$$\int^* d\beta(x)\left(\int^* |f(s^{-1}x)|\, d\,|\mu|(s)\right)^2 \leqq \|\mu\|^2 \cdot (\mathrm{N}_2(f))^2,$$

which proves (14.9.2.1) in this case. Finally, when $p = +\infty$, we remark that μ and the function 1 are convolvable by virtue of (14.6.3), and that $\mu * 1$ is equal to the constant function $\mu(1)$. It follows immediately that μ is convolvable with every function in $\mathscr{L}^{\infty}_{\mathbf{C}}(\mathrm{G}, \beta)$ and that the inequality (14.9.2.1) is again valid (cf. (14.5.3)).

(ii) The fact that the integral $\int f(s^{-1}x)\, d\mu(s)$ exists for all $x \in \mathrm{G}$ follows from (13.20.4). To show that $\mu * f$ is continuous at a point $x_0 \in \mathrm{G}$, we may limit ourselves to the case where $\mu \geqq 0$. For each $\varepsilon > 0$, there exists a compact subset K of G such that $\mu(\complement\mathrm{K}) \leqq \varepsilon$. Let V_0 be a compact neighborhood of x_0. The function f is uniformly continuous on the compact set $\mathrm{K}^{-1}\mathrm{V}_0$; hence there exists a neighborhood $\mathrm{V} \subset \mathrm{V}_0$ of x_0 in G such that the relation $x \in \mathrm{V}$ implies

$$|f(s^{-1}x) - f(s^{-1}x_0)| \leqq \varepsilon/\mu(\mathrm{K})$$

for all $s \in K$ (3.16.5). Hence, for all $x \in V$, we have

$$\left| \int f(s^{-1}x)\, d\mu(s) - \int f(s^{-1}x_0)\, d\mu(s) \right| \leqq 2\,\|f\|\,\mu(\complement K) + \int_K \frac{\varepsilon}{\mu(K)}\, d$$
$$\leqq \varepsilon(1 + 2\,\|f\|),$$

from which (ii) follows.

(iii) Again we may suppose that $\mu \geqq 0$. By hypothesis, for eac there exists a compact subset H of G such that $|f(x)| \leqq \varepsilon$ for all $x \notin$ K as in the proof of (ii) above, and suppose that $x \notin KH$. Then if s have $s^{-1}x \notin H$ and therefore

$$\left| \int f(s^{-1}x)\, d\mu(s) \right| \leqq \left| \int_{G-K} f(s^{-1}x)\, d\mu(s) \right| + \left| \int_K f(s^{-1}x)\, d\mu(s) \right|$$
$$\leqq \|f\| \cdot \mu(\complement K) + \varepsilon \cdot \mu(K) \leqq \varepsilon(\|f\| + \|\mu\|),$$

which proves that $\mu * f \in \mathscr{C}^0_c(G)$.

(14.9.3) *Every measure μ on G is convolvable with every function $f \in$ the integral on the right-hand side of* (14.8.2) *is defined for all $x \in G$, function $x \mapsto \int f(s^{-1}x)\, d\mu(s)$ is continuous on* G.

Since the measure $f \cdot \beta$ has compact support, μ and $f \cdot \beta$ are con (14.6.4), and it is clear that the integral $\int f(s^{-1}x)\, d\mu(s)$ is defined for a The continuity of the function $x \mapsto \int f(s^{-1}x)\, d\mu(s)$ follows from (14.

We shall leave to the reader the task of stating the corresponding tions for the convolution $f * \mu$. It should be noticed in particular tha and its analog for $f * \mu$ prove that if G is *unimodular* (14.3), then $\mathscr{L}$ *left and right module* over the algebra $M^1_c(G)$, and the external laws position of these two module structures are *compatible* by virtue of

(14.9.4) *Let μ, ν be two measures on* G, *and let $f \in \mathscr{K}_c(G)$. Supp $\check{\mu}$ and ν are convolvable. Then the function $\mu * f$ is ν-integrable and*

$$\langle f, \check{\mu} * \nu \rangle = \langle \mu * f, \nu \rangle. \tag{14.9.4.1}$$

Likewise, if μ and ν are bounded and $f \in \mathscr{C}^\infty_c(G)$, the function continuous and bounded (hence ν-integrable) and the formula (14.9.4.1

For the hypothesis implies that the function $(s, x) \mapsto f(s^{-1}x)$ is integrable, and the result therefore follows from the theorem of L Fubini.

PROBLEMS

1. Let G be a locally compact group and μ a positive nonzero bounded measure on G, such that $\mu * \mu = \mu$.
(a) Show that $S = \mathrm{Supp}(\mu)$ is compact. (If $f \in \mathscr{K}_+(G)$ is not identically zero, remark that $\mu * f$ must be constant on S, and use (14.9.2(iii)).)
(b) Show that S is a compact subgroup of G and that μ is the Haar measure on S for which $\mu(S) = 1$. (Use (a), Problem 2 of Section 12.9 and Problem 5 of Section 14.7.)

2. (a) Generalize (14.9.2(i)) to the case where $1 < p < +\infty$, by using Hölder's inequality (Section 13.11, Problem 12).
(b) Let μ be a bounded measure on G. Show that the norm of the continuous endomorphism of $L^1(G, \beta)$ induced by $f \mapsto \mu * f$ (14.9.2) is equal to $\|\mu\|$. (Let (f_n) be a sequence satisfying the conditions of (14.11.1). If the norm in question were $\|\mu\| - \alpha$ with $\alpha > 0$, we should have $N_1(\mu * f_n * g) \leqq (\|\mu\| - \alpha)N_1(f_n * g) \leqq (\|\mu\| - \alpha)N_1(g)$. Deduce that $N_\infty(\mu * f_n) \leqq \|\mu\| - \alpha$, and obtain a contradiction by letting n tend to $+\infty$.)
(c) Under the hypotheses of (b), show that the norm of the continuous endomorphism of $L^\infty(G, \beta)$ induced by $f \mapsto \mu * f$ is equal to $\|\mu\|$. (Reduce to the case where μ has compact support and has a continuous density with respect to $|\mu|$.)
(d) Suppose that G is compact and μ is positive. Show that, for $1 < p < +\infty$, the norm of the continuous endomorphism of $L^p(G, \beta)$ induced by $f \mapsto \mu * f$ is equal to $\|\mu\|$.
(e) Let G be a cyclic group of order 3. Give an example of a measure μ on G such that the norm of the endomorphism of $L^2(G, \beta)$ induced by $f \mapsto \mu * f$ is strictly less than $\|\mu\|$.

3. (a) Let μ be a bounded measure on G. Show that we have $N_1(\mu * f) = N_1(f)$ for all $f \in \mathscr{L}^1(G, \beta)$ if and only if μ is of the form $c \cdot \varepsilon_x$ with $|c| = 1$. (Using Problem 2(b), show that for each $f \in \mathscr{K}(G)$ we must have $\left|\int f \, d\mu\right| = \int |f| \, d|\mu|$. Deduce first that $\mu = c|\mu|$, where c is a constant such that $|c| = 1$, and then that μ is a point-measure.)
(b) Take G to be a cyclic group of order 3. Give an example of a measure μ on G which is not a point-measure and is such that $N_2(\mu * f) = N_2(f)$ for all real-valued functions f on G.

4. Let G be a locally compact group and β a left Haar measure on G. Let f be a bounded real-valued function on G, uniformly continuous with respect to a left-invariant distance on G. If μ is any bounded measure on G, show that $\mu * f$ (relative to β) is uniformly continuous with respect to a left-invariant distance on G.

5. Let G be a locally compact group and β a left Haar measure on G.
(a) With the notation of Section 13.20, Problem 1, let (μ_n) be a sequence of bounded measures on G which converges to μ with respect to the topology $\mathscr{T}_3$, and let (f_n) be a sequence of functions in $\mathscr{L}^1(G, \beta)$ such that the sequence of bounded measures $(f_n \cdot \beta)$ converges to $f \cdot \beta$ with respect to the topology $\mathscr{T}_6$. Show that the sequence of bounded measures $(\mu_n * (f_n \cdot \beta))$ converges to $(\mu_n * (f \cdot \beta))$ with respect to $\mathscr{T}_6$ (cf. Problem 4(d) of Section 14.7). (Use Problems 1 and 2 of Section 13.20.)
(b) Let $E_{5/2}$ denote the space of bounded real-valued functions on G which are uniformly continuous with respect to a left-invariant distance on G, and let $\mathscr{T}_{5/2}$

denote the weak topology on $M^1_{\mathbf{R}}(G)$ corresponding to the vector spac
topology $\mathscr{T}_{5/2}$ is finer than $\mathscr{T}_2$ and coarser than $\mathscr{T}_3$. Give an exam
quence (μ_n) which tends to 0 with respect to $\mathscr{T}_{5/2}$, and a sequence (f_n) of
$\mathscr{L}^1(G, \beta)$ such that the sequence $(f_n \cdot \beta)$ converges to 0 with respect to $\mathscr{T}$
that the sequence $(\mu_n * (f_n \cdot \beta))$ does not tend to 0 with respect to $\mathscr{T}_3$. (
and $f_n(t)$ to be the function which is equal to $\sin nt$ in the interval $[0, \pi]$
elsewhere.)

(c) Let (μ_n) be a sequence of bounded measures on G which converge
respect to $\mathscr{T}_{5/2}$, let $f \in \mathscr{L}^1(G, \beta)$ and let (f_n) be a sequence of functions
such that $N_1(f - f_n) \to 0$. Show that $N_1(\mu_n * f_n - \mu * f) \to 0$. (Reduce
where $\mu = 0$ and $f_n = f$ for all n, and then to the case where $f \in \mathscr{K}(G)$. Sh
sequence $(\mu_n * (f \cdot \beta))$ tends to 0 with respect to $\mathscr{T}_3$, by remarking that if g
continuous function on G, the function $f * g$ is uniformly continuous with
left-invariant distance on G.) Show that the result is no longer valid if $\mathscr{T}_{5/2}$
by $\mathscr{T}_2$.

(d) Let (μ_n) be a sequence of bounded measures on G which tends to 0
to $\mathscr{T}_{5/2}$, and (ν_n) a sequence of bounded measures on G which tends to ν
to $\mathscr{T}_2$. Show that the sequence $(\mu_n * \nu_n)$ tends to 0 with respect to $\mathscr{T}_{5/2}$
(14.11.1), reduce to proving that $\langle f * g, \mu_n * \nu_n \rangle \to 0$ for any $f \in \mathscr{K}(G)$
formly continuous with respect to a left-invariant distance on G. Then use

6. The notation is that of Problem 5.

(a) Let (μ_n) be a sequence of bounded measures on G. Suppose that, for ea
$f \in \mathscr{L}^1(G, \beta)$, the sequence $(\mu_n * (f \cdot \beta))$ tends to 0 with respect to $\mathscr{T}_2$. Sh
sequence of norms $(\|\mu_n\|)$ is bounded. (Apply the Banach–Steinhaus theo
sequence of mappings $f \mapsto (\mu_n * f)^\sim$ of $L^1(G, \beta)$ into $L^1(G, \beta)$, and use Pro
Deduce that the sequence (μ_n) tends to 0 with respect to $\mathscr{T}_2$.

(b) Suppose that, for each $f \in \mathscr{L}^1(G, \beta)$, the sequence $(\mu_n * (f \cdot \beta))$ conver
to 0. Show that the sequence (μ_n) tends vaguely to 0. (If K is any compact s
show as in (a) that the sequence $(|\mu_n|(K))$ is bounded.) Give an example in
sequence $(\|\mu_n\|)$ is not bounded (take $G = \mathbf{Z}$).

(c) Suppose that, for each $f \in \mathscr{L}^1(G, \beta)$, the sequence $(\mu_n * (f \cdot \beta))$ con
with respect to $\mathscr{T}_{5/2}$. Show that $\mu_n \to 0$ with respect to $\mathscr{T}_{5/2}$ (use (a)). Give
where $N_1(\mu_n * f) \to 0$ for each $f \in \mathscr{L}^1(G, \beta)$ but the sequence (μ_n) does no
with respect to $\mathscr{T}_3$.

10. CONVOLUTION OF TWO FUNCTIONS

(14.10.1) *Let f and g be two (complex) locally β-integrable functi
Then the measures $f \cdot \beta$ and $g \cdot \beta$ are convolvable if and only if ther
β-negligible set* N *such that the function $s \mapsto g(s^{-1}x)f(s)$ is β-integra
$x \notin$ N and the function*

$$x \mapsto \int |g(s^{-1}x)f(s)|\, d\beta(s)$$

(defined almost everywhere, relative to β) is locally β-integrable. When this condition is satisfied, the function

$$h(x) = \int g(s^{-1}x)f(s)\,d\beta(s),$$

defined almost everywhere relative to β, is locally β-integrable, and

$$(f \cdot \beta) * (g \cdot \beta) = h \cdot \beta.$$

This is a particular case of (14.8.1).

When the conditions of (14.10.1) are satisfied, the functions f and g are said to be *convolvable* (with respect to β). Any function which is equal almost everywhere (with respect to β) to the function h above is called a *convolution of f and g* (with respect to β) and is written $f * g$ (or $f *^{\beta} g$, where it is necessary to bring β into the notation). Thus for almost all x we have

$$(f * g)(x) = \int g(s^{-1}x)f(s)\,d\beta(s) \tag{14.10.2}$$

and likewise, using (14.8.4),

$$(f * g)(x) = \int f(xs^{-1})g(s)\Delta_G(s^{-1})\,d\beta(s) \tag{14.10.3}$$

almost everywhere with respect to β.

When one of the functions equal to h almost everywhere is *continuous* on G, we adopt the same convention as in (14.8) and call this function *the* convolution of f and g.

In particular, when $f * g$ is continuous, we have

$$(f * g)(e) = \int f(s)g(s^{-1})\,d\beta(s). \tag{14.10.4}$$

It should be remarked that the property of f and g of being convolvable does not depend on the choice of left Haar measure β, but their convolution $f *^{\beta} g$ does. If β is replaced by $a\beta$, where $a > 0$, then we have $f *^{a\beta} g = a \cdot f *^{\beta} g$.

When G is *discrete*, to say that f and g are convolvable signifies that the family $(g(s^{-1}x)f(s))_{s \in G}$ is *absolutely summable* (5.3.3) for all $x \in G$, and we have

$$(f * g)(x) = \sum_{s \in G} g(s^{-1}x)f(s)$$

if the Haar measure β is such that $\beta(\{e\}) = 1$.

From the results of (14.9) we have, in particular:

(14.10.5) *Suppose that f and g are locally β-integrable. If one of the functions f, g is continuous, and if one has compact support, then f and g are convolvable, the right-hand sides of* (14.10.2) *and* (14.10.3) *are defined for all $x \in G$, and the function $f * g$ is continuous. If both f and g belong to $\mathscr{K}_C(G)$, then so does $f * g$.*

This follows from (14.9.1) and (14.9.3).

(14.10.6) *Let f be a β-integrable function.*

(i) *For $p = 1$, 2 or $+\infty$, the function f is convolvable with every function $g \in \mathscr{L}^p_C(G, \beta)$; the function $f * g$ belongs to $\mathscr{L}^p_C(G, \beta)$; and*

$$N_p(f * g) \leqq N_1(f) N_p(g). \tag{14.10.6.1}$$

(ii) *If $p = +\infty$, and $g \in \mathscr{L}^p_C(G, \beta)$, the integral*

$$\int g(s^{-1}x) f(s)\, d\beta(s) = \int f(xs^{-1}) g(s) \Delta(s^{-1})\, d\beta(s)$$

is defined for all $x \in G$, and the function $x \mapsto \int f(xs^{-1}) g(s) \Delta(s^{-1})\, d\beta(s)$ is uniformly continuous with respect to every right-invariant distance on G.

(iii) *If $p = 1$, we have*

$$\int (f * g)(x)\, d\beta(x) = \left(\int f(s)\, d\beta(s) \right) \left(\int g(s)\, d\beta(s) \right). \tag{14.10.6.2}$$

(iv) *If $g \in \mathscr{C}^0_C(G)$, then also $f * g \in \mathscr{C}^0_C(G)$.*

Parts (i) and (iv) follow from (14.9.2) and the relation $\|f \cdot \beta\| = N_1(f)$ (13.20.3). To prove (14.10.6.2), we remark that by virtue of (14.8.1.1) and the fact that the function $(s, x) \mapsto g(x) f(s)$ is $(\beta \otimes \beta)$-integrable (13.21.14), the function $(s, x) \mapsto g(s^{-1}x) f(s)$ is also $(\beta \otimes \beta)$-integrable. The formula (14.10.6.2) then comes immediately from the Lebesgue–Fubini theorem and the left-invariance of β.

As to (ii), for each $x \in G$ the function $s \mapsto g(s^{-1}x)$ belongs to $\mathscr{L}^\infty_C(G)$, and therefore the integral on the right-hand side of (14.10.2) is defined for all $x \in G$. If we put $\nu = \Delta^{-1} \cdot \beta$, then ν is a right Haar measure (14.3.4), and we may write (14.10.3) in the form

$$(f * g)(x) = \int f(xs^{-1}) g(s)\, d\nu(s).$$

Consequently,

$$|(f * g)(x) - (f * g)(x')| \leqq N_\infty(g) \int |f(xs^{-1}) - f(x's^{-1})|\, dv(s)$$
$$= N_\infty(g) \int |f(s^{-1}) - f(x'x^{-1}s^{-1})|\, dv(s).$$

The result will therefore be a consequence of the following more general lemma:

(14.10.6.3) *For $p = 1$ or $p = 2$, every right Haar measure v on G and every function $h \in \mathscr{L}^p_{\mathbf{C}}(G, v)$, the mapping $s \mapsto \delta(s)h$ of G into $\mathscr{L}^p_{\mathbf{C}}(G, v)$ is continuous and satisfies $N_p(\delta(s)h) = N_p(h)$.*

The second assertion is an immediate consequence of the right-invariance of v. To prove the first, suppose first of all that $h \in \mathscr{K}_{\mathbf{C}}(G)$; then the continuity of $s \mapsto \delta(s)h$ follows from (14.1.5.5). In the general case, if (h_n) is a sequence of functions in $\mathscr{K}_{\mathbf{C}}(G)$ which converges to h in $\mathscr{L}^p_{\mathbf{C}}(G, \beta)$ (13.11.6), the relation $N_p(\delta(s)h - \delta(s)h_n) = N_p(h - h_n)$ shows that the sequence of functions $s \mapsto \delta(s)h_n$ converges *uniformly* on G to the function $s \mapsto \delta(s)h$. Hence the result (7.2.1).

(14.10.6.4) In the same way, one shows that if $h \in \mathscr{L}^p_{\mathbf{C}}(G, \beta)$ where $p = 1$ or 2, the mapping $s \mapsto \gamma(s)h$ of G into $\mathscr{L}^p_{\mathbf{C}}(G, \beta)$ is continuous, and that $N_p(\gamma(s)h) = N_p(h)$.

(14.10.7) *Let $f \in \mathscr{L}^2_{\mathbf{C}}(G, \beta)$ and let $g \in \mathscr{L}^2_{\mathbf{C}}(G, \check{\beta})$. Then the integral $\int g(s^{-1}x)(f(s)\, d\beta(s)$ is defined for all $x \in G$, and the function $f * g$ belongs to $\mathscr{C}^0_{\mathbf{C}}(G)$ and satisfies the inequality*

(14.10.7.1) $$\|f * g\| \leqq N_2(f)N_2(\check{g}).$$

For each $x \in G$, the function $s \mapsto g(s^{-1}x)$ belongs to $\mathscr{L}^2_{\mathbf{C}}(G, \beta)$, and therefore the first assertion follows from (13.11.7). Moreover, again from (13.11.7),

$$\int |g(s^{-1}x)f(s)|\, d\beta(s) \leqq \left(\int |f(s)|^2\, d\beta(s)\right)^{1/2} \left(\int |g(s^{-1}x)|^2\, d\beta(s)\right)^{1/2}$$
$$= N_2(f)\left(\int |\check{g}(x^{-1}s)|^2\, d\beta(s)\right)^{1/2}$$
$$= N_2(f)N_2(\check{g})$$

which is (14.10.7.1). If both f and g are in $\mathscr{K}_{\mathbf{C}}(\mathrm{G})$, then so is $f * g$; since the formula (14.10.7.1) shows that the bilinear mapping $(f, g) \mapsto f * g$ of $\mathscr{L}^2_{\mathbf{C}}(\mathrm{G}, \beta) \times \mathscr{L}^2_{\mathbf{C}}(\mathrm{G}, \check{\beta})$ into $\mathscr{B}_{\mathbf{C}}(\mathrm{G})$ is continuous (5.5.1), and since $\mathscr{K}_{\mathbf{C}}(\mathrm{G})$ is dense in $\mathscr{L}^2_{\mathbf{C}}(\mathrm{G}, \beta)$ and in $\mathscr{L}^2_{\mathbf{C}}(\mathrm{G}, \check{\beta})$ (13.11.6), the values of the mapping $(f, g) \mapsto f * g$ belong to the closure of $\mathscr{K}_{\mathbf{C}}(\mathrm{G})$ in $\mathscr{B}_{\mathbf{C}}(\mathrm{G})$ (3.11.4): that is, they belong to $\mathscr{C}^0_{\mathbf{C}}(\mathrm{G})$ (13.20.5).

Proposition (14.10.7) implies the following corollary:

(14.10.8) *Let* A *and* B *be two* β*-integrable sets in* G. *Then the function* $x \mapsto \beta(\mathrm{A} \cap x\mathrm{B})$ *is continuous on* G *and tends to* 0 *at infinity* (13.20.6). *If* B^{-1} *is also* β*-integrable, the function* $x \mapsto \beta(\mathrm{A} \cap x\mathrm{B})$ *is* β*-integrable, and we have*

$$\int \beta(\mathrm{A} \cap x\mathrm{B})\, d\beta(x) = \beta(\mathrm{A})\beta(\mathrm{B}^{-1}).$$

If moreover neither A *nor* B *is* β*-negligible, the set* AB^{-1} *has a nonempty interior. Finally, for every subset* A *which is* β*-measurable and not* β*-negligible, the set* AA^{-1} *is a neighborhood of the neutral element* e *of* G.

We have $\varphi_{\mathrm{A}} \in \mathscr{L}^2(\mathrm{G}, \beta)$ and $\varphi_{\mathrm{B}} \in \mathscr{L}^2(\mathrm{G}, \beta)$, hence we may apply (14.10.7) with $f = \varphi_{\mathrm{A}}$ and $g = \check{\varphi}_{\mathrm{B}}$. Since $\check{\varphi}_{\mathrm{B}}(s^{-1}x) = \varphi_{\mathrm{B}}(x^{-1}s) = \varphi_{x\mathrm{B}}(s)$, we have $(\varphi_{\mathrm{A}} * \check{\varphi}_{\mathrm{B}})(x) = \int \varphi_{\mathrm{A} \cap x\mathrm{B}}(s)\, d\beta(s) = \beta(\mathrm{A} \cap x\mathrm{B})$. This proves the first assertion. If B^{-1} is also β-integrable, then both φ_{A} and $\varphi_{\mathrm{B}^{-1}}$ belong to $\mathscr{L}^1(\mathrm{G}, \beta)$, and we can apply (14.10.6.2), which gives us the formula $\int \beta(\mathrm{A} \cap x\mathrm{B})\, d\beta(x) = \beta(\mathrm{A})\beta(\mathrm{B}^{-1})$. If the right-hand side of this formula is nonzero, it follows that $\beta(\mathrm{A} \cap x\mathrm{B})$ is not identically zero, and hence there exists a nonempty open set U in which the continuous function $\beta(\mathrm{A} \cap x\mathrm{B})$ is > 0. This implies that $\mathrm{U} \subset \mathrm{AB}^{-1}$. Finally, to prove the last assertion, we observe that there exists a compact set $\mathrm{K} \subset \mathrm{A}$ which is not β-negligible, so that we may assume that A is compact; but then $\beta(\mathrm{A}) = (\varphi_{\mathrm{A}} * \check{\varphi}_{\mathrm{A}})(e)$ is > 0, and it follows as above that there exists a neighborhood of e contained in AA^{-1}.

(14.10.9) *Let* f, g *be two locally* β*-integrable functions,* μ *a measure on* G. *The two following conditions are equivalent:*

(a) f *and* $\check{g}$ *are convolvable and the function* $|f| * |\check{g}|$ *is* β*-integrable*;

(b) $\check{\mu}$ *and* f *are convolvable and the function* $g(|\check{\mu}| * |f|)$ *is* β*-integrable.*

If these conditions are satisfied, we have

$$\langle g, \check{\mu} * (f \cdot \beta) \rangle = \langle f * \check{g}, \mu \rangle. \tag{14.10.9.1}$$

From (13.21.9) and (14.8.1), it follows that condition (a) means that the

function $(s, x) \mapsto g(x^{-1}s)f(x) = g(s^{-1}x)f(x)$ is $(\mu \otimes \beta)$-integrable, and the right-hand side of (14.10.9.1) is then equal to

$$\iint g(s^{-1}x)f(x)\, d\mu(s)\, d\beta(x). \tag{14.10.9.2}$$

Similarly, condition (b) means that the function $(s, x) \mapsto g(x)f(sx)$ is $(\mu \otimes \beta)$-integrable, and the left-hand side of (14.10.9.1) is then equal to

$$\iint g(x)f(sx)\, d\mu(s)\, d\beta(x). \tag{14.10.9.3}$$

But from the theorem of Lebesgue–Fubini and the left-invariance of β, it follows that one of the integrals (14.10.9.2), (14.10.9.3) exists if and only if the other one exists, and then both are equal.

PROBLEMS

1. (a) Let g be a β-measurable function of *compact support* on G. Show that f and g are convolvable and that $f * g$ is *continuous* on G in the following two cases:

(i) g is essentially bounded and f is locally β-integrable;

(ii) $g \in \mathscr{L}^2_{\mathbf{C}}(\mathrm{G}, \beta)$, f is β-measurable and f^2 is locally β-integrable.

(b) Assume that G is unimodular. Show that if $f \in \mathscr{L}^q_{\mathbf{C}}(\mathrm{G}, \beta)$ and $g \in \mathscr{L}^p_{\mathbf{C}}(\mathrm{G}, \beta)$, where $p \geqq 1, q \geqq 1$ and $\frac{1}{p} + \frac{1}{q} \geqq 1$, then f and g are convolvable, $f * g \in \mathscr{L}^r_{\mathbf{C}}(\mathrm{G}, \beta)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, and $\mathrm{N}_r(f * g) \leqq \mathrm{N}_q(f)\mathrm{N}_p(g)$ (W. Young's inequality). (Consider first the case where $\frac{1}{p} + \frac{1}{q} = 1$, then use (14.10.6.1) and the Riesz–Thorin theorem (Section 13.17, Problem 7).)

2. Let G be a compact group, β the Haar measure on G for which $\beta(\mathrm{G}) = 1$, and let A, B be two β-integrable sets. Show that for each $\varepsilon > 0$ there exists $x \in \mathrm{G}$ such that $\beta(\mathrm{A} \cap x\mathrm{B}) \leqq (1 + \varepsilon)\beta(\mathrm{A})\beta(\mathrm{B})$. Deduce that, for each integer n such that $\beta(\mathrm{A}) \geqq 1/n$, there exist n points $x_1, \ldots, x_n$ in G such that $\beta(x_1\mathrm{A} \cup x_2\mathrm{A} \cup \cdots \cup x_n\mathrm{A}) \geqq \frac{1}{2}$. (Consider the sets $x_j \cdot \complement\mathrm{A}$.)

3. Let G be a compact commutative group, β the Haar measure on G for which $\beta(\mathrm{G}) = 1$, and g a β-measurable function on G. Let (r_n) be the orthonormal system of Rademacher functions (Section 13.21, Problem 10). Let $\alpha > 0$ and let A be the set of $x \in \mathrm{G}$ such that $|g(x)| > \alpha$.

Let n be an integer $\geqq 1$ such that $n \cdot \beta(\mathrm{A}) \geqq 1$. Show that there exists a β-measurable set B such that $\beta(\mathrm{B}) \geqq \frac{1}{2}$, and n points $s_1, \ldots, s_n$ in G such that, if we put $\mathrm{F}(x, t) = \sum_{k=1}^{n} r_k(t)g(s_k x)$, the following is true: for each $x \in \mathrm{B}$, there exists a finite union $\mathrm{I}(x)$ of intervals in $[0, 1]$ such that (i) $\lambda(\mathrm{I}(x)) \geqq \frac{1}{2}$ (λ being Lebesgue measure); (ii) for all

$t \in \mathrm{I}(x)$, $\mathrm{F}(x, t) > \alpha$. (Use Problem 2, and observe that, by virtue of the relation $r_n(1-t) = -r_n(t)$, for a given integer $h \in [1, n]$ the set of $t \in [0, 1]$ such that the numbers $r_h(t)g(s_h x)$ and $\sum_{k \neq h} r_k(t)g(s_k x)$ have the same sign, has measure $\geqq \frac{1}{2}$.)

4. Let G be a compact commutative group, β the Haar measure on G for which $\beta(\mathrm{G}) = 1$. Let (U_n) be a sequence of continuous endomorphisms of $\mathrm{L}^2_{\mathbf{R}}(\mathrm{G}, \beta)$, each of which commutes with all translations $\tilde{f} \mapsto (\gamma(s)f)^{\sim}$ for all $s \in \mathrm{G}$. For each $f \in \mathscr{L}^2_{\mathbf{R}}(\mathrm{G}, \beta)$ we denote by $U_n \cdot f$ any function belonging to the class $U_n \cdot \tilde{f}$, and we put

$$U^* \cdot f = \sup_n U_n \cdot f.$$

For each $\alpha > 0$, let $\mathrm{E}_\alpha(f)$ be the set of points $x \in \mathrm{G}$ such that $(U^* \cdot f)(x) > \alpha$.

(a) Let $f \in \mathscr{L}^2_{\mathbf{R}}(\mathrm{G}, \beta)$ be such that $\mathrm{N}_2(f) \leqq 1$, and let n be an integer such that $n \cdot \beta(\mathrm{E}_\alpha(f)) > 1$. Let $s_1, \ldots, s_n$ be points of G such that the union B of the sets $s_j^{-1}\mathrm{E}_\alpha(f)$ has measure $\beta(\mathrm{B}) \geqq \frac{1}{2}$ (Problem 2), and let $\mathrm{F}(x, t) = \sum_{k=1}^{n} r_k(t) f(s_k x)$. Show that for each $x \in \mathrm{B}$ there exists a finite union of intervals $\mathrm{I}(x) \subset [0, 1]$ such that $\lambda(\mathrm{I}(x)) \geqq \frac{1}{2}$ and such that $x \in \mathrm{E}_\alpha(\mathrm{F}(\cdot, t))$ for all $t \in \mathrm{I}(x)$. (Observe that if $x \in \mathrm{B}$ there exists an integer m and an integer $j \in [1, n]$ such that $(U_m \cdot f)(s_j x) = 0$, and apply Problem 3 to $g = U_m \cdot f$.)

(b) Let $\mathrm{S} \subset [0, 1]$ be a λ-integrable set such that $\lambda(\mathrm{S}) \geqq \frac{3}{4}$. Show that there exists $t \in \mathrm{S}$ such that $\beta(\mathrm{E}_\alpha(\mathrm{F}(\cdot, t))) > \frac{1}{8}$. (If H is the set of $(x, t) \in \mathrm{G} \times [0, 1]$ such that $(U^* \cdot \mathrm{F}(\cdot, t))(x) > \alpha$, remark that for each $x \in \mathrm{B}$ we have $\lambda(\mathrm{H}(x)) \geqq \frac{1}{4}$, and deduce that $(\beta \otimes \lambda)(\mathrm{H}) > \frac{1}{8}$.)

(c) For each $\mathrm{M} > 0$, let S_M be the set of $t \in [0, 1]$ such that $\mathrm{N}_2(\mathrm{F}(\cdot, t)) \leqq \mathrm{M}$. Show that $\lambda(\mathrm{S}_\mathrm{M}) \geqq 1 - n/\mathrm{M}^2$. Deduce from (b) that if $\mathrm{M}^2 \geqq 4n$ there exists $t \in [0, 1]$ such that both $\mathrm{N}_2(\mathrm{F}(\cdot, t)) \leqq \mathrm{M}$ and $\beta(\mathrm{E}_\alpha(\mathrm{F}(\cdot\, t))) > \frac{1}{8}$.

(d) Suppose that for each $f \in \mathscr{L}^2_{\mathbf{R}}(\mathrm{G}, \beta)$ the function $U^* \cdot f$ is finite almost everywhere. Show that there exists a constant $\mathrm{C} > 0$ such that, for each $f \in \mathscr{L}^2_{\mathbf{R}}(\mathrm{G}, \beta)$ and each $\alpha > 0$, we have $(\beta(\mathrm{E}_\alpha(f)))^{1/2} \leqq \mathrm{C}\alpha^{-1}\mathrm{N}_2(f)$ (E. Stein's theorem). (Deduce from Section 13.12, Problem 12 that there exists a constant $c > 0$ such that for each function $h \in \mathscr{L}^2_{\mathbf{R}}(\mathrm{G}, \beta)$ satisfying $\mathrm{N}_2(h) \leqq \mathrm{M}$, we have $\beta(\mathrm{E}_{c\mathrm{M}}(h)) < \frac{1}{8}$. Then make use of (c) above, taking $\mathrm{M} = \alpha/c$, $h = \mathrm{F}(\cdot, t)$ and $n = [\frac{1}{4}\mathrm{M}^2]$.)

5. Let λ be Lebesgue measure on $\mathbf{R}$ and let f be a compactly supported λ-integrable function. Put

$$\theta_\varepsilon(f)(x) = \sup_{h > \varepsilon} \frac{1}{2h} \int_{-h}^{h} |f(x-t)|\, dt$$

for each $\varepsilon > 0$, and

$$\theta(f)(x) = \sup_{h > 0} \frac{1}{2h} \int_{-h}^{h} |f(x-t)|\, dt;$$

these are lower semicontinuous functions on $\mathbf{R}$ (Problem 1) with compact support. $\theta(f)$ is the *Hardy–Littlewood maximal function* relative to f.

(a) For each $\alpha > 0$, let $\mathrm{E}_{\varepsilon,\alpha}(f)$ be the set of $x \in \mathbf{R}$ such that $\theta_\varepsilon(f)(x) > \alpha$. Every compact set $\mathrm{K} \subset \mathrm{E}_{\varepsilon,\alpha}(f)$ is contained in the union of a finite number of compact intervals I_k $(1 \leqq k \leqq n)$ such that $\alpha \cdot \lambda(\mathrm{I}_k) \leqq \int_{\mathrm{I}_k} |f(t)|\, dt$.

(b) Show that there exists a sequence of indices $(k_j)_{1 \leqslant j \leqslant m}$ such that the intervals I_{k_j} are mutually disjoint and such that

$$\lambda\left(\bigcup_{1 \leqslant k \leqslant n} I_k\right) \leqq 2 \sum_{j=1}^{m} \lambda(I_{k_j}).$$

(We may assume that no I_k is contained in the union of the I_h such that $h \neq k$. Show that if we put $I_k = [a_k, b_k]$ and arrange the I_k so that $a_k \leqq a_{k+1}$, then necessarily $a_k < a_{k+1}$, $b_{2k-1} < a_{2k+1}$ and $b_{2k} < a_{2k+2}$, by considering three intervals with consecutive indices. Deduce that the intervals I_{2k-1} are mutually disjoint, the I_{2k} are mutually disjoint and that the k_j may therefore be taken to be either the even indices or the odd indices.)

(c) Deduce from (a) and (b) that if $E_\alpha(f)$ is the set of $x \in \mathbf{R}$ such that $\theta(f)(x) > \alpha$, then

$$\lambda(E_\alpha)(f) \leqq \frac{2}{\alpha} \int_{-\infty}^{+\infty} |f(t)|\, dt.$$

(d) Let g be a nonnegative function in $\mathscr{L}^1_{\mathbf{R}}(\lambda)$ such that (i) $g(-t) = g(t)$; (ii) g is decreasing on $[0, +\infty[$; (iii) $\int g(t)\, dt = 1$. Prove that $|(g * f)(x)| \leqq \theta(f)(x)$ for all $x \in \mathbf{R}$. (For each $\alpha > 0$, let $]-h(\alpha), h(\alpha)[$ be the largest open interval in which $g(t) > \alpha$. Show that

$$(g * f)(x) = \int_0^{+\infty} d\alpha \int_{-h(\alpha)}^{h(\alpha)} f(x - t)\, dt$$

and observe that $\int_0^{+\infty} h(\alpha)\, d\alpha = \frac{1}{2}$.)

(e) State and prove analogous results for functions which are integrable with respect to Haar measure on the torus $\mathbf{T}$.

11. REGULARIZATION

(14.11.1) *Let (f_n) be a sequence of β-integrable functions which satisfy the following conditions:*

(a) *the sequence of integrals $\int |f_n(x)|\, d\beta(x)$ is bounded;*

(b) *the sequence of integrals $\int f_n(x)\, d\beta(x)$ tends to 1;*

(c) *for each neighborhood V of e, the sequence of integrals $\int_{\complement V} |f_n(x)|\, d\beta(x)$ tends to 0.*

Then

(i) *For each bounded continuous function g on G, the sequence $(f_n * g)$ converges uniformly to g on every compact subset of G. If g is uniformly continuous with respect to a right-invariant distance on G, then the sequence $(f_n * g)$ converges uniformly to g in G.*

(ii) *If* $p = 1$ or 2 *and* $g \in \mathscr{L}^p(G)$, *the sequence of norms* $N_p((f_n * g) - g)$ *tends to* 0 *as* n *tends to* $+\infty$. *For each function* $g \in L_c^\infty(G)$, *the sequence* $((f_n \cdot g)^\sim)$ *converges weakly to* $\tilde{g}$ *in* $L_c^\infty(G)$, *considered as the dual of* $L_c^1(G)$ (12.15 and 13.17).

(iii) *Suppose in addition that the supports of the* f_n *are contained in a fixed compact subset of* G. *Then for each measure* μ *on* G, *the sequence of measures*

$$\mu * (f_n \cdot \beta) = (\mu * f_n) \cdot \beta$$

converges vaguely to μ (13.4).

(i) For each $x \in G$ and each compact neighborhood V of e we have from the definitions

$$\begin{aligned} g(x) - (f_n * g)(x) = g(x)\left(1 - \int_V f_n(s)\, d\beta(s)\right) \\ + \int_V f_n(s)(g(x) - g(s^{-1}x))\, d\beta(s) \\ - \int_{\complement V} f_n(s) g(s^{-1}x)\, d\beta(s). \end{aligned}$$

Let V_0 be a compact neighborhood of e and let L be a compact subset of G. Since $V_0^{-1}L$ is compact (12.10.5), the restriction of g to $V_0^{-1}L$ is uniformly continuous with respect to a right-invariant distance on G (3.16.5). Hence, for each $\varepsilon > 0$, there exists a compact neighborhood $V \subset V_0$ of e such that $|g(x) - g(s^{-1}x)| \leqq \varepsilon$ for all $x \in L$ and all $s \in V$. Now choose n_0 so that $\int_{\complement V} |f_n(s)|\, d\beta(s) \leqq \varepsilon$ and $|1 - \int f_n(s)\, d\beta(s)| \leqq \varepsilon$ for all $n \geqq n_0$. We have then

$$\left|1 - \int_V f_n(s)\, d\beta(s)\right| \leqq 2\varepsilon,$$

and therefore, for all $x \in L$,

$$\left|g(x)\left(1 - \int_V f_n(s)\, d\beta(s)\right)\right| \leqq 2\|g\|\varepsilon,$$

$$\left|\int_V f_n(s)(g(x) - g(s^{-1}x))\, d\beta(s)\right| \leqq a\varepsilon,$$

$$\left|\int_{\complement V} f_n(s) g(s^{-1}x)\, d\beta(s)\right| \leqq \|g\|\varepsilon,$$

where $a = \sup_n \int |f_n(s)|\, d\beta(s)$. Hence

$$|g(x) - (f_n * g)(x)| \leqq (a + 3\|g\|)\varepsilon.$$

If g is uniformly continuous with respect to a right-invariant distance on G, the same argument applies, taking $L = G$.

(ii) There exists a function $h \in \mathscr{K}_C(G)$ such that $N_p(g - h) \leqq \varepsilon$ (13.11.6). By (14.10.6.1), it follows that $N_p((f_n * g) - (f_n * h)) \leqq aN_p(g - h) \leqq a\varepsilon$. Thus we are reduced to proving (ii) when $g \in \mathscr{K}_C(G)$. Let S be the support of g. If V is a compact neighborhood of e, we have seen in (i) that $f_n * g - g$ converges uniformly to 0 in the compact set $K = S \cup VS$. Next, if $x \notin K$, we have

$$(f_n * g)(x) - g(x) = \int_{\complement V} f_n(s) g(s^{-1}x)\, d\beta(s)$$

whence, by virtue of the Lebesgue–Fubini theorem and the invariance of Haar measure

$$\begin{aligned}\int_{\complement K} |(f_n * g)(x) - g(x)|\, d\beta(x) &\leqq \int_G d\beta(x) \int_{\complement V} |f_n(s)| \cdot |g(s^{-1}x)|\, d\beta(s) \\ &= \int_{\complement V} d\beta(s) \int_G |f_n(s)| \cdot |g(s^{-1}x)|\, d\beta(x) \\ &= N_1(g) \int_{\complement V} |f_n(s)|\, d\beta(s).\end{aligned}$$

Since, on the other hand, the integral $\int_K |(f_n * g)(x) - g(x)|\, d\beta(x)$ tends to 0 with $1/n$, we see that

$$N_1(f_n * g - g) \to 0,$$

and as $N_\infty(f_n * g - g) \leqq (a + 1)\|g\|$, the same is true of

$$N_2(f_n * g - g)^2 \leqq (a + 1)\|g\|\, N_1(f_n * g - g).$$

Suppose now that $g \in \mathscr{L}^\infty_C(G)$, and let $h \in \mathscr{L}^1_C(G)$. If V is any compact neighborhood of e in G, we have

$$\begin{aligned}\langle h, f_n * g - g\rangle = &\left(1 - \int_V f_n(s)\, d\beta(s)\right)\langle h, g\rangle \\ &- \int_{\complement V} f_n(s)\, d\beta(s) \int_G g(s^{-1}x) h(x)\, d\beta(x) \\ &+ \int_V f_n(s)\, d\beta(s) \int_G h(x)(g(x) - g(s^{-1}x))\, d\beta(x).\end{aligned}$$

But we have

$$\int h(x)(g(x) - g(s^{-1}x))\, d\beta(x) = \int g(x)(h(x) - h(sx))\, d\beta(x),$$

and therefore,

$$\left| \int h(x)(g(x) - g(s^{-1}x))\, d\beta(x) \right| \leqq \mathrm{N}_\infty(g)\mathrm{N}_1(h - \gamma(s^{-1})h).$$

For each $\varepsilon > 0$, it follows from (14.10.6.4) that there exists a compact neighborhood V of e in G such that $\mathrm{N}_1(h - \gamma(s^{-1})h) \leqq \varepsilon$ for all $s \in \mathrm{V}$, and hence,

$$\left| \int_{\mathrm{V}} f_n(s)\, d\beta(s) \int_{\mathrm{G}} h(x)(g(x) - g(s^{-1}x))\, d\beta(x) \right| \leqq a\mathrm{N}_\infty(g)\varepsilon.$$

Choose n_0 such that $\int_{\complement \mathrm{V}} |f_n(s)|\, d\beta(s) \leqq \varepsilon$ and $\left| 1 - \int f_n(s)\, d\beta(s) \right| \leqq \varepsilon$ for all $n \geqq n_0$; then we have

$$\left| \left(1 - \int_{\mathrm{V}} f_n(s)\, d\beta(s) \right) \langle h, g \rangle \right| \leqq 2|\langle h, g \rangle|\varepsilon,$$

and

$$\left| \int_{\complement \mathrm{V}} f_n(s)\, d\beta(s) \int_{\mathrm{G}} g(s^{-1}x)h(x)\, d\beta(x) \right| \leqq \mathrm{N}_\infty(g)\mathrm{N}_1(h)\varepsilon,$$

and therefore, finally

$$|\langle h, f_n * g - g \rangle| \leqq (2|\langle h, g \rangle| + \mathrm{N}_\infty(g)\mathrm{N}_1(h) + a\mathrm{N}_\infty(g))\varepsilon$$

for all $n \geqq n_0$, which shows that $(f_n * g)^\sim$ converges weakly to $\tilde{g}$.

Examples

(14.11.2) Let (V_n) be a fundamental system of neighborhoods of the neutral element e in G. Since the support of β is the whole of G, there exists for each n a function $f_n \in \mathscr{K}_+(\mathrm{G})$ with support contained in V_n and such that $\int f_n(x)\, d\beta(x) \neq 0$ (13.19.1). Multiplying each f_n by a suitable constant, we may assume that $\int f_n(x)\, d\beta(x) = 1$, and the sequence (f_n) now satisfies the conditions of (14.11.1). We see therefore that *every measure* μ on G may be approximated (in the sense of the vague topology) by a sequence of "regularizations" which are measures having a *continuous density* with respect to β (14.9.3).

(14.11.3) Take $G = \mathbf{R}$, and β to be Lebesgue measure. Put

$$g_n(x) = \begin{cases} (1-x^2)^n & \text{if } x \in [-1, 1], \\ 0 & \text{if } |x| > 1. \end{cases} \tag{14.11.3.1}$$

Let $a_n = \int_{-1}^{1} g_n(x)\,dx$, and $f_n = a_n^{-1} g_n$. The sequence (f_n) satisfies the conditions of (14.11.1). For $1 - x^2 \geqq 1 - |x|$ for $1 \leqq x \leqq 1$, hence

$$a_n \geqq 2\int_0^1 (1-x)^n\,dx = \frac{2}{n+1},$$

and therefore $f_n(x) \leqq (n+1)(1-x^2)^n$ for all $x \in [-1, 1]$, which proves that $f_n(x) \to 0$ uniformly on every compact interval not containing 0. Let μ be a measure on $\mathbf{R}$ with support contained in $[-\frac{1}{2}, \frac{1}{2}]$. Then we have

$$(\mu * f_n)(x) = a_n^{-1} \int_{-1/2}^{1/2} g_n(x-y)\,d\mu(y),$$

and if $x \in [-\frac{1}{2}, \frac{1}{2}]$ this gives

$$(\mu * f_n)(x) = a_n^{-1} \int_{-1/2}^{1/2} (1-(x-y)^2)^n\,d\mu(y),$$

showing that the function $\mu * f_n$ is equal to a *polynomial* function on $[-\frac{1}{2}, \frac{1}{2}]$. In particular, if $\mu = h \cdot \beta$, where h is a continuous function with support contained in $[-\frac{1}{2}, \frac{1}{2}]$, we obtain from (14.11.1(i)) the theorem of Weierstrass on uniform approximation of continuous functions by polynomials on a compact interval (7.4.1).

PROBLEMS

1. If a locally compact group G is such that the algebra $\mathscr{K}(G)$ is commutative with respect to convolution, show that G is commutative. (Show by regularization that the algebra of measures with compact support is commutative.)

2. Let G be a locally compact group, β a left Haar measure on G. Show that the algebra $L^1(G, \beta)$ has a unit element if and only if G is discrete. (Suppose that G is not discrete, and let $f_0 \in \mathscr{L}^1(G, \beta)$; then there exists a compact neighborhood V of e such that $\int_V |f_0(x)|\,d\beta(x) < 1$. Show that, if U is a compact symmetric neighborhood of e such that $U^2 \subset V$, then $|(\varphi_U * f_0)(x)| < 1$ for almost all $x \in U$, and hence that $\check{f}_0$ cannot be the unit element of $L^1(G, \beta)$.)

3. Let G be a locally compact group, β a left Haar measure on G. If $G \neq \{e\}$, the algebra $L^1(G, \beta)$ has zero-divisors $\neq 0$. To construct two nonnegligible functions f, g in $\mathscr{L}^1(G, \beta)$ such that $f * g$ is negligible, we may proceed as follows:

(1) The case where G has a compact subgroup $H \neq \{e\}$. Take for f a characteristic function φ_A and for g a function of the form $\varphi_{sB} - \varphi_B$, where A, B, and s are suitably chosen, and remark that $\Delta_G(x) = 1$ for all $x \in H$.

(2) The case $G = \mathbf{Z}$. Show that we may take

$$f(n) = \frac{1}{2n-1} - \frac{1}{2n+1}$$

for all $n \in \mathbf{Z}$, and $g = f$.

(3) The general case. Prove first of all that there exists $a \neq e$ in G such that $\Delta(a) = 1$. The closure H in G of the subgroup generated by a is then either compact or isomorphic to Z (Section 12.9, Problem 10). In the former case, use the result of case (1); in the latter, take

$$f(t) = \sum_{n=-\infty}^{+\infty} \alpha_n \varphi_{Ua^{-n}}(t), \qquad g(t) = \sum_{n=-\infty}^{\infty} \beta_n \varphi_{a^n U}(t)$$

where the set U and the sequences (α_n), (β_n) are suitably chosen with the help of case (2).

4. Let G be a locally compact group, β a left Haar measure on G. In order that a subset H of $\mathscr{L}^p(G, \beta)$ $(1 \leqq p < +\infty)$ should have a relatively compact image $\tilde{H}$ in $L^p(G, \beta)$, it is necessary and sufficient that the following conditions should be satisfied: (1) $\tilde{H}$ is bounded in $L^p(G, \beta)$; (2) for each $\varepsilon > 0$, there exists a compact subset K of G such that $N_p(f\varphi_{G-K}) \leqq \varepsilon$ for all $f \in H$; (3) for each $\varepsilon > 0$, there exists a neighborhood V of e in G such that $N_p((\gamma(s)f) - f) \leqq \varepsilon$ for all $f \in H$ and all $s \in V$. (To prove that these conditions are sufficient, observe that if $g \in \mathscr{K}(G)$ and if L is a compact subset of G, then the image, under the mapping $f \mapsto g * f$, of the set of restrictions to L of functions belonging to H is an equicontinuous subset of $\mathscr{K}(G)$.)

5. Let G be a locally compact group, β a left Haar measure on G, and μ a positive measure on G. Show that if A is a μ-integrable set and B is a universally measurable set in G, the function $u : s \mapsto \mu(A \cap sB)$ is β-measurable on G. If moreover B^{-1} is β-integrable, then so is u and we have

$$\int \mu(A \cap sB)\, d\beta(s) = \mu(A)\beta(B^{-1}).$$

(Use the Lebesgue–Fubini theorem and Problem 21 of Section 13.9.) Give an example of a measure μ such that the function $s \mapsto \mu(A \cap sB)$ is not continuous. If μ is a measure with basis β and if A is μ-integrable and B is universally measurable, then the function $s \mapsto \mu(A \cap sB)$ is continuous on G.

6. Let G be a locally compact group, G′ a topological group, f a homomorphism of G into G′ which is β-measurable, where β is a left Haar measure on G. Show that f is *continuous*. (Observe that if we put $g(x) = f(x^{-1})$, there exists a compact non-β-negligible subset K of G such that the restrictions of f and g to K are continuous. Deduce that the restriction of f to $K \cdot K^{-1}$ is continuous, by using (12.3.8); then apply (14.10.8).)

7. Let G be a locally compact group. For each $t \in \mathbf{R}_+^*$, let $\mu_t \neq 0$ be a positive bounded measure on G. Suppose that the mapping $t \mapsto \mu_t$ is continuous with respect to the topology $\mathscr{T}_2$ (in the notation of Section 13.20, Problem 1), and that $\mu_{s+t} = \mu_s * \mu_t$ for all s, t in $\mathbf{R}_+^*$.
(a) Show that there exists a real number e such that $\|\mu_t\| = e^{ct}$. (Observe that the mapping $t \mapsto \|\mu_t\|$ is lower semicontinuous, and apply Problem 6.)
(b) Show that, as $t \to 0$, μ_t converges (with respect to $\mathscr{T}_2$) to a Haar measure of total mass 1 on a compact subgroup of G. (Use (12.15.9) to show that there exists a sequence (s_n) tending to 0 such that the sequence (μ_{s_n}) tends to a limit μ in the topology $\mathscr{T}_2$, and that $\mu_t * \mu = \mu_t = \mu * \mu_t$ for all $t > 0$. Deduce that $\mu_s \to \mu$ as $s \to 0$, and that $\mu * \mu = \mu$. Complete the proof by using Problem 5 of Section 14.7.)

8. Let λ denote Lebesgue measure on $\mathbf{R}^n$. Let A be a bounded convex open set in $\mathbf{R}^n$ (Section 8.5, Problem 8), and let $\mathrm{D(A)} = \mathrm{A} - \mathrm{A}$ be the set of all $x - y$ where $x, y \in \mathrm{A}$. The set D(A) is convex, open, symmetric and bounded. For each $x \neq 0$ in D(A), let $\rho(x)$ be the unique real number, belonging to $]0, 1[$, such that $\rho(x)^{-1}x$ lies in the frontier of D(A). If we put $\rho(0) = 0$, then ρ is a continuous function on D(A) (Section 12.14, Problem 12).
(a) For each $x \in \mathrm{D(A)}$, show that

$$\lambda(\mathrm{A} \cap (\mathrm{A} + x)) \geqq (1 - \rho(x))^n \lambda(\mathrm{A}).$$

(Observe that if $x \neq 0$ and $\rho(x)^{-1}x = b - a$ where $a, b \in \mathrm{A}$, then

$$(1 - \rho(x))\mathrm{A} + \rho(x)b = (1 - \rho(x))\mathrm{A} + \rho(x)a + x).$$

(b) Show that

$$(\lambda(\mathrm{A}))^2 = \int_{\mathrm{D(A)}} (\varphi_{\mathrm{A}} * \check{\varphi}_{\mathrm{A}})(x)\, d\lambda(x).$$

(c) Deduce from (a) and (b) that

$$\lambda(\mathrm{A}) \geqq \int_{\mathrm{D(A)}} (1 - \rho(x))^n\, d\lambda(x).$$

Prove that this integral is equal to

$$\lambda(\mathrm{D(A)}) \int_0^1 nt^{n-1}(1 - t)^n\, dt = \frac{(n!)^2}{(2n)!} \lambda(\mathrm{D(A)}).$$

(Split up the interval $[0, 1]$ into m parts by means of an increasing sequence $(t_k)_{0 \leqslant k \leqslant m}$, and consider D(A) as the union of the sets defined by $t_k < \rho(x) \leqq t_{k+1}$; then let $m \to \infty$.)
Hence show that

$$\lambda(\mathrm{D(A)}) \leqq \binom{2n}{n} \lambda(\mathrm{A}).$$

9. Let G be a locally compact group, β a left Haar measure on G, and let V be a symmetric compact neighborhood of e in G *which does not contain any subgroup other than* $\{e\}$.

(a) Let f be a continuous function on G, with values in [0, 1] and support contained in V, and such that $f(e) > 0$. Put

$$g(x) = \int f(xs) f(s)\, d\beta(s).$$

Show that for each $z \in G$ we have $\|\gamma(z)g - g\| \leqq c\|\gamma(z)f - f\|$, where $c = \int f(s)\, d\beta(s)$, and that $g(z) < g(e)$ if $z \neq e$. (If not, we should have $\gamma(z^{-1})f = f$, which would imply that $z^k \in V$ for all k.)

(b) Let (f_n) be a uniformly bounded equicontinuous family of positive functions with support contained in V, and suppose that for each n there exists a subset $A_n \subset V$ such that $\|\gamma(z)f_n - f_n\| \leqq M$ for all n and all $z \in A_n$. Show that the set of functions

$$g_n(x) = \int f_n(xs) f_n(s)\, d\beta(s)$$

is equicontinuous, and that the same is true of the set of functions $\gamma(z)g_n - g_n$ where $z \in A_n$, for each n.

10. Let G be a locally compact group and let u, v be two positive bounded functions on G. For each $x \in G$ let

$$w(x) = \sup_{y \in G} u(y)v(y^{-1}x).$$

(a) Show that $w(x) = \sup_{y \in G} u(xy^{-1})v(y)$, and that for each $s \in G$ we have

$$\|\gamma(s)w - w\| \leqq \|v\| \cdot \|\gamma(s)u - u\|.$$

If A, B are the supports of u, v respectively, show that the support of w is contained in the closure of AB.

(b) Suppose that $v(x^{-1}) = v(x)$. Show that, for all x, x' in G, we have

$$|w(x) - w(x')| \leqq \|\gamma(x)v - \gamma(x')v\| \cdot \|u\|.$$

(Observe that for each $y \in G$ we have

$$\begin{aligned}\|u\| \cdot \|\gamma(x)v - \gamma(x')v\| &\geqq u(y)(v(y^{-1}x) - v(y^{-1}x')) \\ &\geqq u(y)v(y^{-1}x) - w(x').)\end{aligned}$$

In particular, if v is continuous and compactly supported, then so is w.

11. Let G be a locally compact group, and V a symmetric compact neighborhood of e in G such that V^2 contains no subgroup $\neq \{e\}$. For each integer i, let U_i be the set of points x such that $x^k \in V$ for $1 \leqq k \leqq i$. As $i \to +\infty$, the sets U_i form a fundamental system of neighborhoods of e in G (Section 12.9, Problem 6b)). Let n_i be the largest integer n such that $U_i^n \subset V$. Define a function u_i on G as follows:

$$u_i(e) = 1, \qquad u_i(x) = 1 - \frac{k}{n_i} \quad \text{if} \quad x \in U_i^k - U_i^{k-1} \quad (1 \leqq k \leqq n_i),$$

and $u_i(x) = 0$ if $x \notin U_i^{n_i}$.

(a) Show that for all $x \in U_i$ we have

$$\|\gamma(x)u_i - u_i\| \leqq 1/n_i.$$

(b) With the help of Problem 10, construct a sequence (w_i) of continuous positive functions, with supports contained in V^2, and such that $\|w_i\| \geqq 1$ for all i, the sequence (w_i) being uniformly bounded and equicontinuous and such that the set of functions $n_i(\gamma(z_i)w_i - w_i)$ is uniformly bounded as z_i runs through U_i for each $i \geqq 1$.
(c) With the help of Problem 9, construct a sequence (g_i) of positive functions with supports contained in V^4, which is uniformly bounded and equicontinuous and such that the set of functions $n_i(\gamma(z_i)g_i - g_i)$ is uniformly bounded and equicontinuous as z_i runs through U_i for each $i \geqq 1$; and such that for each function f which is the uniform limit of a subsequence of the sequence (g_i), we have $f(x) < f(e)$ whenever $x \neq e$.

12. Let G be a locally compact group with no small subgroups, and let V be a symmetric compact neighborhood of e, containing no subgroup of G other than $\{e\}$, and such that for $x, y \in V$ the relation $x^2 = y^2$ implies that $x = y$ (Section 12.9, Problem 6(a)). A continuous homomorphism $r \mapsto X(r)$ of $\mathbf{R}$ into G is called a *one-parameter subgroup* of G (Section 12.9, Problem 7). The constant one-parameter subgroup $r \mapsto e$ is denoted by 0. For each $t \in \mathbf{R}$, the one-parameter subgroup $r \mapsto X(tr)$ is denoted by tX. If X is a one-parameter subgroup of G, a continuous real-valued function f on G is said to be *X-differentiable* if the function $r^{-1}(\gamma(X(r))f - f)$ converges uniformly on G to a limit $D_X f$ as $r \to 0$ in $\mathbf{R}$. If so, then f is also tX-differentiable for all real t, and we have $D_{tX} f = tD_X f$. If f is X-differentiable, then for each $x \in G$ the function $r \mapsto f(X(-r)x)$ is differentiable on $\mathbf{R}$, and its derivative is the function $r \mapsto D_X f(X(-r)x)$.

A continuous bounded real-valued function f on G is said to be *adequate* if there exists a neighborhood V of e in G such that $\|\gamma(z)f - f\| \neq 0$ for all $z \in V$. The functions f constructed in Problem 11(c) as limits of subsequences of the sequence (g_i) are adequate. Show that, if f is adequate and if $D_X f$ exists for some one-parameter subgroup $X \neq 0$, then $D_X f \neq 0$. If on the contrary f is not adequate, then there exists a one-parameter subgroup $X \neq 0$ such that $D_X f = 0$. (Consider a sequence (a_j) of elements $\neq e$ in G, tending to e, and such that $\|\gamma(a_j)f - f\| = 0$. Show that it can be assumed that there exists a sequence of integers m_j with the property that for all $r \in \mathbf{R}$ the sequence $a_j^{[rm_j]}$ converges to $X(r)$, where $X \neq 0$ (Section 12.9, Problem 7).)

13. The notation and assumptions on G and V remain the same as in Problem 12. Let (f_j) be a sequence of bounded continuous functions on G which converges uniformly on G to a function f. Let (a_j) be a sequence of elements of G tending to e, and (m_j) a sequence of integers > 0.
(a) Suppose that f is adequate and that the functions $m_j(\gamma(a_j)f_j - f_j)$ are uniformly bounded. Then, for each neighborhood U of e, there exists a number $\varepsilon > 0$ such that the relation $|r| < \varepsilon$ implies $a_j^{[rm_j]} \in U$. (Suppose that $U^2 \subset V$. Let

$$\alpha = \sup_j m_j \|\gamma(a_j)f_j - f_j\|;$$

by taking j sufficiently large, we may suppose that $a_j \in U$ and $\|\gamma(z)f_j - f_j\| \geqq \beta > 0$ for all $z \in V - U$. Show that we cannot have $a_j^k \in V - U$ for $k < \alpha m_j/\beta$, by using the inequality

$$(*) \qquad \|\gamma(st)f - f\| \leqq \|\gamma(s)f - f\| + \|\gamma(t)f - f\|.)$$

(b) Suppose that the functions $m_j(\gamma(a_j)f_j - f_j)$ are uniformly bounded and uniformly equicontinuous, and also that for each $r \in \mathbf{R}$ the sequence $(a_j^{[rm_j]})$ tends to $X(r)$, where X is a one-parameter subgroup. Show that $D_X f$ exists and is the limit of the

sequence $(m_j(\gamma(a_j)f_j - f_j))$. (It is enough to prove that if the sequence of functions $(m_j(\gamma(a_j)f_j - f_j))$ converges to a function F, then $D_X f$ exists and is equal to F. Fix a neighborhood W of e such that $\|\gamma(z)F - F\| \leqq \frac{1}{2}\varepsilon$ for all $z \in W$, then fix a real number $\delta > 0$ such that $a_j^k \in W$ for $0 \leqq k < \delta m_j$. Now show that for large j, and $k < \delta m_j$, we have

$$\|k^{-1}m_j(\gamma(a_j^k)f_j - f_j) - F\| \leqq \varepsilon,$$

and deduce that $\|r^{-1}(\gamma(X(r))f - f) - F\| \leqq \varepsilon$ for $0 \leqq r < \delta$.)

In particular, if the sequence $(a_j^{[rm_j]})$ tends to e for all $r \in \mathbf{R}$, then the sequence $(m_j(\gamma(a_j)f_j - f_j))$ tends uniformly to 0.

14. (a) Consider the sequence (g_i) constructed in Problem 11(c). Show that the sequence of numbers i/n_i is bounded above. (Argue by contradiction, by supposing that there exists a subsequence $(i(k))_{k \in \mathbf{N}}$ such that $\lim_{k \to \infty} n_{i(k)}/i(k) = 0$. Reduce to the case where the sequence $(g_{i(k)})$ converges uniformly to a limit g in G, and there exists a sequence of points $c_{i(k)} \in U_{i(k)}^{n_{i(k)}}$ converging to an element $c \neq e$. Then there exists an element $a_{i(k)} \in U_{i(k)}$ such that

$$\|\gamma(c_{i(k)})g_{i(k)} - g_{i(k)}\| \leqq n_{i(k)}\|\gamma(a_{i(k)})g_{i(k)} - g_{i(k)}\|.$$

Now use the hypothesis $\lim_{k \to \infty} n_{i(k)}/i(k) = 0$, Problem 7 of Section 12.9, and the definition of the U_i to deduce that for each $r \in \mathbf{R}$ the sequence $(a_{i(k)}^{[rn_{i(k)}]})$ tends to e. Using Problem 13(b), show that this implies that $\|\gamma(c)g - g\| = 0$, contrary to the fact that g is adequate.)

(b) Show that if g is the uniform limit of a subsequence of the sequence (g_i), then g is *X-differentiable* for *every* one-parameter subgroup X. (Using (a) and Problem 11(c), show that the functions $i(\gamma(X(1/i))g_i - g_i)$ form a uniformly bounded and uniformly equicontinuous set, and apply Problem 13(b).)

15. Let G be a unimodular locally compact group and β a Haar measure on G. Show that a function $f \in \mathscr{L}^\infty_{\mathbf{C}}(G, \beta)$ is equal almost everywhere to a function which is uniformly continuous with respect to a right-invariant distance on G if and only if the function $s \mapsto \gamma(s)f$, with values in $\mathscr{L}^\infty_{\mathbf{C}}(G, \beta)$, is continuous at the point e. (To show that the condition is sufficient, consider a sequence (u_n) of functions belonging to $\mathscr{K}_+(G)$ such that $\mathrm{Supp}(u_n) \subset V_n$ (notation of (14.11.2)) and $\int u_n(x)\, d\beta(x) = 1$. Prove that $N_\infty(u_n * f - f)$ tends to 0 with $1/n$, by using (13.17.1) and the Lebesgue–Fubini theorem.)

16. If μ is any bounded measure on $\mathbf{R}$, the integral

$$F_\mu(z) = \int \frac{d\mu(t)}{t - z}$$

is defined for every $z \in \mathbf{C}$ such that $\mathscr{I}z \neq 0$, and is an analytic function of z in each of the half-planes $\mathscr{I}z > 0$, $\mathscr{I}z < 0$. It is analytic also at all $x \in \mathbf{R}$ which do not lie in the support of μ. The function F_μ is called the *Stieltjes transform* of μ.

(a) Let $x_0 \in \mathbf{R}$. Suppose that the restriction of μ to an open neighborhood V of 0 has a *continuous* density g with respect to Lebesgue measure on V. Show that as y tends to 0 through positive values, $\mathscr{I}(F_\mu(x_0 + iy))$ tends to a limit equal to $\pi g(x_0)$.

(b) Show that if the analytic function F_μ is identically 0 on the half-plane $\mathscr{I}z > 0$, then $\mu = 0$. (Replacing μ by a regularization $\mu * (f \cdot \beta)$, show that

$$F_{\mu * f}(z) = \iint \frac{f(u)\, du\, d\mu(t)}{t + u - z} = 0$$

and use (a) to conclude that $\mu * (f \cdot \beta)$ is zero.)

17. Let G be a holomorphic function in the disk $B: |z| < 1$. Show that $\mathscr{R}G(z) \geqq 0$ for all $z \in B$ if and only if there exists a positive measure ν on the interval $[0, 2\pi]$ such that

$$G(z) = c \cdot i + \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z}\, d\nu(\varphi) \qquad (z \in B)$$

where $c \in \mathbf{R}$. (To show that the condition is necessary, note that for $|z| < r < 1$ we have

$$G(z) = i \cdot \mathscr{I}G(0) + \frac{1}{2\pi} \int_0^{2\pi} \frac{re^{i\varphi} + z}{re^{i\varphi} - z}\, p_r(\varphi)\, d\varphi$$

where $p_r(\varphi) = \mathscr{R}G(re^{i\varphi})$. Observe that the measure $p_r \cdot \lambda$ (where λ is Lebesgue measure on $[0, 2\pi]$) is positive and of total mass $\mathscr{R}G(0)$, and use (13.4.3).)

Deduce that for a function F holomorphic in the upper half-plane $\mathscr{I}z > 0$ to be such that $\mathscr{R}F(z) \geqq 0$ for all z in this half-plane, it is necessary and sufficient that there should exist a *bounded positive* measure μ on $\mathbf{R}$ and two constants $a \geqq 0$, $b \in \mathbf{R}$ such that

$$F(z) = az + b(1 + z^2) \int \frac{d\mu(t)}{t - z}$$

for $\mathscr{I}z > 0$. (Map B onto the upper half-plane by means of $z \mapsto i(1 + z)/(1 - z)$).

18. (a) Let λ be Lebesgue measure on $\mathbf{R}$, and let f be a λ-integrable function. For each $h > 0$, put

$$f_h(x) = \frac{1}{2h} \int_{-h}^{h} f(x - t)\, dt.$$

Show that, for almost all $x \in \mathbf{R}$, $f_h(x) \to f(x)$ as $h \to 0+$ (Lebesgue's theorem). (If we put (Section 12.7, Problem 8)

$$\mathbf{R}(f)(x) = \limsup_{h \to 0} |f_h(x) - f(x)|$$

then $\mathbf{R}(f - g) = \mathbf{R}(f)$ for all $g \in \mathscr{K}(\mathbf{R})$; also $\mathbf{R}(f) \leqq \theta(f) + |f|$, with the notation of Section 14.10, Problem 5. Use Section 14.10, Problem 5(c) to deduce that the set of $x \in \mathbf{R}$ such that $\mathbf{R}(f)(x) > \alpha$ has measure zero, for all $\alpha > 0$.)

(b) Let (g_n) be a sequence of nonnegative functions in $\mathscr{K}(\mathbf{R})$, satisfying conditions (a), (b), and (c) of (14.11.1); assume also that $g_n(-t) = g_n(t)$ and that g_n is decreasing in $[0, +\infty[$. Show that as $n \to \infty$, $(g_n * f)(x) \to f(x)$ almost everywhere in $\mathbf{R}$ (Lebesgue's theorem). (Same method, using Section 14.10, Problem 5(d).)

(c) State and prove analogous results for Haar measure on the torus $\mathbf{T}$.

CHAPTER XV

NORMED ALGEBRAS AND SPECTRAL THEORY

The spectral theory of operators, which we have already encountered in an elementary aspect in Chapter XI, is one of the masterpieces of modern analysis. Its main object is to obtain, for linear operators on a Hilbert or prehilbert space satisfying suitable continuity conditions, an analog of the classical theorem of algebra which assigns a canonical form (by means of "Jordan matrices") to an $n \times n$ matrix over $\mathbf{C}$ (or, equivalently, to an endomorphism of a *finite*-dimensional complex vector space). In Chapter XI we have seen how this result, suitably modified, can be extended to *compact* operators. But there is another much less obvious generalization, due to Hilbert and his successors, which applies to a wider class of operators: in particular, to continuous *self-adjoint* operators (11.5), and more generally to *normal* operators (15.11). Just as in the classical case a self-adjoint (or normal) matrix over $\mathbf{C}$ has a canonical form which is a *diagonal* matrix, the continuous normal operators can all be described in terms of a *single model*: namely multiplication $M_u(u) : \tilde{f} \mapsto (uf)^{\sim}$ in a space $L^2_{\mathbf{C}}(\mu)$ by the class of an *essentially bounded* function u (15.10). The Lebesgue theory intervenes here in an essential way (even if we begin with a self-adjoint operator which comes from a differential equation as regular as we please (Chapter XXIII)), and it is no exaggeration to say that the occurrence of Lebesgue theory in spectral theory and related fields such as harmonic analysis and the theory of representations of locally compact groups is the principal reason of its importance in Analysis.

A modern account of spectral theory does not follow the path mapped out by Hilbert, but uses a much more elegant and powerful method based on the theory of normed algebras inaugurated by Gelfand and his school. In this chapter we shall concentrate mainly on normed algebras with involution (15.4), because these are the ones which arise in spectral theory. But the general theory of normed algebras, and especially the fundamental concepts of

spectrum and *Gelfand transformation* (15.3), have found many other applications in modern analysis, notably in the theory of analytic functions. Some of these applications are touched on in the problems, and the reader interested in this aspect is referred to [35] and [29].

The central part of this chapter is the study of *representations* of algebras with involution, which enables such an algebra, given "abstractly," to be "realized" as an *algebra of operators* on a Hilbert space.

The essential notion in the modern development of this theory is that of a *Hilbert form*, which is closely related to that of a *Hilbert algebra* (15.7). We shall study in detail only two particular aspects of the theory of Hilbert algebras: the first (15.8) prepares the way for the theory of representations of compact groups (Chapter XXI), and the second (15.9) for spectral theory and harmonic analysis (Chapter XXII). The reader who wishes to go further (notably in view of the deep and difficult theory of representations of locally compact groups) is warmly recommended to read the two beautiful volumes by J. Dixmier ([24] and [25]) which dominate the subject.

The Hilbert spectral theory (15.10 and 15.11) appears in our treatment as an immediate particular case of the general theorem of Bochner–Godement (15.9). It can be reached more directly and rapidly (Section 15.10, Problem 2) from the Gelfand–Neumark theorem (15.4), but it seemed to us to be more instructive to deduce it from a much more powerful theorem which is the cornerstone of harmonic analysis, even if this requires a small additional effort.

The applications of spectral theory are not limited to those referred to above. Among the most celebrated we should mention at least the following: (1) one of the most elegant theories in Analysis, namely the "moment problem" inaugurated by Stieltjes, with its many ramifications (analytic functions, orthogonal polynomials, Jacobi matrices, continued fractions, etc), which fits admirably into the theory of unbounded Hermitian operators; (2) the interesting relations between ergodic theory and spectral theory; (3) perturbation theory. Some of the important results of these theories are mentioned in the problems, and the reader is referred for more ample information to the works [20], [28], [30], and [32] in the References.

1. NORMED ALGEBRAS

When we speak of *algebras* in this chapter, we shall always mean algebras over the *field of complex numbers* **C**. A *normed algebra* is defined to be an algebra A endowed with a norm $x \mapsto \|x\|$ (5.1) satisfying the inequality

$$\|xy\| \leqq \|x\| \cdot \|y\| \tag{15.1.1}$$

for all x, y in A.

If also A has a unit element e and is not the zero algebra (in other words, if $e \neq 0$) we shall always assume that the norm satisfies the additional condition

(15.1.2) $$\|e\| = 1.$$

The inequality (15.1.1) shows that the bilinear mapping $(x, y) \mapsto xy$ of $A \times A$ into A is *continuous* (5.5.1).

A *complete* normed algebra (i.e., a normed algebra in which the underlying normed vector space is a Banach space) is called a *Banach algebra.*

It is clear that every subalgebra B of a normed algebra A (if A has a unit element e, we require that B contains e), endowed with the restriction to B of the norm $\|x\|$, is a normed algebra. If $\mathfrak{m}$ is a closed two-sided ideal of A, then the quotient algebra $A/\mathfrak{m}$, endowed with the norm (12.14.10.1) induced by $\|x\|$, is again a normed algebra. For if $\dot{x}$, $\dot{y}$ are two elements of $A/\mathfrak{m}$, then for each $\varepsilon > 0$ there exist $x \in \dot{x}$ and $y \in \dot{y}$ such that

$$\|x\| \leqq \|\dot{x}\| + \varepsilon, \qquad \|y\| \leqq \|\dot{y}\| + \varepsilon,$$

from which it follows that $\|xy\| \leqq (\|\dot{x}\| + \varepsilon)(\|\dot{y}\| + \varepsilon)$; since $xy \in \dot{x}\dot{y}$ and since ε was arbitrary, this shows that $\|\dot{x}\dot{y}\| \leqq \|\dot{x}\| \cdot \|\dot{y}\|$. If moreover A has a unit element e, and if $\mathfrak{m} \neq A$, then $\dot{e}$ is the unit element of $A/\mathfrak{m}$ and is $\neq 0$, hence $\|\dot{e}\| \leqq \|e\| = 1$ by definition, and on the other hand $\|\dot{e}\| = \|\dot{e}^2\| \leqq \|\dot{e}\|^2$, which implies that $\|\dot{e}\| \geqq 1$. Hence $\|e\| = 1$, and so $A/\mathfrak{m}$ is a normed algebra.

(15.1.3) *Let* A *be a normed algebra. Then the closure in* A *of a subalgebra of* A (resp. *of a commutative subalgebra, a left ideal, a right ideal*) *is a subalgebra* (resp. *a commutative subalgebra, a left ideal, a right ideal*).

By virtue of (5.4.1) and the principle of extension of identities, this follows directly from the continuity of multiplication in A, by the same proof as in (5.4.1).

If A, B are two normed algebras, an *algebra* isomorphism $u : A \to B$ is said to be a *topological isomorphism* if it is *bicontinuous*, that is if (5.5.1) there exist two real numbers $a > 0$ and $b > 0$ such that $a\|x\| \leqq \|u(x)\| \leqq b\|x\|$ for all $x \in A$ (5.5.1). The isomorphism u is said to be *isometric* if in addition we have $\|u(x)\| = \|x\|$ for all $x \in A$.

Examples of Normed Algebras

(15.1.4) For each nonempty set X, the set $\mathscr{B}_{\mathbf{C}}(X)$ of bounded complex-valued functions on X is a commutative Banach algebra with respect to the norm (7.1.1) and ordinary multiplication of functions: the inequality (15.1.1) is clearly satisfied, the constant function 1 is the unit element of $\mathscr{B}_{\mathbf{C}}(X)$ and satisfies (15.1.2). If X is a topological space the subspace $\mathscr{C}^{\infty}_{\mathbf{C}}(X)$ of bounded continuous functions on X is a closed subalgebra of $\mathscr{B}_{\mathbf{C}}(X)$. If X is metrizable, separable and locally compact, the space $\mathscr{C}^{0}_{\mathbf{C}}(X)$ (13.20.5) of continuous functions which tend to 0 at infinity, and the space $\mathscr{K}_{\mathbf{C}}(X)$ of continuous functions with compact support, are ideals in the algebra $\mathscr{C}^{\infty}_{\mathbf{C}}(X)$, the former being the closure of the latter.

(15.1.5) If X is the closed disk $|z| \leqq 1$ in $\mathbf{C}$, then the set $\mathscr{A}(X) \subset \mathscr{C}_{\mathbf{C}}(X)$ of continuous functions on X which are *analytic* in the interior $|z| < 1$ of X is a *closed* subalgebra of $\mathscr{C}_{\mathbf{C}}(X)$, by virtue of the theorem of convergence of analytic functions (9.12.1).

(15.1.6) We have already seen (11.1) that the algebra $\mathscr{L}(E)$ of continuous endomorphisms of a complex normed space E is a normed algebra (in general noncommutative) with the identity mapping 1_E of E as unit element (the inequality (15.1.1) in this case is just (5.7.5), and the relation (15.1.2) is obvious). Furthermore, if E is a Banach space, then $\mathscr{L}(E)$ is a Banach algebra (5.7.3).

The set of *compact* operators in $\mathscr{L}(E)$ is a *closed two-sided ideal* by virtue of (11.2.6) and (11.2.10).

(15.1.7) Let G be a separable metrizable locally compact group. The set $M^1_{\mathbf{C}}(G)$ of bounded measures on G, endowed with the convolution product $(\lambda, \mu) \mapsto \lambda * \mu$ and the norm (13.20.1) is a *Banach algebra.* For in this case the inequality (15.1.1) is just (14.6.2.1) with $n = 2$; the unit element ε_e (the Dirac measure at the neutral element e of G) is such that $\|\varepsilon_e\| = 1$; and $M^1_{\mathbf{C}}(G)$, being the dual of the normed space $\mathscr{C}^{0}_{\mathbf{C}}(G)$ (13.20.6), is complete (5.7.3). The subspace of $M^1_{\mathbf{C}}(G)$ consisting of measures with *base* a left Haar measure β on G can be identified (together with its norm) with the *Banach space* $L^1_{\mathbf{C}}(G, \beta)$, because $N_1(f) = \|f \cdot \beta\|$ (13.20.3), and it follows from (14.9.2) that if G is unimodular, $L^1_{\mathbf{C}}(G, \beta)$ is a *closed two-sided ideal* in $M^1_{\mathbf{C}}(G)$. We shall always identify $L^1_{\mathbf{C}}(G, \beta)$ with this ideal (the identification of course depends on the choice of β). Under this identification, $\mathscr{K}_{\mathbf{C}}(G)$ is identified with a *subalgebra* (14.10.5) of $M^1_{\mathbf{C}}(G)$, which is not, in general, an ideal (unless G is compact (14.9.1)) and, in general, is not closed, because its closure is $L^1_{\mathbf{C}}(G, \beta)$ (13.11.6).

Remark

(15.1.8) An algebra A over **C**, endowed with a topology, is said to b *normable* if its topology can be defined by a norm with respect to which A becomes a normed algebra in the sense defined above. For this it is necessary and sufficient that the topology of A should be definable by a norm, and should be compatible with the vector space structure of A, and that the mapping $(x, y) \mapsto xy$ of $A \times A$ into A should be *continuous* for this topology Clearly these conditions are necessary. To see they are sufficient, observe that if $\|x\|$ is a norm defining the topology of A, then there exists a constant $a > 0$ such that $\|xy\| \leqq a \cdot \|x\| \cdot \|y\|$ for all x, y in A (5.5.1). If we replace the norm $\|x\|$ by the equivalent norm (5.6.1) $a\|x\| = \|x\|_1$, then clearly $\|xy\|_1 \leqq \|x\|_1 \cdot \|y\|_1$. This establishes our assertion in the case where A has no unit element. If A has a unit element $e \neq 0$, we consider for each $x \in A$ the left translation $L_x : y \mapsto xy$; this is a continuous endomorphism of the normed space A, and we have $L_{x+x'} = L_x + L_{x'}$ and $L_{\lambda x} = \lambda L_x$ for all $\lambda \in \mathbf{C}$, and $L_{xx'} = L_x \circ L_{x'}$. In other words, the mapping $x \mapsto L_x$ is a homomorphism of the algebra A into the algebra $\mathscr{L}(A)$ of continuous endomorphisms of the normed *space* A. Moreover, this homomorphism is *injective*, because the relation $L_x = 0$ implies $x = xe = L_x(e) = 0$. Now we have $\|L_x\| \leqq a\|x\|$ (5.7) and on the other hand, $\|x\| = \|xe\| \leqq \|L_x\| \cdot \|e\|$. It follows that if we put $\|x\|_2 = \|L_x\|$, then $\|x\|_2$ is a norm on A which is *equivalent* to the given norm $\|x\|$ (5.6.1), and for which A is a normed algebra.

PROBLEMS

1. Let A be the **C**-algebra of complex-valued functions defined and k times continuously differentiable on [0, 1]. Show that the function

$$\|x\| = \sum_{h=0}^{k} \frac{1}{h!} \sup_{0 \leqslant t \leqslant 1} |x^{(h)}(t)|$$

on A is a norm for which A is a Banach algebra.

2. Let (α_n) be a sequence of real numbers >0 such that $\alpha_0 = 1$, $\alpha_{m+n} \leqq \alpha_m \alpha_n$ and $\lim_{n \to \infty} \alpha_n^{1/n} = 0$. Let A denote the set of all formal power series $x = \sum_{n=0}^{\infty} \xi_n T^n$ with the complex coefficients ξ_n such that $\sum_{n=0}^{\infty} \alpha_n |\xi_n| < +\infty$. Show that A is a subalgebra of the algebra **C**[[T]] of formal power series in T, and that $\|x\| = \sum_{n=0}^{\infty} \alpha_n |\xi_n|$ is a norm on A for which A is a Banach algebra with unit element.

3. Let $\mathscr{A}(X)$ be the set of continuous functions on the closed unit disk $X : |z| \leqq 1$ in $\mathbf{C}$ which are analytic in the interior $|z| < 1$ of X. For each pair of elements $x, y \in \mathscr{A}(X)$ we define

$$(x * y)(\zeta) = \zeta \int_0^1 x(\zeta - t\zeta)y(t\zeta)\, dt$$

for all $\zeta \in X$. Show that this product makes $\mathscr{A}(X)$ a commutative algebra without a unit element, and that with respect to the norm induced by the norm on $\mathscr{C}_{\mathbf{C}}(X)$ the algebra $\mathscr{A}(X)$ is a Banach algebra.

4. (a) Let Ω be the set of finite real-valued functions ω on $\mathbf{R}$ belonging to $L^1(\mathbf{R}, \lambda)$ (where λ is Lebesgue measure) such that $\omega(-t) = \omega(t)$, $\omega(t) \geqq 0$ for all $t \in \mathbf{R}$, and such that ω is *decreasing* on $[0, +\infty[$. Every function $\omega \in \Omega$ is the uniform limit of an increasing sequence (ω_n) of step functions belonging to Ω. Put

$$V(\omega) = \omega(0) + \int_{-\infty}^{+\infty} \omega(t)\, dt.$$

If $\omega_1, \omega_2 \in \Omega$, show that $\omega_1 * \omega_2 \in \Omega$ and that $V(\omega_1 * \omega_2) \leqq V(\omega_1) \cdot V(\omega_2)$ (consider first the case where one of ω_1, ω_2 is a step function).
(b) Let $\mathscr{A}$ denote the set of λ-measurable complex-valued functions f on $\mathbf{R}$ such that, for at least one function $\omega \in \Omega$, the function $|f|^2\omega^{-1}$ is λ-integrable (with the convention that $0 \cdot (+\infty) = 0$). We have $\Omega \subset \mathscr{A}$. For each function $f \in \mathscr{A}$, put

$$N_B(f) = \left(\inf_{\omega \in \Omega} V(\omega) \int_{-\infty}^{+\infty} \frac{|f(t)|^2}{\omega(t)}\, dt \right)^{1/2}.$$

Show that also

$$N_B(f) = \frac{1}{2} \inf_{\omega \in \Omega} \left(V(\omega) + \int_{-\infty}^{+\infty} \frac{|f(t)|^2}{\omega(t)}\, dt \right).$$

Deduce that $\mathscr{A}$ is a complex vector space and that N_B is a seminorm on A. (Observe that if a, b, α, β are real and >0, then

$$\frac{(a+b)^2}{\alpha+\beta} \leqq \frac{a^2}{\alpha} + \frac{b^2}{\beta}.)$$

Show that $\mathscr{A} \subset \mathscr{L}^1_{\mathbf{C}}(\mathbf{R}, \lambda) \cap \mathscr{L}^2_{\mathbf{C}}(\mathbf{R}, \lambda)$ and that $N_1(f) \leqq N_B(f)$ and $N_2(f) \leqq N_B(f)$.
(c) Let $f \in \mathscr{L}^1 \cap \mathscr{L}^2$ be the function defined as follows: $f(x) = 1/n^2$ if $2^n - 1 \leqq x \leqq 2^n$ (n an integer $\geqq 1$), $f(x) = 0$ otherwise. Show that $f \notin \mathscr{A}$.
(d) For each function $f \in \mathscr{A}$, show that there exists $\omega_f \in \Omega$ such that

$$V(\omega_f) = \int \frac{|f(t)|^2}{\omega_f(t)}\, dt = N_B(f).$$

(Consider a sequence (ω_n) of functions belonging to Ω such that $V(\omega_n) = N_B(f)$ and such that the sequence of integrals $\int |f|^2\omega_n^{-1}\, d\lambda$ tends to $N_B(f)$. Then consider the function $\limsup_{n\to\infty} \omega_n$, and use Fatou's lemma.)
(e) The subspace $\mathscr{N}$ of λ-negligible functions is contained in $\mathscr{A}$. By passing to the quotient, N_B induces a norm on the space $A = \mathscr{A}/\mathscr{N}$. Show that, with respect to this norm (denoted by $N_B(\tilde{f})$ or $\|\tilde{f}\|$), the space A is a *Banach space*. (If (f_n) is a sequence in

$\mathscr{A}$ such that $(\check{f}_n)$ is a Cauchy sequence in A, show that the sequence (ω_{f_n}) converges in $\mathscr{D}'$, and that the sequence of classes of the functions $|f_n|^2\omega\check{f}_n^{-1}$ converges in L^1.)

(f) Let I_n be the interval $[-n, n]$ in $\mathbf{R}$ and $\mathscr{A} \mid I_n$ the set of functions belonging to $\mathscr{A}$ whose support is contained in I_n. Show that, for each $f \in \mathscr{A}$, the sequence of functions $f\varphi_{I_n}$ tends to f in $\mathscr{A}$. The topologies defined on $\mathscr{A} \mid I_n$ by N_B and by the seminorm N_2 are equivalent. Deduce that the space $\mathscr{K}(\mathbf{R})$ is dense in $\mathscr{A}$, and that the Banach space A is separable.

(g) If f and g are two functions belonging to $\mathscr{A}$, show that

$$\frac{|f * g|^2}{\omega_f * \omega_g} \leqq \frac{|f|^2}{\omega_f} \cdot \frac{|g|^2}{\omega_g}.$$

Deduce that $N_B(f * g) \leqq N_B(f)N_B(g)$ and hence that A is a commutative Banach algebra (called *Beurling's algebra*). It has no unit element.

5. Let A be a normed algebra with no unit element. On the set $\tilde{A} = A \times \mathbf{C}$, define a $\mathbf{C}$-algebra structure as follows:

$$(x, \lambda) + (x', \lambda') = (x + x', \lambda + \lambda'),$$
$$(x, \lambda)(x', \lambda') = (xx' + \lambda x' + \lambda' x, \lambda\lambda'),$$
$$\mu(x, \lambda) = (\mu x, \mu\lambda).$$

Show that $(0, 1)$ is the unit element of $\tilde{A}$, and that $\|(x, \lambda)\| = \|x\| + |\lambda|$ is a norm on $\tilde{A}$ for which $\tilde{A}$ is a normed algebra, and a Banach algebra if A is a Banach algebra.

2. SPECTRUM OF AN ELEMENT OF A NORMED ALGEBRA

In this section we shall assume that A is a *normed algebra with unit element* $e \neq 0$. For each $x \in A$, a complex number ζ is said to be a *regular value for* x if $x - \zeta e$ has an *inverse* in A. The complex numbers ζ that are not regular values for x are called the *spectral values* of x. The set of spectral values of x is called the *spectrum of* x (in A) and is denoted by $Sp_A(x)$ or $Sp(x)$. When A is the algebra $\mathscr{L}(E)$ of continuous endomorphisms of a normed space $E \neq \{0\}$, these definitions agree with those of (11.1).

Examples

(15.2.1) For each $\lambda \in \mathbf{C}$, it follows from the definition that $Sp_A(\lambda e) = \{\lambda\}$ and that $Sp_A(x + \lambda e) = Sp_A(x) + \lambda$, for all $x \in A$.

(15.2.2) Every $\zeta \in \mathbf{C}$ such that $x - \zeta e$ is a (left or right) *zero-divisor* in A is a spectral value of x, but the converse is not true (Section 11.1, Problem 3).

(15.2.3) Let $P(X) = \alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n$ be a nonconstant polynomial with complex coefficients, with $\alpha_n \neq 0$. For each $x \in A$, put

$$P(x) = \alpha_0 e + \alpha_1 x + \cdots + \alpha_n x^n \in A.$$

For each complex number μ, we can write $P(X) - \mu$ in the form

$$P(X) - \mu = \alpha_n(X - \lambda_1) \cdots (X - \lambda_n),$$

where the λ_j are complex numbers (9.11.2). Since the subalgebra of A generated by e and x is commutative, we have

$$P(x) - \mu e = \alpha_n(x - \lambda_1 e) \cdots (x - \lambda_n e).$$

Hence $P(x) - \mu e$ is invertible in A if and only if all the $x - \lambda_j e$ are invertible in A; consequently $\mu \in \mathrm{Sp}_A(P(x))$ if and only if $\lambda_j \in \mathrm{Sp}_A(x)$ for at least one value of j, and since $\mu = P(\lambda_j)$ for each j, this condition is equivalent to $\mu \in P(\mathrm{Sp}_A(x))$. Therefore

$$\mathrm{Sp}_A(P(x)) = P(\mathrm{Sp}_A(x)). \tag{15.2.3.1}$$

Suppose now that $0 \notin P(\mathrm{Sp}_A(x))$, so that $P(x)$ is *invertible* in A. Let $Q(X)$ be another polynomial with complex coefficients, and put

$$R(X) = Q(X)/P(X).$$

Then we may form the element $Q(x)(P(x))^{-1} = (P(x))^{-1}Q(x)$ of A, which we denote by $R(x)$. Then we have

$$\mathrm{Sp}_A(R(x)) = R(\mathrm{Sp}_A(x)). \tag{15.2.3.2}$$

For, replacing R by $R - \mu$ for some $\mu \in C$ if necessary, we have to show that $R(x)$ is invertible if and only if $0 \notin R(\mathrm{Sp}_A(x))$; or equivalently that $Q(x)$ is invertible if and only if $0 \notin Q(\mathrm{Sp}_A(x))$. But this follows from (15.12.3) if Q is not a constant, and it is obvious if Q is a constant.

(15.2.4) *Let* A *be a Banach algebra with unit element* $e \neq 0$.

(i) *The multiplicative group* G *of invertible elements of* A *is open in* A, *and contains the ball* $\|x - e\| < 1$. *The topology induced on* G *by that of* A *is compatible with the group structure of* G.

(ii) *For each* $x \in A$, *the set* $R_x = C - \mathrm{Sp}_A(x)$ *of regular values for* x *is open in* C, *and the mapping* $\zeta \mapsto (x - \zeta e)^{-1}$ *of* R_x *into* A *is analytic.*

(iii) *The set* $\mathrm{Sp}_A(x)$ *is compact and nonempty, and is contained in the ball* $|\zeta| \leqq \|x\|$.

For (i), we have only to copy the proof of (8.3.2), replacing $\mathscr{L}(E; F)$ by A, and H by G. This proof shows also that the mapping $x \mapsto x^{-1}$ of G into G is *differentiable*. Again, the proofs of (ii) and (iii) are identical (apart from notation) with those of (11.1.2) and (11.1.3).

(15.2.5) (Gelfand–Mazur theorem) *Let* A *be a Banach algebra. If* A *is a field, then* $A = \mathbf{C}e$.

If $x \in A$, there exists $\lambda \in \mathbf{C}$ such that $x - \lambda e$ is not invertible, by virtue of (15.2.4(iii)). But since A is a field, this implies that $x = \lambda e$. Hence $A = \mathbf{C}e$.

(15.2.6) *Let* A *be a Banach algebra with unit element* $e \neq 0$, *and let* x *be an invertible element of* A *such that* $\|x\| = \|x^{-1}\| = 1$. *Then* $\mathrm{Sp}_A(x)$ *is contained in the unit circle* **U**.

If B is the disk $|\zeta| \leqq 1$ in **C**, it follows from (15.2.4(iii)) that $\mathrm{Sp}(x) \subset B$ and $\mathrm{Sp}(x^{-1}) \subset B$. Since $\mathrm{Sp}(x^{-1}) = (\mathrm{Sp}(x))^{-1}$ by (15.2.3.2), the result follows.

(15.2.7) (i) *Let* A *be a normed algebra. For each* $x \in A$, *the sequence* $(\|x^n\|^{1/n})$ *is convergent, and its limit to equal to* $\inf_n (\|x^n\|^{1/n})$.

(ii) *If* A *is a Banach algebra with unit element* $e \neq 0$, *then the number* $\rho(x) = \lim_{n \to \infty} (\|x^n\|^{1/n})$ *is equal to the radius of the smallest disk with center* 0 *which contains* $\mathrm{Sp}_A(x)$.

(i) Put $\alpha_n = \|x^n\|$. Since the result is obvious if x is nilpotent, we may assume that $\alpha_n > 0$ for all n. Clearly we have $\alpha_{n+p} \leqq \alpha_n \alpha_p$ by virtue of (15.1.1). Let m be an integer > 0. Then every integer $n > 0$ is uniquely of the form $n = p(n)m + q(n)$, where $p(n)$ and $q(n)$ are integers and $0 \leqq q(n) < m$, and we have $\lim_{n \to \infty} p(n)/n = 1/m$. Hence from above we see that $\alpha_n^{1/n} \leqq \alpha_m^{p(n)/n} \alpha_{q(n)}^{1/n}$, and since $q(n)$ takes only m distinct values as $n \to \infty$, it follows that $\limsup_{n \to \infty} \alpha_n^{1/n} \leqq \alpha_m^{1/m}$. This is true for all integers m, and therefore

$$\limsup_{n \to \infty} \alpha_n^{1/n} \leqq \inf_n \alpha_n^{1/n} \leqq \liminf_{n \to \infty} \alpha_n^{1/n},$$

which proves (i) (12.7.11).

(ii) If $|\zeta| > \rho(x)$, then for each r satisfying $\rho(x) < r < |\zeta|$ we have $\|(\zeta^{-1}x)^n\| \leqq (r/|\zeta|)^n$ for all sufficiently large n; hence the series with general term $(\zeta^{-1}x)^n$ is convergent, and its sum is $(e - \zeta^{-1}x)^{-1}$ (8.3.2.1). Consequently $\zeta \notin \mathrm{Sp}_A(x)$. Conversely, if $0 < r < \rho(x)$, the function $(e - \zeta^{-1}x)^{-1}$

cannot be defined and analytic in the region $|\zeta| > r$, because the function $\xi \mapsto (e - \xi x)^{-1}$, which is analytic and equal to the sum of the series $\sum_n (\xi x)^n$ for all sufficiently small $|\xi|$, would then be equal to the sum of this series for $|\xi| < r^{-1}$ by virtue of (9.9.1) and (9.9.2). But this is absurd, because if $\rho(x)^{-1} < r'^{-1} < r^{-1}$, we have $\|(\xi x)^n\| \geqq (|\xi| r')^n$ for all sufficiently large n, and $(|\xi| r')^n$ tends to $+\infty$ as n tends to $+\infty$ when $|\xi| > r'^{-1}$.

The number $\rho(x)$ is called the *spectral radius* of x. We have

$$\rho(x) \leqq \|x\|, \tag{15.2.7.1}$$

$$\rho(x^k) = (\rho(x))^k \tag{15.2.7.2}$$

for all integers $k \geqq 0$, by virtue of (15.2.4(iii)) and (15.2.3.1).

(15.2.8) (i) *Let* A, B *be two Banach algebras with nonzero unit elements* e, e' *respectively, and let* $u : A \to B$ *be an algebra homomorphism such that* $u(e) = e'$. *Then* $\mathrm{Sp}_B(u(x)) \subset \mathrm{Sp}_A(x)$ *for all* $x \in A$.

(ii) *In particular, suppose that* A *is a closed subalgebra of* B, *having the same unit element. Then* $\mathrm{Sp}_B(x) \subset \mathrm{Sp}_A(x)$, *and every frontier point of* $\mathrm{Sp}_A(x)$ *belongs to* $\mathrm{Sp}_B(x)$. *Hence, if* $\mathrm{Sp}_A(x)$ *has empty interior, we have* $\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x)$.

(i) If $\zeta \in \mathbf{C}$ is such that $x < \zeta e$ is invertible in A, then $u(x - \zeta e) = u(x) - \zeta e'$ is invertible in B, whence the result.

(ii) The first assertion is a particular case of (i). To prove the second assertion, it is enough to show that if λ_0 is a frontier point of $\mathrm{Sp}_A(x)$, then $x - \lambda_0 e$ is not invertible in B. Now, by hypothesis there exists a sequence (λ_n) of regular values for x (in A) tending to λ_0. For each $n \geqq 1$, the inverse $(x - \lambda_n e)^{-1}$ of $x - \lambda_n e$ in A exists, and is therefore also the inverse of $x - \lambda_n e$ *in* B. If $\lambda_0 \notin \mathrm{Sp}_B(x)$, the sequence $((x - \lambda_n e)^{-1})$ would tend in B to the inverse y of $x - \lambda_0 e$ in B (15.2.4(ii)). But since A is closed in B, we should have $y \in A$, so that y would also be the inverse of $x - \lambda_0 e$ in A, which is absurd.

PROBLEMS

1. Let A be a normed algebra with no unit element. Then A can be considered as a closed two-sided ideal in the normed algebra $\tilde{A}$ defined in Problem 5 of Section 15.1. For each $x \in A$, the *spectrum of* x in A (denoted by $\mathrm{Sp}(x)$ or $\mathrm{Sp}_A(x)$) is defined to be the spectrum of x in $\tilde{A}$, and $\rho(x)$ is the spectral radius of x considered as an element of $\tilde{A}$.

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(a) Let A be a Banach algebra, and put $b = \inf_{x \neq 0} (\|x^2\|/\|x\|^2)$. Show that

$$b \leqq \inf_{x \neq 0} \frac{\rho(x)}{\|x\|} \leqq \sqrt{b}$$

for all $x \in A$.
(b) If x, y are two elements of a Banach algebra A, show that the intersections of $\mathrm{Sp}(xy)$ and $\mathrm{Sp}(yx)$ with $\mathbf{C} - \{0\}$ are equal, and hence that $\rho(xy) = \rho(yx)$ (cf. Section 11.1, Problem 2).

2. Let A be a Banach algebra with unit element.
(a) If $x \in A$ has a left (resp. right) inverse y, then each $x' \in A$ such that

$$\|x' - x\| < \|y\|^{-1}$$

has a left (resp. right) inverse.
(b) Let (x_n) be a sequence of elements of A, each of which has a left (resp. right) inverse y_n. If the sequence (x_n) converges to x, and if the sequence (y_n) is bounded, then x has a left (resp. right) inverse.

3. Let A be a normed algebra. For each $x \in A$, let L_x (resp. R_x) denote the continuous linear mapping $y \mapsto xy$ (resp. $y \mapsto yx$) of A into A. The element x is said to be a *left* (resp. *right*) *topological zero-divisor* if L_x (resp. R_x) is not a homeomorphism of A onto its image.
(a) Give an example of a left topological zero-divisor which is not a left zero-divisor (cf. Section 11.1, Problem 4).
(b) For each $x \in A$, put

$$\lambda(x) = \inf_{y \neq 0} \|xy\|/\|y\|, \qquad \lambda'(x) = \inf_{y \neq 0} \|yx\|/\|y\|.$$

Show that

$$|\lambda(x) - \lambda(y)| \leqq \|x - y\|, \qquad |\lambda'(x) - \lambda'(y)| \leqq \|x - y\|,$$

$$\lambda(x)\lambda(y) \leqq \lambda(xy) \leqq \|x\|\lambda(y), \qquad \lambda'(x)\lambda'(y) \leqq \lambda'(xy) \leqq \lambda'(x)\|y\|.$$

Deduce that the set of left (resp. right) topological zero-divisors is closed in A.
(c) Let A be a Banach algebra with unit element. If $x \in A$ is not left-invertible but is the limit of a sequence (x_n) of left-invertible elements, then x is a right topological zero divisor (use Problem 2(b)). Deduce that the set of elements of A which are neither invertible nor topological zero-divisors is open in A.
(d) In the algebra $\mathscr{A}(X)$ (15.1.5), show that the identity function 1_X is neither invertible nor a topological zero-divisor.

4. Let A be a Banach algebra with unit element e, in which the only right topological zero-divisor is 0. Show that $A = \mathbf{C}e$. (Consider the frontier points of $\mathrm{Sp}(x)$ and use Problem 3(c).)

5. (a) In a Banach algebra A with unit element, an element x is said to be *topologically nilpotent* if the sequence $(x^n)_{n \geqslant 1}$ tends to 0. Show that x is topologically nilpotent if and only if $\rho(x) < 1$.
(b) An element $x \in A$ is said to be *quasi-nilpotent* if $\rho(x) = 0$, or equivalently if $(\lambda x)^n$ tends to 0 as n tends to $+\infty$, for all scalars λ. A quasi-nilpotent element is a left and right topological zero-divisor.
(c) Let u be a continuous endomorphism of a Banach space E. If $\lim_{n \to \infty} \|u^n(t)\|^{1/n} = 0$ for all $t \in E$, show that u is quasi-nilpotent in the Banach algebra $A = \mathscr{L}(E)$. (Observe

that, if $\lambda \in \mathbf{C} - \{0\}$, the series with general term $u^n(t)/\lambda^{n+1}$ is convergent for all $t \in \mathrm{E}$, and use the Banach–Steinhaus theorem (12.16.5).)

6. Let E be a separable Hilbert space and let $(e_n)_{n \geqslant 1}$ be a Hilbert basis of E. Let A be the Banach algebra $\mathscr{L}(\mathrm{E})$.

(a) Let $u \in \mathrm{A}$ be the endomorphism such that $u(e_n) = 2^{-n} e_{n+1}$. Show that u is quasi-nilpotent, but not nilpotent.

(b) Define a sequence (α_n) as follows: if n is the product of 2^k and an odd number, then $\alpha_n = e^{-k}$. Let $v \in \mathrm{A}$ be the endomorphism defined by $v(e_n) = \alpha_n e_{n+1}$. Show that v is not quasi-nilpotent. (Observe that $\|v^k\| = \sup_m (\alpha_m \alpha_{m+1} \cdots \alpha_{m+k-1})$, and evaluate $\alpha_1 \alpha_2 \cdots \alpha_{2r-1}$.)

(c) For each integer k, let $v_k \in \mathrm{A}$ be the endomorphism defined as follows: $v_k(e_n) = 0$ if n is the product of 2^k and an odd integer, $v_k(e_n) = \alpha_n e_{n+1}$ otherwise. Show that v_k is nilpotent and that $\|v - v_k\|$ tends to 0 in A as k tends to $+\infty$. Hence the set of quasi-nilpotent elements of A is not closed.

7. Let A be a Banach algebra with unit element e. The *radical* of A is defined to be the set $\mathfrak{R}$ of elements $x \in \mathrm{A}$ such that $\rho(ax) = 0$ for all $a \in \mathrm{A}$. If $x \in \mathfrak{R}$, then also $\rho(xa) = 0$ for all $a \in \mathrm{A}$, and $\mathfrak{R}$ is a closed two-sided ideal, equal to the set of all x such that $e - ax$ is invertible for all $a \in \mathrm{A}$. (Observe that if u is invertible and $x \in \mathfrak{R}$, then $u - ax$ is also invertible for all $a \in \mathrm{A}$.) Every element of $\mathfrak{R}$ is quasi-nilpotent. If $\mathfrak{J}$ is a left ideal all of whose elements are quasi-nilpotent, then $\mathfrak{J} \subset \mathfrak{R}$. If $\mathfrak{R} = \{0\}$, the algebra A is said to be *without radical*. In any case, $\mathrm{A}/\mathfrak{R}$ is without radical. If $\pi : \mathrm{A} \to \mathrm{A}/\mathfrak{R}$ is the canonical homomorphism, then $\mathrm{Sp}_{\mathrm{A}}(x) = \mathrm{Sp}_{\mathrm{A}/\mathfrak{R}}(\pi(x))$ for all $x \in \mathrm{A}$.

8. Let E be a Hilbert space and let $\mathrm{A} = \mathscr{L}(\mathrm{E})$ be the Banach algebra of continuous endomorphisms of E.

(a) Let $u \in \mathrm{A}$. If u is not injective, then u is a left zero-divisor. If $u(\mathrm{E})$ is not dense in E, then u is a right zero-divisor. If u is injective and $u(\mathrm{E})$ is dense in E but not equal to E, then u is a left and right topological zero-divisor, but not a zero-divisor. (Remark that there exists a sequence (x_n) of elements of E such that $\|x_n\| = 1$ for all n, and such that $\lim_{n \to \infty} u(x_n) = 0$.)

(b) If u is surjective but not injective, or if u is injective and $u(\mathrm{E})$ is closed in E but distinct from E, then u is an interior point of the set of non-invertible elements of A (use Problem 3(c).).

(c) Show that A is without radical (show that if the radical were $\neq \{0\}$ it would contain an orthogonal projection of rank 1).

9. Let A be a Banach algebra with unit element e. For each $x \in \mathrm{A}$ the series

$$e + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

is convergent; its sum is denoted by $\exp(x)$. If x, y are two elements of A which commute, then $\exp(x + y) = \exp(x) \exp(y)$. In particular, the image of C under the mapping $\zeta \mapsto \exp(\zeta x)$ is a connected subgroup of the group G of invertible elements of A.

For each $x \in \mathrm{A}$ such that $\|e - x\| < 1$, the series with general term $-(e - x)^n/n$ ($n \geqq 1$) is convergent. If its sum is denoted by $\log x$, then we have $\exp(\log x) = x$.

Show that the subgroup of G generated by $\exp(\mathrm{A})$ is the connected component of the neutral element of G. Consider in particular the case where A is commutative.

An element x of A is of the form $\exp(y)$ for some $y \in A$ if and only if x is contained in a connected commutative subgroup of G. (To prove that this condition is sufficient, reduce to the case where A is commutative.)

10. Let A be a Banach algebra with unit element e. Let G be the group of invertible elements of A, and G_0 the connected component of the neutral element of G.
(a) If $x \in A$ is such that $x^n = e$ for some integer $n > 0$, show that $x \in G_0$. (Observe that the spectrum of x is finite, and deduce that the set of complex numbers λ such that $\lambda x + (1 - \lambda)e$ is invertible is connected.)
(b) Suppose that A is commutative. Show that every element of G/G_0 (other than the neutral element) has infinite order. (Remark that if $x \in G$ and $x^n \in G_0$, then $x^n = \exp(y)$ for some $y \in A$, whence $(x \exp(-(1/n)y))^n = e$; now use (a).)

11. Let A be a commutative Banach algebra with unit element e. Let $x \in A$ and let $K = Sp_A(x)$. Suppose that K has the following property: there exists a fundamental system of neighborhoods U_n of K such that the frontier of each U_n is the union of a certain number of disjoint simple closed curves which are images of circuits γ_{nk} $(1 \leqq k \leqq p_n)$; the frontier of U_{n+1} is contained in U_n, and if a function F is analytic in a neighborhood of $\bar{U}_n - U_{n+1}$, with values in a complex Banach space, then

$$\sum_{k=1}^{p_n} \int_{\gamma_{nk}} F(\zeta)\, d\zeta = \sum_{k=1}^{p_{n+1}} \int_{\gamma_{n+1,k}} F(\zeta)\, d\zeta;$$

finally, if F is analytic in U_n, then $\sum_{k=1}^{p_n} \int_{\gamma_{nk}} F(\zeta)\, d\zeta = 0$, and for all $z \in U_n$,

$$F(z) = \frac{1}{2\pi i} \sum_{k=1}^{p_n} \int_{\gamma_{nk}} \frac{F(\zeta)\, d\zeta}{\zeta e - z}.$$

(These properties can be proved by using Problem 4(b) of the Appendix to Chapter IX; see Chapter XXIV for another proof.)
(a) Let V be a neighborhood of K, and f an analytic function on V with values in C. Show that, for each n such that $U_n \subset V$, the element

$$\frac{1}{2\pi i} \sum_{k=1}^{p_n} \int_{\gamma_{nk}} f(\zeta)(\zeta e - x)^{-1}\, d\zeta$$

of A is independent of n. We denote it by $f(x)$. If W is another neighborhood of K, and g an analytic function in W with values in C, and if $f(\zeta) = g(\zeta)$ for all ζ in some neighborhood of K contained in $V \cap W$, then $f(x) = g(x)$.
(b) If $f(\zeta) = 1$, then $f(x) = e$. If $f(\zeta) = \zeta$, then $f(x) = x$. (Use Cauchy's theorem (9.6.3) to reduce the problem to integration around a circle with center 0 and radius $> \rho(x)$, so as to be able to expand $(\zeta e - x)^{-1}$ as a power series in ζ^{-1}.)
(c) If $h = f + g$ (resp. $h = fg$), where f and g are analytic in a neighborhood of K, then $h(x) = f(x) + g(x)$ (resp. $h(x) = f(x)g(x)$). (For the second assertion, express $f(x)$ in terms of a neighborhood U_n and $g(x)$ in terms of U_{n+1}; observe that

$$(\lambda e - x)^{-1}(\mu e - x)^{-1} = \frac{1}{\lambda - \mu}\left((\mu e - x)^{-1} - (\lambda e - x)^{-1}\right)$$

and use the Lebesgue–Fubini theorem.)
(d) If f is analytic in a neighborhood of K, with values in C, then $Sp(f(x)) = f(Sp(x))$. (Observe that if $\lambda \in Sp(x)$ we have $f(\lambda)e - f(x) = (\lambda e - x)g(x)$ for a suitably chosen function g.) In particular, $f(x)$ is invertible in A if and only if $f(\zeta) \neq 0$ on $Sp(x)$.

(e) Let f be an analytic function on a neighborhood of $\mathrm{Sp}(x)$, and g an analytic function on a neighborhood of $\mathrm{Sp}(f(x))$. Show that, by restricting the open set on which f is defined, if necessary, the function $h = g \circ f$ is defined, and that $h(x) = g(f(x))$.
(f) In particular, if $\mathrm{Sp}(x)$ is contained in the disk $|\zeta| < 1$, then for each $\alpha \in \mathbf{R}$ the series

$$e + \sum_{n=1}^{\infty} \binom{\alpha}{n} x^n$$

converges in A to an element y which commutes with x, and $\mathrm{Sp}(y)$ is the image of $\mathrm{Sp}(x)$ under the mapping $\zeta \mapsto (1+\zeta)^\alpha$ (with the determination $(1+\zeta)^\alpha = 1$ when $\zeta = 0$). In particular, if $\alpha = 1/m$ where m is an integer > 0, then $y^m = 1 + x$, and we write

$$y = (e + x)^{1/m}.$$

12. With the same hypotheses as in Problem 11, let $(\mathrm{K}_j)_{1 \leqslant j \leqslant n}$ be a partition of $\mathrm{Sp}(x)$ into compact sets. For each j, let U_j be a neighborhood of K_j, such that the U_j are pairwise disjoint. Let f_j be the analytic function on $\bigcup_{k=1}^{n} \mathrm{U}_k$ which takes the value 1 on U_j and 0 on U_k for $k \neq j$. Show that $e_j = f_j(x)$ is idempotent in A, that $e_j e_k = 0$ whenever $k \neq j$, and that $\sum_{j=1}^{n} e_j = e$.
(b) Let λ be an isolated point of $\mathrm{Sp}(x)$, and let

$$(\zeta e - x)^{-1} = \sum_{n=-\infty}^{+\infty} (\zeta - \lambda)^n a_n \qquad (a_n \in \mathrm{A})$$

be the Laurent expansion of $(\zeta e - x)^{-1}$ in a neighborhood of λ. If e_λ is the idempotent corresponding to $\{\lambda\}$ in the partition of $\mathrm{Sp}(x)$ consisting of $\{\lambda\}$ and $\mathrm{Sp}(x) - \{\lambda\}$, then $e_\lambda(\lambda e - x)^k = (-1)^k a_{-k-1}$ for all $k \geqq 0$.
(c) Let B be the subalgebra of A generated by e and x. In order that $\mathrm{Sp}(x)$ should consist of only a finite number of points, which are poles of $(\zeta e - x)^{-1}$, it is necessary and sufficient that B should be finite-dimensional. (If $(\zeta e - x)^{-1} = \sum_{j=1}^{r} \mathrm{F}_j(\zeta) a_j$, where the F_j are scalar rational functions, then all powers of x are linear combinations of the a_j. Conversely, if B is finite-dimensional, there exists a polynomial f, not identically zero, such that $f(x) = 0$. Then $\mathrm{Sp}(x)$ is contained in the set of zeros of f; if λ is a zero of f, of order p, show also that $(\lambda e - x)^p e_\lambda = 0$.)
(d) Let U be an open neighborhood of $\mathrm{Sp}(x)$, having only finitely many connected components U_j, and let f be an analytic function on U. Show that $f(x) = 0$ if and only if the following conditions are satisfied: (1) for each index j such that $\mathrm{Sp}(x) \cap \mathrm{U}_j$ is either infinite or else contains an essential singularity of $(\zeta e - x)^{-1}$, we have $f|\mathrm{U}_j = 0$; (2) every pole of order p of $(\zeta e - x)^{-1}$ is a zero of f, of order $\geqq p$. (For the necessity of (2), argue as in (c).)

13. Let A be a Banach algebra with unit element, and U an open set in $\mathbf{C}$. Show that the set Ω of elements $x \in \mathrm{A}$ such that $\mathrm{Sp}_\mathrm{A}(x) \subset \mathrm{U}$ is open in A. If f is analytic on U, then the mapping $x \mapsto f(x)$ of Ω into A is indefinitely differentiable.

14. Let A be a Banach algebra with unit element denoted by 1.
(a) For each pair x, y of elements of A and each $\zeta \in \mathbf{C}$, show that $\rho(e^{\zeta x} y e^{-\zeta x}) = \rho(y)$.

(b) If $x, y \in A$ are such that

$$\|e^{\zeta x} y\, e^{-\zeta x}\| \leqq k\|y\| \qquad (k \text{ a constant} > 0)$$

for *all* $\zeta \in \mathbf{C}$, show that $xy = yx$. (Use Liouville's theorem and the Taylor of the function $\zeta \mapsto e^{\zeta x} y e^{-\zeta x}$ at $\zeta = 0$.)

(c) If $\|x^2\| = \|x\|^2$ for all $x \in A$, show that A is commutative. (Observe th $\rho(x) = \|x\|$ for all $x \in A$, and apply this relation to $e^{\zeta x} y\, e^{-\zeta x}$; then use (

(d) Show that, if there exists a constant $k > 0$ such that $\|yx\| \leqq k\|xy\|$ in A, then A is commutative. (Apply the inequality with $e^{-\zeta x}$ in place of in place of y, and use (b).)

(e) Let $a \in A$ be such that

$$\|(a + \zeta \cdot 1)x\| \leqq \|x(a + \zeta \cdot 1)\|$$

for all $\zeta \in \mathbf{C}$ and all $x \in A$. Prove that a belongs to the center of A. (Pro each $\zeta \in \mathbf{C}$ and all integers $n \geqq n_0$ (where n_0 depends on ζ),

$$\left\| \left(1 + \frac{\zeta}{n} a\right)^n y \left(1 + \frac{\zeta}{n} a\right)^{-n} \right\| \leqq \|y\|$$

for all $y \in A$; then let n tends to $+\infty$ and use (b).)

15. Let A be a Banach algebra with unit element e, and let C be a commutat subalgebra containing e. Show that there exists a commutative Banach B of A containing C such that $\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x)$ for each $x \in C$.

3. CHARACTERS AND SPECTRUM OF A COMMUTATIVE BANA ALGEBRA. THE GELFAND TRANSFORMATION

Let A be a *commutative* algebra (over $\mathbf{C}$). A *character* of A to be any homomorphism χ of the algebra A into $\mathbf{C}$ which is *cally zero*. Because $\chi(\lambda x) = \lambda\chi(x)$ for all scalars λ, this co equivalent to $\chi(A) = \mathbf{C}$. If A has a unit element $e \neq 0$, then (otherwise $\chi(x) = \chi(ex) = \chi(e)\chi(x) = 0$ for all $x \in A$); it follows that because $\chi(e)^2 = \chi(e^2) = \chi(e)$.

An ideal $\mathfrak{m} \neq A$ in A is said to be *maximal* if there exists no id that $\mathfrak{m} \neq \mathfrak{n}$, $\mathfrak{n} \neq A$ and $\mathfrak{m} \subset \mathfrak{n}$. If A has a unit element, $\mathfrak{m}$ is maxi only if $A/\mathfrak{m}$ is a (commutative) *field.*

(15.3.1) *Let* A *be a commutative Banach algebra with unit element*

(i) *For each* $x \in A$ *and each character* χ *of* A, *we have* $\chi(x) \in$

(ii) *Every character* χ *of* A *is a continuous linear form with no*

(iii) *The mapping* $\chi \mapsto \chi^{-1}(0)$ *is a bijection of the set of charac onto the set of maximal ideals of* A (which are therefore *closed*).

Assertion (i) is a particular case of (15.2.8(i)). It follows (15.2. $|\chi(x)| \leqq \|x\|$, which shows (5.5.1) that χ is a continuous linear norm $\leqq 1$; and since $\chi(e) = 1$ and $\|e\| = 1$, we have $\|\chi\| = 1$. Fin

$\chi(A) = \mathbf{C}$, the quotient algebra $A/\mathfrak{m}$ is a field isomorphic to $\mathbf{C}$, hence $\mathfrak{m}$ is maximal. Conversely, let $\mathfrak{n}$ be a maximal ideal of A. The fact that $\mathfrak{n}$ is closed is a consequence of the following lemma:

(15.3.1.1) *If* $\mathfrak{a} \neq A$ *is an ideal in* A, *then its closure* $\bar{\mathfrak{a}}$ *is an ideal* $\neq A$.

For the complement $\complement\mathfrak{a}$ contains the set G of invertible elements of A, which is open in A (15.2.4(i)), and hence G is contained in the complement of $\bar{\mathfrak{a}}$.

Applied to the maximal ideal $\mathfrak{n}$, this lemma shows that $\bar{\mathfrak{n}} = \mathfrak{n}$, because $\bar{\mathfrak{n}} \supset \mathfrak{n}$ and $\bar{\mathfrak{n}} \neq A$. The quotient normed algebra (15.1) $A/\mathfrak{n}$ is therefore a Banach algebra (12.14.9) which is a field because $\mathfrak{n}$ is maximal, hence by the Gelfand–Mazur theorem (15.2.5) is isomorphic to $\mathbf{C}$. In other words, there exists a unique homomorphism $\chi : A \to \mathbf{C}$ such that $\chi(e) = 1$ and $\chi^{-1}(0) = \mathfrak{n}$, and the proof of (iii) is complete.

(15.3.2) *Let* A *be a Banach algebra with unit element* $e \neq 0$. *The set* $\mathbf{X}(A)$ *of characters of* A *is a subset of the unit ball* $\|x'\| \leqq 1$ *in the dual* A′ *of the Banach space* A, *and is closed in* A′ *with respect to the weak topology* (12.15). *If* A *is separable then* $\mathbf{X}(A)$ *is metrizable and compact for the weak topology.*

In view of (12.15.9) we need only prove that $\mathbf{X}(A)$ is a closed subset of A′, or equivalently that if u lies in the closure of $\mathbf{X}(A)$ in A′ for the weak topology, then $\langle xy, u\rangle = \langle x, u\rangle\langle y, u\rangle$ for all $x, y \in A$, and $\langle e, u\rangle = 1$. But this follows from the continuity of the mapping $u \mapsto \langle x, u\rangle$ of A′ into $\mathbf{C}$ (for each $x \in A$) and the principle of extension of identities.

The set $\mathbf{X}(A)$, endowed with the topology induced by the weak topology on A′, is called the *spectrum* of A. For each $x \in A$, the mapping $\chi \mapsto \chi(x)$ of $\mathbf{X}(A)$ into $\mathbf{C}$ is denoted by $\mathscr{G}x$ or $\mathscr{G}_A x$, and is called the *Gelfand transform* of x; the mapping $x \mapsto \mathscr{G}x$ of A into $\mathbf{C}^{\mathbf{X}(A)}$ is called the *Gelfand transformation*. Hence we have by definition, for all $x \in A$ and all $\chi \in \mathbf{X}(A)$,

$$(\mathscr{G}x)(\chi) = \chi(x). \tag{15.3.3}$$

(15.3.4) *Let* A *be a separable commutative Banach algebra with unit element* $e \neq 0$.

(i) *The Gelfand transformation* $x \mapsto \mathscr{G}x$ *is a continuous homomorphism of the Banach algebra* A *into the Banach algebra* $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$ *such that* $\|\mathscr{G}x\| = \rho(x) \leqq \|x\|$ *and such that* $\mathscr{G}e$ *is the constant function* 1.

(ii) *The set of values of the continuous function* $\mathscr{G}x$ *on* $\mathbf{X}(A)$ *is equal to* $\mathrm{Sp}_A(x)$. *In particular, for* x *to be invertible it is necessary and sufficient that* $\mathscr{G}x$ *should not vanish on* $\mathbf{X}(A)$.

For each $x \in A$, the mapping $\chi \mapsto (\mathscr{G}x)(\chi) = \chi(x)$ is continuous on $\mathbf{X}(A)$, by the definition of the weak topology. Moreover, we have

$$(\mathscr{G}(xy))(\chi) = \chi(xy) = \chi(x)\chi(y) = ((\mathscr{G}x)(\chi))((\mathscr{G}y)(\chi))$$

and therefore $\mathscr{G}(xy) = (\mathscr{G}x)(\mathscr{G}y)$. The relation $\mathscr{G}e = 1$ is obvious. The equality $\|\mathscr{G}x\| = \rho(x)$ will follow from (ii) and the definition of $\rho(x)$ (15.2.7). Consider therefore the first assertion in (ii). We know already that $\chi(x) \in \mathrm{Sp}(x)$ for all $x \in A$ and $\chi \in \mathbf{X}(A)$ (15.3.1). Let us show conversely that, for each $\lambda \in \mathrm{Sp}(x)$, there exists a character χ such that $\chi(x) = \lambda$, or equivalently that $\chi(x - \lambda e) = 0$. Since $x - \lambda e$ is not invertible, the ideal $A(x - \lambda e)$ generated by this element is distinct from A. In view of (15.3.1(iii)) we are therefore reduced to proving the following lemma:

(15.3.4.1) *In a separable commutative Banach algebra* A *with unit element* $e \neq 0$, *every ideal* $\mathfrak{a} \neq A$ *is contained in a maximal ideal.*

Let (x_n) be a dense sequence in A. We define inductively an increasing sequence $(\mathfrak{a}_n)$ of closed ideals $\neq A$ $(n \geqq 0)$ as follows. Take $\mathfrak{a}_0 = \bar{\mathfrak{a}}$ (which is distinct from A, by (15.3.1.1)). In the quotient Banach algebra $A/\mathfrak{a}_{n-1}$ consider the image $\bar{x}_n$ of x_n and an element λ_n of its spectrum (which by (15.2.4) is not empty), so that $\bar{x}_n - \lambda_n \bar{e}$ (where $\bar{e}$ is the canonical image of e in $A/\mathfrak{a}_{n-1}$) is not invertible in $A/\mathfrak{a}_{n-1}$. It follows that the ideal $\mathfrak{a}'_n$ in A generated by $\mathfrak{a}_{n-1}$ and $x_n - \lambda_n e$ is distinct from A, and therefore $\mathfrak{a}_n = \bar{\mathfrak{a}}'_n$ is distinct from A. The union $\mathfrak{m}'$ of the ideals $\mathfrak{a}_n$ is therefore also an ideal $\neq A$, and hence the closure $\mathfrak{m}$ of $\mathfrak{m}'$ is also an ideal $\neq A$. We shall show that $\mathfrak{m}$ is *maximal*, which will complete the proof. If e'' is the unit element of $A/\mathfrak{m}$, and x''_n the image of x_n in $A/\mathfrak{m}$, then by construction we have $x''_n = \lambda_n e''$; in other words, all the x''_n belong to the closed subalgebra (5.9.2) $\mathbf{C}e''$ of $A'' = A/\mathfrak{m}$. Since they form a dense subset of A'' (3.11.4), it follows that $A'' = \mathbf{C}e''$, and therefore $\mathfrak{m}$ is maximal.

Remarks

(15.3.5) It should be noted that the Gelfand transformation $x \mapsto \mathscr{G}x$ is not necessarily injective. For example, if x is nilpotent but not zero, say $x^k = 0$, then the image of $\mathrm{Sp}(x)$ under $\zeta \mapsto \zeta^k$ is $\{0\}$ ((15.2.3.1) and (15.2.1)) and therefore $\mathrm{Sp}(x) = \{0\}$, hence $\chi(x) = 0$ for all characters χ, although $x \neq 0$. This example also shows that $\|\mathscr{G}x\|$ need not be equal to $\|x\|$. But even if $\mathscr{G}$ is isometric (in which case the image $\mathscr{G}(A)$ is *closed* in $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$ (3.14.4)), $\mathscr{G}(A)$ may not be equal to $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$ (15.3.8).

(15.3.6) *Suppose that there exists an element* $x_0 \in A$ *which, together with* e, *generates a dense subalgebra of* A (in other words, as P runs through the set $\mathbf{C}[X]$ of polynomials in X with complex coefficients, the subalgebra consisting of the $P(x_0)$ is dense in A). *Then the mapping* $\chi \mapsto \chi(x_0)$ *is a homeomorphism of* $\mathbf{X}(A)$ *onto* $\mathrm{Sp}_A(x_0)$.

Notice first that the elements $P(x_0)$ for which the coefficients of the polynomial P are of the form $\alpha + \beta i$ with α and β rational, form a dense subset of A, and therefore A is separable. The mapping $\chi \mapsto \chi(x_0)$ is continuous and surjective (15.3.4), and $\mathbf{X}(A)$ is compact and metrizable (15.3.2); hence it is enough to show that this mapping is also injective (3.17.12). But if $\chi_1(x_0) = \chi_2(x_0)$ for two characters χ_1, χ_2, then also $\chi_1(P(x_0)) = \chi_2(P(x_0))$ for all polynomials P; since χ_1 and χ_2 are continuous on A, we deduce that $\chi_1 = \chi_2$ (3.15.2), and the proof is complete.

Examples of Spectra and Gelfand Transformations

(15.3.7) Let X be a metrizable compact space and $A = \mathscr{C}_{\mathbf{C}}(X)$ the Banach algebra of continuous complex-valued functions on X. Then the characters of A are the *Dirac measures* ε_x ($x \in X$) (13.1.3), and the mapping $x \mapsto \varepsilon_x$ is a *homeomorphism* of X onto the spectrum $\mathbf{X}(A)$. For in this situation we have $A' = \mathrm{M}_{\mathbf{C}}(X)$, since every measure on X is bounded (13.20), and the weak topology on A' is by definition the *vague* topology (13.4). A character is therefore a measure $\mu \neq 0$ (because $\mu(1) = 1$). We shall show that $\mathrm{Supp}(\mu)$ consists of a single point. If $\mathrm{Supp}(\mu)$ contained two points $a \neq b$, then a and b would have disjoint neighborhoods U and V, respectively, hence we should have functions f and g belonging to A, with supports contained in U and V, respectively, and such that $\mu(f) \neq 0$ and $\mu(g) \neq 0$. But then $fg = 0$ so that $\mu(f)\mu(g) = \mu(fg) = 0$, giving a contradiction. If $\mathrm{Supp}(\mu) = \{x\}$, then the relation $\varepsilon_x(f) = 0$ implies $\mu(f) = 0$, and therefore (A.4.15) there exists a scalar α such that $\mu = \alpha\varepsilon_x$; since $\mu(1) = 1$, it follows that $\alpha = 1$. The continuity of the mapping $x \mapsto \varepsilon_x$ is immediately seen: for each $f \in A$, we have $\langle f, \varepsilon_x - \varepsilon_{x_0} \rangle = f(x) - f(x_0)$, and since f is continuous, for each $\delta > 0$ there exists a neighborhood V of x_0 in X such that $|f(x) - f(x_0)| \leqq \delta$ for all $x \in V$. In order to prove that $x \mapsto \varepsilon_x$ is a homeomorphism, it is enough (12.3.6) to show that this mapping is injective; and this is clear, because ε_{x_1} and ε_{x_2} have distinct supports if $x_1 \neq x_2$. We have $(\mathscr{G}f)(\varepsilon_x) = \varepsilon_x(f) = f(x)$ for all $x \in X$ and all $f \in A$, and therefore when we identify X and $\mathbf{X}(A)$ by means of the mapping $x \mapsto \varepsilon_x$, the Gelfand transformation becomes the identity mapping.

(15.3.8) Let D be the closed unit disk $|\zeta| \leqq 1$ in $\mathbf{C}$, and let $A = \mathscr{A}(D)$ be the closed subalgebra of $\mathscr{C}_{\mathbf{C}}(D)$ consisting of the continuous functions on D which

are analytic in the interior of D (15.1.5). We shall show that the characters of A are again in this case the restrictions to A of the *Dirac measures* $\varepsilon_x (x \in D)$. From this it will follow (15.3.7) that $x \mapsto \varepsilon_x$ is again a homeomorphism of D onto the spectrum $\mathbf{X}(A)$. Let f_0 be the canonical injection of D in $\mathbf{C}$; then it is enough to show that f_0 and the unit element 1 of A generate a *dense* subalgebra of A (for it is clear that $\mathrm{Sp}_A(f_0) = D$, and the result will then follow by (15.3.6)). Now, let A_0 denote the subalgebra of A consisting of the functions which are restrictions to D of functions analytic on some *neighborhood* of D in $\mathbf{C}$. This subalgebra is *dense* in A, because for each integer $n > 0$ and each function $f \in A$, the function f_n which is the restriction to D of the mapping

$$\zeta \mapsto f((n/(n+1))\zeta)$$

of $((n+1)/n)D$ into $\mathbf{C}$ clearly belongs to A_0, and $f_n \to f$ uniformly on D as $n \to +\infty$ (3.16.5). On the other hand, every function $f \in A_0$ is the uniform limit (in D) of a sequence of polynomials P_n, namely the partial sums of the Taylor series for f at the point 0 (9.9.1 and 9.9.2). Hence the set of polynomials in ζ is dense in A, and our assertion is proved. We remark that here the Gelfand transformation is *isometric*, but $\mathscr{G}(A)$ is distinct from $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$.

(15.3.9) Let B be the Banach algebra consisting of the restrictions to the unit circle $\mathbf{U} : |\zeta| = 1$ of the functions belonging to the algebra A of (15.3.8). Then the restriction mapping $f \mapsto f|\mathbf{U}$ is an *isometric isomorphism* of A onto B: this is an immediate consequence of the maximum modulus principle (9.5.9), which implies that $\|f\| = \|f|\mathbf{U}\|$. Here the mapping $x \mapsto \varepsilon_x$ of $\mathbf{U}$ into $\mathbf{X}(B)$ is again continuous and injective; but it is no longer a homeomorphism, because all the characters of A can be identified with characters of B, and therefore $\mathbf{X}(B) = \mathbf{X}(A) = D$.

PROBLEMS

1. Let X be a metrizable compact space and B a subalgebra of $\mathscr{C}_{\mathbf{C}}(X)$ containing the unit element. Assume that B is equipped with a norm $\|x\|_B$ which makes it a Banach algebra.
   (a) Show that $\|x\|_B \geqq \|x\|$ (the norm on $\mathscr{C}_{\mathbf{C}}(X)$), by remarking that $\varphi(t) : x \mapsto x(t)$ is a character of B for all $t \in X$. Deduce that 0 is the only quasi-nilpotent element of B.
   (b) Show that φ is a continuous mapping of X into $\mathbf{X}(B)$. If B separates the points of X (7.3) then φ is a homeomorphism of X onto a closed subset of $\mathbf{X}(B)$.
   (c) Suppose that, for each function $x \in B$, the conjugate $\bar{x}$ belongs to B, and that if $x \in B$ is such that $x(t) \neq 0$ for all $t \in X$, then the inverse x^{-1} of x in $\mathscr{C}_{\mathbf{C}}(X)$ belongs to B. Show that in these conditions the mapping φ is surjective. (Let $\mathfrak{m}$ be a maximal ideal of B, and let Z be the set of all $t \in X$ such that $x(t) = 0$ for all $x \in \mathfrak{m}$. Show that

Z is not empty: if Z were empty there would exist a finite open covering (V_j) of X, and for each j a function $x_j \in \mathfrak{m}$ such that $x_j(t) \neq 0$ for all $t \in V_j$; now consider the function $\sum_j x_j \bar{x}_j$, which belongs to $\mathfrak{m}$.)

(d) Deduce from (b) and (c) that the spectrum $\mathbf{X}(A)$ of the algebra of Section 15.1, Problem 1 can be canonically identified with the interval [0, 1].

2. Show that the spectrum of the algebra A of Section 15.1, Problem 2 consists of a single point, and that the unique maximal ideal of A is the radical of A (Section 15.2, Problem 7).

3. In the algebra $\mathscr{A}(X)$ of Section 15.1, Problem 3, show that $\|x^n\| \leqq \|x\|^n/(n-1)!$, and deduce that $\mathscr{A}(X)$ is equal to its radical. Deduce that $\mathscr{A}(X)$ has no characters (use Problem 5 of Section 15.1).

4. Let B be the Banach subalgebra of $\mathscr{C}_C(D)$ (notation of (15.3.8)) generated by $\mathscr{A}(D)$ and the function $|f_0| : \zeta \mapsto |\zeta|$. Show that $\mathbf{X}(B)$ is homeomorphic to the set of points $(x_1, x_2, x_3) \in \mathbf{R}^3$ such that $x_1^2 + x_2^2 \leqq x_3^2$ and $0 \leqq x_3 \leqq 1$. (Remark that B contains all functions of the form $\zeta \mapsto g(|\zeta|)$, where g is a continuous function on [0, 1], and that B also contains the subalgebra A_0 of $\mathscr{A}(D)$ introduced in (15.3.8). To show that $|\chi(f_0)| \leqq \chi(|f_0|)$ for all characters χ of B, consider the function $\zeta(\zeta - (|\zeta| + \varepsilon))^{-1}$, which belongs to B for all $\varepsilon > 0$.

5. The space l^2_C (6.5) becomes a Banach algebra without unit element under the multiplication $(\xi_n)(\eta_n) = (\xi_n \eta_n)$. Let A be the Banach algebra obtained by adjoining a unit element to l^2_C (Section 15.1, Problem 5). Show that $\mathbf{X}(A)$ may be identified with the compact subset of $\bar{\mathbf{R}}$ consisting of $+\infty$ and the integers $\geqq 1$ (use the fact that l^2_C can be canonically identified with its dual). The Gelfand transformation $\mathscr{G}_A$ then becomes the identity mapping, and $\mathscr{G}_A(A)$ is a nonclosed dense subalgebra of $\mathscr{C}_C(\mathbf{X}(A))$.

6. (a) Show that the dual of the underlying Banach space of the Beurling algebra A (Section 15.1, Problem 4) can be identified with the space of classes of λ-measurable complex-valued functions g such that

$$\|\tilde{g}\|^2 = \sup \frac{1}{V(\omega)} \int_{-\infty}^{+\infty} |g(t)|^2 \omega(t)\, dt < +\infty,$$

where the supremum is taken over the set of functions $\omega \in \Omega$ such that $V(\omega) > 0$. Then $\|\tilde{g}\|$ is the norm on the dual A′ of A. Show that

$$\|\tilde{g}\|^2 = \sup_n \frac{1}{2n+1} \int_{-n}^{+n} |g(t)|^2\, dt.$$

The canonical bilinear form $\langle \tilde{f}, \tilde{g} \rangle$ (where $\tilde{f} \in A$ and $\tilde{g} \in A'$) is then equal to

$$\int_{-\infty}^{+\infty} f(t)g(t)\, dt.$$

(b) Deduce from (a) that the characters of the Banach algebra $\tilde{A}$ obtained by adjoining a unit element to A (Section 15.1, Problem 5) have as their restrictions to A

the linear forms $\tilde{f} \mapsto \int_{-\infty}^{+\infty} f(t)e^{i\xi t}\,dt$, where ξ is real. (A character corresponds to the class of a λ-measurable function g such that $g(s+t) = g(s) + g(t)$ almost everywhere with respect to the Lebesgue measure $\lambda \otimes \lambda$ on $\mathbf{R}^2$. Deduce first that g is continuous, by observing that if $g_1 = g * \rho$ is a regularization of g, then $g_1(s+t) = g(s)g_1(t)$. Deduce then that $g(t) = e^{\zeta t}$, with $\zeta \in \mathbf{C}$, and show that ζ must be pure imaginary.)

7. Let $\mathscr{A}$ be the complex vector space of λ-measurable functions x on $[0, +\infty[$ such that $\|\tilde{x}\| = \int_0^{\infty} |x(t)| \operatorname{sh} 2t\,dt < +\infty$.

(a) Show that, for all x, y in $\mathscr{A}$, the function

$$(x \odot y)(u) = \int_0^{+\infty} dt \int_{|u-t|}^{u+t} x(s)y(t)\,ds$$

belongs to $\mathscr{A}$ (use the Lebesgue–Fubini theorem). The mapping $(x, y) \mapsto x \odot y$ defines on $\mathscr{A}$ the structure of a commutative algebra. Let $\mathscr{N}$ be the subspace of λ-negligible functions; then $\mathrm{A} = \mathscr{A}/\mathscr{N}$ is a commutative Banach algebra without unit element.

(b) Every continuous linear form on A is of the type

$$\tilde{x} \mapsto \int_0^{+\infty} x(t)\omega(t)\,dt$$

where ω is λ-measurable and such that $|\omega(t)| \leqq \mathrm{M} \operatorname{sh} 2t$ almost everywhere, M being a suitable positive constant. Deduce that every character χ of the algebra $\tilde{\mathrm{A}}$ obtained by adjoining a unit element to A, such that $\chi \mid \mathrm{A}$ is not identically zero, corresponds to a function ω such that

$$\omega(s)\omega(t) = \int_{|t-s|}^{t+s} \omega(u)\,du$$

almost everywhere with respect to Lebesgue measure on $[0, +\infty[\times [0, +\infty[$. Show that, by replacing ω by an equivalent function, we may suppose that ω is continuous and indefinitely differentiable and satisfies the relations $\omega(0) = 0$ and

$$\omega'(s)\omega(t) = \omega(t+s) + \omega(t-s)$$

whenever $s < t$. Deduce that $\omega(t) = 2(\sin \rho t)/\rho$ where ρ is a complex number, and that the condition $|\omega(t)| \leqq \mathrm{M} \operatorname{sh} 2t$ imposes the restriction $|\mathscr{I}\rho| \leqq 2$. The complement in $\mathbf{X}(\mathrm{A})$ of the character which vanishes on A can be identified with the orbit space of the group consisting of the identity and the symmetry $\zeta \mapsto -\zeta$ acting on the locally compact subspace of C consisting of the complex numbers ρ with $|\mathscr{I}\rho| \leqq 2$.

8. Let X be a compact space and B a Banach subalgebra of $\mathscr{C}_{\mathbf{C}}(\mathrm{X})$ (with respect to the induced norm) which contains the unit element and separates the points of X. Then the compact space X can be canonically identified with a closed subset of $\mathbf{X}(\mathrm{B})$ (Problem 1).

For every function $f \in \mathrm{B}$, the function $\mathscr{G}_{\mathrm{B}} f$ extends f and is such that $\|f\| = \sup_{\chi \in \mathbf{X}(\mathrm{B})} |(\mathscr{G}_{\mathrm{B}} f)(\chi)|$: in other words, $\mathscr{G}_{\mathrm{B}}$ is an isometric isomorphism of B onto a Banach subalgebra of $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(\mathrm{B}))$ (cf. Problem 1). Show that if $\mathscr{R}f \geqq 0$ then $\mathscr{R}(\mathscr{G}_{\mathrm{B}} f) \geqq 0$. (Consider the function $e^{-f} \in \mathrm{B}$.)

9. With the hypotheses of Problem 8, a *representative measure* of a character $\chi \in B$ is a positive measure μ on X such that $\chi(f) = \int_X f\,d\mu$ for all $f \in B$; such a measure μ must have total mass 1. A *quasi-representative measure* of χ is a complex measure θ on X such that $\int_X f\,d\theta = c\chi(f)$ for all $f \in B$, where c is a constant, or equivalently such that $\int_X f\,d\theta = 0$ for all functions $f \in \mathrm{Ker}(\chi)$.

If μ is a quasi-representative measure for χ, then $v \cdot \mu$ is a quasi-representative measure, for any $v \in B$.

A *Jensen measure* for χ is defined to be a positive measure μ on X such that $\log|\chi(f)| \leqq \int_X^* \log|f|\,d\mu$ for all $f \in B$. (Since $\log|f|$ is μ-measurable and bounded above on X, the above integral is either finite (if $\log|f|$ is μ-integrable) or equal to $-\infty$; we make the convention that $\log|\chi(f)| = -\infty$ if $\chi(f) = 0$.)

(a) For each $f \in \mathscr{C}_{\mathbf{R}}(X)$, put $M(f) = \sup(\log|\chi(u)|)$, taken over the set of $u \in B$ such that $|u| \leqq e^f$. Then we have

$$\inf_{x \in X} f(x) \leqq M(f) \leqq \sup_{x \in X} f(x);$$

$M(a) = a$ if $a \in \mathbf{R}$ is a constant; and $M(f) + M(g) \leqq M(f+g)$. Also, for each invertible function u in B, we have $M(\log|u|) = \log|\chi(u)|$, and $M(\mathscr{R}u) = \mathscr{R}\chi(u)$ for all $u \in B$. For each $f \in \mathscr{C}_{\mathbf{R}}(X)$, the limit $Q(f) = \lim_{t \to +\infty} t^{-1}M(tf)$ exists and is finite (Section 12.7, Problem 10) and has the following properties: $\inf_{x \in X} f(x) \leqq Q(f) \leqq \sup_{x \in X} f(x)$; $Q(tf) = tQ(f)$ for all $t \geqq 0$; $Q(f+g) \geqq Q(f) + Q(g)$. For μ to be a Jensen measure for χ, it is necessary and sufficient that $Q(f) \leqq \int_X f\,d\mu$ for all $f \in \mathscr{C}_{\mathbf{R}}(X)$. Hence deduce the existence of at least one Jensen measure for χ (Section 13.3, Problem 2).

(b) Every Jensen measure for the character χ is a representative measure for χ. (Observe that if $f \in B$ is invertible in B, then $\log|\chi(f)| = \int_X \log|f|\,d\mu$, and replace f by e^f or e^{if}.) The set $\mathfrak{R}(\chi)$ of representative measures for χ, and the set $\mathfrak{J}(\chi)$ of Jensen measures for χ are convex, vaguely compact subsets of $M_{\mathbf{R}}(X)$.

(c) If the real parts of the functions belonging to B are *dense* in $\mathscr{C}_{\mathbf{R}}(X)$, then there is only one representative measure for χ (which is therefore the unique Jensen measure for χ). In this case B is said to be a *Dirichlet algebra*.

(d) The algebra B defined in (15.3.9) is a Dirichlet algebra (use (7.4.2)). For the character $f \mapsto f(0)$, the normalized Haar measure on U is the unique representative measure.

10. With the hypotheses and notation of Problem 8, put

$$D_p(\mu) = \inf_{u \in B,\ \chi(u)=1} \int |u|^p\,d\mu$$

where μ is any positive measure on X, and $0 < p < +\infty$.

(a) If $0 < p < r < +\infty$, then

$$D_p(\mu) \leqq (D_r(\mu))^{p/r}(\mu(X))^{(r-p)/r}$$

(use Hölder's inequality). Deduce that the function $p \mapsto D_p(\mu)$ is continuous on the right on $]0, +\infty[$. (Use the fact that this function is upper semicontinuous.) If $\mu(X) = 1$, then $0 \leqq D_p(\mu) \leqq 1$ for $0 < p < +\infty$.

If $p > 1$ and if $f \in \mathscr{L}^p(\mu)$ and $g \in \mathscr{L}^q(\mu)$ (where $1/p + 1/q = 1$) are positive functions, then

$$D_1((fg) \cdot \mu) \leqq (D_p(f^p \cdot \mu))^{1/p} \cdot (D_q(g^q \cdot \mu))^{1/q}.$$

If $\mu(X) = 1$, then for each positive function $f \in \mathscr{L}^1(\mu)$, we have

$$D_1(f \cdot \mu) \leqq (D_p(f^p \cdot \mu))^{1/p}.$$

(b) If $D_p(\mu) = 1$ for some $p > 0$, show that μ is a representative measure for χ. (Observe that, if $v \in B$ is such that $\chi(v) = 0$, then $\chi(\exp(tv/p)) = 1$ for all $t > 0$; show that $\int \mathscr{R}v \, d\mu \geqq 0$ by making t tend to 0, and deduce that $\int \mathscr{R}v \, d\mu = 0$.)

(c) Conversely, if μ is a representative measure for χ, then $D_p(\mu) = 1$ for all $p \geqq 1$. Show that $D_p(\mu) = 1$ for all $p > 0$ if and only if μ is a Jensen measure for χ (use Section 13.11, Problem 12(e)).

(d) Let p be such that $1 \leqq p < +\infty$ and $D_p(\mu) > 0$. Show that there exists a function $h \in \mathscr{L}^q(\mu)$ (where $(1/p) + (1/q) = 1$) such that $N_q(h) = 1$ and

$$(D_p(\mu))^{1/p} \chi(u) = \int uh \, d\mu$$

for all $u \in B$. (Use the Hahn–Banach theorem, Proposition (13.17.1) and Problem 1 of (13.17).) If $p > 1$, the measure $|h|^q \cdot \mu$ is a representative measure for χ.

11. With the hypotheses and notation of Problem 8, for every function $f \geqq 0$ belonging to $\mathscr{L}^1(\mu)$, put $J_\mu(f) = \exp\left(\int^* \log |f| \, d\mu\right)$ (with $\exp(-\infty) = 0$). A representative measure μ for χ is said to be *hyperextremal* if every representative measure for χ *with base* μ is necessarily equal to μ; it is then an extremal point of the convex set $\mathfrak{R}(\chi)$ (Problem 9).

(a) Let $p \geqq 1$. For a positive measure μ of mass 1 on X to be a Jensen measure for χ, it is necessary and sufficient that $D_p(f \cdot \mu) \geqq J_\mu(f)$ for all functions $f \geqq 0$ belonging to $\mathscr{L}^1(\mu)$. (To show that the condition is sufficient, consider the function $(|u| + \varepsilon)^{-p}$, where $u \in B$ and $\varepsilon > 0$.)

(b) Let $p \geqq 1$ and let μ be a positive measure of mass 1 on X. Then the following conditions are equivalent:

- (α) μ is a hyperextremal representative measure for χ.
- (β) μ is a representative measure for χ, and $D_p(f \cdot \mu) \leqq J_\mu(f)$ for all functions $f \geqq 0$ belonging to $\mathscr{L}^1(\mu)$.
- (γ) μ is a representative measure for χ, and for each $f \in \mathscr{L}^1(\mu)$ such that $f \cdot \mu$ is a quasi-representative measure for χ, we have $\left|\int f \, d\mu\right| \leqq J_\mu(|f|)$.

Moreover, if these conditions are satisfied, then μ is a Jensen measure for χ, and $D_p(f \cdot \mu) = J_\mu(f)$ for all $p \geqq 1$ and all functions $f \geqq 0$ belonging to $\mathscr{L}^1(\mu)$.

(To show that (α) implies (β), prove that if μ is hyperextremal and $p \geqq 1$ and $f \geqq 0$ belongs to $\mathscr{L}^p(\mu)$, then $(D_p(f^p \cdot \mu))^{1/p} \leqq D_1(f \cdot \mu)$: for this, use Problem 10(d). Then use Problem 10(a) to deduce that if f, g are $\geqq 0$ and belong to $\mathscr{L}^1(\mu)$, then $D_1(fg \cdot \mu) \leqq D_1(f \cdot \mu) D_1(g \cdot \mu)$. Conclude that, for all functions $f \geqq 0$ in $\mathscr{L}^1(\mu)$, we

have $D_1(f \cdot \mu) \leqq \left(\int f^{2^{-n}}\, d\mu\right)^{2^n}$ for all $n \geqq 1$, and use Problem 12(e) of Section 13.11. Use this problem also to show that (γ) implies (α).)

(c) Let $p \geqq 1$ and let μ be a positive measure on X. For each measure θ on X, let $d\theta/d\mu$ denote the function h (defined up to equivalence) which features in the Lebesgue decomposition (13.18.4) $\theta = h \cdot \mu + \sigma$, where $|\sigma|$ is disjoint from μ. Show that every representative measure for χ has base μ if and only if

$$D_p(\nu) = D_p\left(\frac{d\nu}{d\mu} \cdot \mu\right)$$

for every positive measure ν on X. (To show that the condition is necessary, reduce to the case $p > 1$ and $D_p(\nu) > 0$. Then apply Problem 10(d) and Hölder's inequality.)

(d) Let $p \geqq 1$ and let μ be a positive measure on X with total mass 1. Then the following conditions are equivalent:

- (α) μ is the unique representative measure for χ.
- (β) $D_p(\nu) = J_\mu(d\nu/d\mu)$ for all positive measures ν on X.
- (γ) $D_p(\nu) \leqq J_\mu(d\nu/d\mu)$ for all positive measures ν on X.
- (δ) $\left|\int d\theta\right| \leqq J_\mu(|d\theta/d\mu|)$ for all quasi-representative measures θ for χ. ("Szegö-Kolmogoroff-Krein theorem").

(To prove that (γ) implies (α), use (c) to show that μ is hyperextremal, and then use (b).)

12. With the hypotheses and notation of Problem 8, let μ be a representative measure for χ. Let $K(\mu)$ denote the set of functions $f \in \mathscr{L}^1(\mu)$ such that $f \cdot \mu$ is a quasi-representative measure for χ, in other words such that $\int uf\, d\mu = \left(\int f\, d\mu\right)\left(\int u\, d\mu\right)$ for all $u \in B$. Then $\left|\int f\, d\mu\right| \leqq D_1(|f| \cdot \mu) = J_\mu(|f|)$. We have $B \subset K(\mu)$.

(a) For each function $g \geqq 0$ in $\mathscr{L}^1(\mu)$, show that there exists a function $f \in K(\mu)$ such that $|f| \leqq g$ and $\int f\, d\mu = D_1(|f| \cdot \mu) = D_1(g \cdot \mu)$ (use Problem 10(d)).

(b) A function $f \in K(\mu)$ is said to be *exterior* if $\left|\int f\, d\mu\right| = D_1(|f| \cdot \mu) > 0$, and *interior* if $|f(x)| = 1$ almost everywhere with respect to μ. Let $E(\mu)$ (resp. $I(\mu)$) be the set of exterior (resp. interior) functions. The intersection $E(\mu) \cap I(\mu)$ consists of functions equivalent to *constants* c such that $|c| = 1$.

(c) Let $g \geqq 0$ be a function in $\mathscr{L}^1(\mu)$ such that $D_1(g \cdot \mu) = J_\mu(g) > 0$. Show that there exists a function $f \in E(\mu)$ such that $|f| = g$. (Use (a) by writing $f = gh$, where $h \in \mathscr{L}^\infty(\mu)$ and $J_\mu(|h|) = 1$, and use Problem 12(e) of Section 13.11.)

13. With the hypotheses and notation of Problem 8, let μ be a *hyperextremal* representative measure for χ. Then $D_1(f \cdot \mu) = J_\mu(f)$ for all $f \geqq 0$ in $\mathscr{L}^1(\mu)$ (Problem 11(b)).

(a) Let $f \in K(\mu)$. For f to be an exterior function (Problem 12), it is necessary and sufficient that $\inf_{u \in B} \int |1 - uf|\, du = 0$. (To show that this condition is sufficient, prove that it implies that there exists a sequence (u_n) in B such that $\chi(u_n) = 1$ for all n and such that $\int |u_n f|\, d\mu$ tends to $\left|\int f\, d\mu\right|$. To show that the condition is necessary, we

may assume that $\left|\int f\,d\mu\right| = 1$, and then there exists a sequence (u_n) with the properties above. Use the relation $\mathrm{D}_1(|u_n f| \cdot \mu) = \mathrm{J}_\mu(|u_n f|)$ to deduce that the sequence of integrals $\int |u_n f|^{1/2}\,du$ tends to 1, and use Problem 4 of Section 13.12.)

(b) Let $f \in \mathrm{E}(\mu)$, and let $g \in \mathscr{L}^\infty(\mu)$ be such that $fg \in \mathrm{K}(\mu)$. Show that

$$g \in \mathrm{K}(\mu) \cap \mathscr{L}^\infty(\mu)$$

and that $\int fg\,d\mu = \left(\int f\,d\mu\right)\left(\int g\,d\mu\right)$. (Consider a sequence (u_n) in B such that $u_n f \to 1$ in the mean, and for each $v \in \mathrm{B}$ consider the integrals $\int u_n vfg\,d\mu$.)

(c) If f, g in $\mathrm{E}(\mu)$ are such that $|f(x)| = |g(x)|$ almost everywhere, then $f = cg$ almost everywhere, where c is a constant. (Use (b) to show that the function f/g belongs to $\mathrm{E}(\mu)$ and to $\mathrm{I}(\mu)$.)

(d) For each function $f \in \mathrm{K}(\mu)$ such that $\mathrm{J}_\mu(|f|) > 0$, show that $f = gh$, where $g \in \mathrm{E}(\mu)$ and $h \in \mathrm{I}(\mu)$, and that g, h are unique up to a constant factor of absolute value 1. (Use Problem 12(c), and (b) and (c) above.)

(e) If f, g belongs to $\mathrm{K}(\mu) \cap \mathscr{L}^\infty(\mu)$, then $\int fg\,d\mu = \left(\int f\,d\mu\right)\left(\int g\,d\mu\right)$. (Notice first that if $h \in \mathrm{K}(\mu) \cap \mathscr{L}^\infty(\mu)$ is such that $\int h\,d\mu = 0$, then $\int h^2\,d\mu = 0$. To prove this, use the inequality

$$1 = \left|\int (1 + zh)\,d\mu\right| \leqq \mathrm{D}_1(|1 + zh| \cdot \mu) = \mathrm{J}_\mu(|1 + zh|)$$

valid for all $z \in \mathbf{C}$, and deduce that $1 \leqq \int |1 - z^2h^2|\,d\mu$; then use Problem 7 of Section 13.12. To deal with the general case, consider the function $h = f + zg - \int (f + zg)\,d\mu$, where $z \in \mathbf{C}$.)

(f) If f, g in $\mathrm{K}(\mu)$ are such that $fg \in \mathscr{L}^1(\mu)$, show that $fg \in \mathrm{K}(\mu)$ and that

$$fg\,d\mu = \left(\int f\,d\mu\right)\left(\int g\,d\mu\right).$$

(Assume first that $g \in \mathrm{E}(\mu)$, and show that, for each $u \in \mathrm{K}(\mu)$ such that $ug \in \mathscr{L}^1(\mu)$, we have $\mathrm{D}_1(|g| \cdot \mu)\int |u\,d\mu| \leqq \int |ug|\,d\mu$. Deduce as in Problem 10(d) that there exists a function $v \in \mathscr{L}^\infty(\mu)$ such that $|v| \leqq 1$ and $\mathrm{D}_1(|g| \cdot \mu)\int u\,d\mu = \int ugv\,d\mu$ for all $u \in \mathrm{K}(\mu)$ such that $ug \in \mathscr{L}^1(\mu)$. Using (b), conclude that v is a constant of absolute value 1. In the general case, show first that, by addition of a suitable constant to f and g, we may assume that $\int f\,d\mu \neq 0$ and $\int g\,d\mu \neq 0$. Then use the canonical decomposition of (d) above, and show with the help of (e) that if f and g are in $\mathrm{I}(\mu)$ then so is fg.)

(g) Deduce from (f) that if $f, g \in \mathrm{E}(\mu)$ are such that $fg \in \mathscr{L}^1(\mu)$, then $fg \in \mathrm{E}(\mu)$.

(h) If $f \in \mathscr{L}^1(\mu)$ and $g \in \mathrm{E}(\mu)$ and $fg \in \mathrm{K}(\mu)$, show that $f \in \mathrm{K}(\mu)$. (By adding a suitable constant to f, reduce to the case where $\int fg\,d\mu \neq 0$, and deduce that $\mathrm{J}_\mu(|f|) = \mathrm{D}_1(|f| \cdot \mu) > 0$. Then show, by using the results above, that $f = uv$ with

$v \in \mathrm{E}(\mu)$ and $|u(x)| = 1$ almost everywhere. Deduce that $u \in \mathrm{I}(\mu)$ and then use (f).)

(i) Let $f \in \mathrm{K}(\mu)$ be such that $\int f\bar{u}\, d\mu = 0$ for all $u \in \mathrm{I}(\mu)$ for which $\int u\, d\mu \neq 0$. Show that f is μ-negligible. (By applying the canonical decomposition of (d) above to the function $1 + zf$, where z is a complex constant, and by using Problem 6 of Section 13.12, show that $1 + zf \in \mathrm{E}(\mu)$; then apply Problem 17 of Section 13.21.)

(j) Show that every *real* function $f \in \mathrm{K}(\mu)$ is equivalent to a constant. (Reduce to the case where $\int f\, d\mu = 0$, and apply (i).)

(k) Deduce from (j) that if $\bar{\mathrm{B}}$ denotes the set of functions $\bar{u}$ such that $u \in \mathrm{B}$, then $\mathrm{B} + \bar{\mathrm{B}}$ is *dense* in $\mathscr{L}^p(\mu)$ for $1 \leqq p < +\infty$. (If $(1/p) + (1/q) = 1$ and if $f \in \mathscr{L}^p(\mu)$ is such that $\int fu\, d\mu = 0$ and $\int f\bar{u}\, d\mu = 0$ for all $u \in \mathrm{B}$, observe that $\int (\mathscr{R}f)u\, d\mu > 0$, and use (j).)

14. With the notation and hypotheses of Problem 8, assume in addition that there exists *only one* representative measure μ for χ. Let ν be a quasi-representative measure for χ, and write $\nu = \nu' + \nu''$, where ν' is a measure with base μ, and ν'' is disjoint from μ. Then ν' and ν'' are quasi-representative measures for χ, and $\int d\nu'' = 0$ ("theorem of F. and M. Riesz"). (Assume that $\int d\nu \neq 0$, by replacing ν by $\nu + \mu$ if necessary. If $\rho = |\nu| = |\nu'| + |\nu''|$, then $\mathrm{D}_1(\rho) > 0$ and therefore $\mathrm{D}_1(\rho) = \mathrm{J}_\mu(|h|)$, where $\nu' = h \cdot \mu$, by virtue of Problem 11(d). Hence there exists a sequence (v_n) in B such that $\chi(v_n) = 1$ for all n and such that $\int |v_n|\, d\rho \to \mathrm{J}_\mu(|h|)$. Since $\mathrm{J}_\mu(|h|) \leqq \int |v_n h|\, d\mu$, the integrals $\int |v_n h|\, d\mu$ also tend to $\mathrm{J}_\mu(|h|)$, and $\int |v_n|\, d|\nu''| \to 0$. Write $h = fg$, where $f \in \mathrm{E}(\mu)$ and $|f| = |h|$, so that $g \in \mathscr{L}^\infty(\mu)$ and $|g| = 1$; as in Problem 13(a), show that fv_n tends to $\int f\, d\mu$ in the mean (with respect to μ). From the hypothesis that ν is quasi-representative, deduce that $g \in \mathrm{I}(\mu)$, and apply Problem 13(f).)

15. With the hypotheses and notation of Problem 8, suppose that μ is a *hyperextremal* representative measure for χ. For $1 \leqq p \leqq +\infty$, let $\mathscr{H}^p(\mu)$ denote the set of functions $f \in \mathscr{L}^p(\mu)$ such that

$$\int fg\, d\mu = \left(\int d\mu\right)\left(\int g\, d\mu\right)$$

for all $g \in \mathrm{K}(\mu) \cap \mathscr{L}^q(\mu)$, where $(1/p) + (1/q) = 1$. Then $\mathscr{H}^p(\mu) \subset \mathrm{K}(\mu)$, and by virtue of Problem 13(f) we have

$$\mathscr{H}^p(\mu) = \mathrm{K}(\mu) \cap \mathscr{L}^p(\mu)$$

for $1 \leqq p \leqq +\infty$. The canonical image of $\mathscr{H}^p(\mu)$ in $\mathrm{L}^p(\mu)$ is denoted by $\mathrm{H}^p(\mu)$ and called the *Hardy space*.

(a) For $1 \leqq p < +\infty$, show that $\mathscr{H}^p(\mu)$ is the closure of B in $\mathscr{L}^p(\mu)$. When $p = +\infty$, $\mathscr{H}^\infty(\mu)$ is the closure of B in $\mathscr{L}^\infty(\mu)$ with respect to the weak topology. (In each case, show that if $g \in \mathscr{L}^q(\mu)$ is such that $\int ug\, d\mu = 0$ for all $u \in \mathrm{B}$, then $g \in \mathscr{L}^q(\mu) \cap \mathrm{K}(\mu)$ and $\int g\, d\mu = 0$, and consequently $\int fg\, d\mu = 0$ for all $f \in \mathscr{H}^p(\mu)$.)

(b) Let $1 \leqq p < +\infty$, and let $f \in \mathscr{H}^p(\mu)$. Show that $f\mathrm{B}$ is dense in $\mathscr{H}^p(\mu)$ if and only if f is an *exterior* function. (The necessity of the condition follows from Problem 13(a). To show that it is sufficient, observe that if $g \in \mathscr{L}^q(\mu)$ is such that $\int gfu\, d\mu = 0$ for all $u \in \mathrm{B}$, then $gf \in \mathrm{K}(\mu)$, and deduce from Problem 13(h) that $g \in \mathrm{K}(\mu)$ and that $\int gf\, d\mu = 0$. Apply Problem 13(f) to show that $\int g\, d\mu = 0$ and consequently that $\int gu\, du = 0$ for all $u \in \mathrm{B}$.)

(c) The space $\mathscr{H}^p(\mu)$ is a *closed* B-*submodule* of $\mathscr{L}^p(\mu)$. If B_0 is the closed ideal $\mathrm{Ker}(\chi)$ of B, then a closed B-submodule M of $\mathscr{L}^p(\mu)$ is said to be *proper* if $\mathrm{B}_0\mathrm{M}$ is not dense in M. Show that $\mathscr{H}^p(\mu)$ is a proper B-module. Show that every proper closed B-submodule of $\mathscr{L}^2(\mu)$ is of the form $q\mathscr{H}^2(\mu)$, where $|q| = 1$. (First remark that there exists in M a nonnegligible function q which is orthogonal to $\mathrm{B}_0\mathrm{M}$ with respect to the hermitian scalar product $(f, g) \mapsto \int f\bar{g}\, d\mu$. By expressing that q is orthogonal to $q\mathrm{B}_0$, deduce that (up to a constant factor) we may assume that $|q| = 1$ (use Problem 13(j)). Then $q\mathscr{H}^2(\mu)$ is a closed B-module contained in M. Show that in fact $q\mathscr{H}^2(\mu) = \mathrm{M}$, by remarking that if $g \in \mathrm{M}$ is orthogonal to $q\mathscr{H}^2(\mu)$, then $g\bar{q}$ is orthogonal to B, and q is orthogonal to $g\mathrm{B}_0$, hence $g\bar{q}$ is orthogonal to B_0; then use Problem 13(k).)

Show that the class of q is determined up to a constant factor (use (b)).

(d) Deduce from (c) that the Hilbert space $\mathrm{L}^2(\mu)$ is the *Hilbert sum* of $\mathrm{H}^2(\mu)$ (the canonical image of $\mathscr{H}^2(\mu)$) and $\overline{\mathrm{H}^2_0}(\mu)$ (the canonical image of the closed subspace $\overline{\mathscr{H}^2_0}(\mu)$ consisting of the functions f such that $\bar{f} \in \mathscr{H}^2(\mu)$ and $\int \bar{f}\, d\mu = 0$). (Consider the subspace M of $\mathscr{L}^2(\mu)$ consisting of the functions orthogonal to $\overline{\mathscr{H}^2_0}(\mu)$. Show that M is a proper closed B-module, and therefore of the form $q\mathscr{H}^2(\mu)$ for some function q such that $|q| = 1$. Also M contains the constants and hence $\bar{q} \in \mathscr{H}^2(\mu)$; conclude that q is equivalent to a constant.)

(e) Let $f \in \mathscr{L}^1(\mu)$ be such that f does not lie in the closure of $f\mathrm{B}_0$ in $\mathscr{L}^1(\mu)$. Show that $f = gh$ where $h \in \mathscr{H}^2(\mu)$ is an *exterior* function such that $|h|^2 = f$, and $g \in \mathscr{L}^2$ lies in the closure of $f\mathrm{B}$ in $\mathscr{L}^1(\mu)$. (Apply Problem 12(c) to the function $|f|^{1/2}$.) If also $f \in \mathscr{H}^1(\mu)$, then $f = gh$ where g, h belong to $\mathscr{H}^2$ and $|g| = |h|$.

Conversely, if $f \in \mathscr{L}^1(\mu)$ is of the form gh^2, where $|q| = 1$ and $h \in \mathscr{H}^2(\mu)$ is an exterior function, then f does not belong to the closure of $f\mathrm{B}_0$ in $\mathscr{L}^1(\mu)$ (observe that $h^2 \in \mathrm{E}(\mu)$, and use Problem 13(a)). Equivalently, $\mathrm{J}_\mu(|f|) > 0$.

(f) Let M be a proper closed B-submodule of $\mathscr{L}^1(\mu)$. Show that M is of the form $q\mathscr{H}^1(\mu)$, where $|q| = 1$. (Let $\mathrm{N} = \mathrm{M} \cap \mathscr{L}^2(\mu)$, which is a closed B-submodule of $\mathscr{L}^2(\mu)$. If $f \in \mathrm{M}$ is not in the closure of $\mathrm{B}_0\mathrm{M}$ in $\mathscr{L}^1(\mu)$, deduce from (e) that $f = gh$, with $g \in \mathrm{N}$ and $h \in \mathscr{H}^2(\mu)$; moreover g is not in the closure of $\mathrm{B}_0\mathrm{N}$ in $\mathscr{L}^2(\mu)$. Use (c) to show that $\mathrm{N} = q\mathscr{H}^2(\mu)$ with $|q| = 1$, and deduce that $f \in q\mathscr{H}^1(\mu)$. Applying this result to $f + f_1$, where f_1 is in the closure of $\mathrm{B}_0\mathrm{M}$ in $\mathscr{L}^1(\mu)$, deduce that $f_1 \in q\mathscr{H}^1(\mu)$ and hence that $\mathrm{M} \subset q\mathscr{H}^1(\mu)$.)

16. We shall apply the results of Problems 10 to 15 to the Dirichlet algebra B of (15.3.9), where μ is the normalized Haar measure $d\mu(\theta) = (2\pi)^{-1}\, d\theta$ on the unit circle U, and χ is the character $f \mapsto f(0)$, so that μ is the *unique* representative measure for χ (Problem 9(d)).

(a) B is the closure in $\mathscr{C}_{\mathbf{C}}(\mathrm{U})$ of the algebra of trigonometric polynomials $\sum_{k=0}^{n} c_k e^{ki\theta}$

with exponents $k \geqq 0$. To each function $f \in \mathscr{H}^p(\mu)$ there corresponds the analytic function in the disk $|z| < 1$ defined by

$$F(z) = \frac{1}{2\pi i} \int_U \frac{f(\zeta)\, d\mu(\zeta)}{\zeta - z}.$$

For $1 \leqq p = < \infty$ (resp. $p = +\infty$) f is the limit (resp. weak limit) in $\mathscr{L}^p(\mu)$ of the functions $\theta \mapsto F(re^{i\theta})$ as $r \mapsto 1$, so that $H^p(\mu)$ may be identified with the space of analytic functions F corresponding to the functions $f \in \mathscr{H}^p(\mu)$. (Use the fact that $\mathscr{H}^p(\mu)$ is the closure (resp. weak closure) of B in $\mathscr{L}^p(\mu)$.) In particular, $H^2(\mu)$ is identified with the space of functions

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

which are analytic for $|z| < 1$ and such that $\sum_{n=0}^{\infty} |c_n|^2 < +\infty$ (cf. the problem in Section 9.13).

(b) Let ν be a measure *disjoint from* μ which is quasi-representative for χ. Show that $e^{-i\theta} \cdot \nu$ is also a quasi-representative measure for χ, and hence by induction that $e^{-ni\theta} \cdot \nu$ is a quasi-representative measure for χ, for all integers $n > 0$. Hence show that $\nu = 0$ and consequently (Problem 14) that every quasi-representative measure for χ *has base* μ.

(c) Let $f \in \mathscr{H}^1(\mu)$ be nonnegligible (with respect to μ). Then the function $\log|f|$ is μ-integrable, or equivalently, $J_\mu(|f|) > 0$, and consequently f is the product of an exterior function and an interior function. (This follows from Problem 11(b) if $\int f\, d\mu \neq 0$. If $\int f\, d\mu = 0$ then f, identified with an analytic function on the unit disk, is of the form $f(z) = z^n g(z)$ where $n > 0$ and $g(0) \neq 0$. We have $g \in \mathscr{H}^1(\mu)$ and $\log|f| = \log|g|$ on U.) In particular, a function belonging to $\mathscr{H}^1(\mu)$ which vanishes on a non-μ-negligible subset of U is μ-negligible. Every function in $\mathscr{H}^1(\mu)$ is the product of two functions in $\mathscr{H}^2(\mu)$ and conversely (Problem 15(e)). A function $f \geqq 0$ belonging to $\mathscr{L}^1(\mu)$ is equivalent to a function of the form $|g|^2$, where $g \in \mathscr{H}^2(\mu)$, if and only if $J_\mu(f) > 0$ (or equivalently if and only if $\log|f|$ is μ-*integrable*). (See Section 22.19, Problem 19.)

17. Let B be the Banach algebra considered in Problem 16, and B_1 the Banach subalgebra of B consisting of functions f continuous on $|z| \leqq 1$ and analytic on $|z| < 1$, and such that $f'(0) = 0$. For each complex number c such that $|c| \leqq 1$, let μ_c denote the measure $F_c \cdot \mu$, where $F_c(z) = 1 - \mathscr{R}(\bar{c}z)$. Show that the measures μ_c are representative measures for the character $\chi : f \mapsto f(0)$; the representative measures which are extremal points of $\mathfrak{R}(\chi)$ are the μ_c such that $|c| = 1$; the measure $\mu_0 = \mu$ is the unique Jensen measure for χ; and hence there exists no hyperextremal representative measure for χ.

18. Let A be a commutative Banach algebra with unit element.

(a) Let $u \in A$ be such that $\rho(u) < 1$. Then for each character χ of A the element $1 - \overline{\chi(u)}\, u$ is invertible. If $v = (\chi(u) - u)(1 - \overline{\chi(u)}\, u)^{-1}$, then $|\chi(v)| < 1$.

(b) If χ_1, χ_2 are two characters of A, let $\sigma(\chi_1, \chi_2)$ denote the least upper bound of the number $|\chi_2(u)|$ as u runs through the set of elements of A such that $\rho(u) \leqq 1$ and $\chi_1(u) = 0$. Show that if $\rho(u) \leqq 1$ we have

$$|\chi_1(u) - \chi_2(u)| \leqq \sigma(\chi_1, \chi_2)|1 - \overline{\chi_1(u)}\, \chi_2(u)|$$

(use (a)). Deduce that $\sigma(\chi_1, \chi_2) = \sigma(\chi_2, \chi_1)$ and that

$$(*) \qquad |\chi_2(u)| \leqq \frac{\sigma(\chi_1, \chi_2) + |\chi_1(u)|}{1 + \sigma(\chi_1, \chi_2)|\chi_1(u)|}.$$

Deduce from this inequality that, if χ_1, χ_2, χ_3 are three characters of A, we h

$$\sigma(\chi_1, \chi_3) \leqq \frac{\sigma(\chi_1, \chi_2) + \sigma(\chi_2, \chi_3)}{1 + \sigma(\chi_1, \chi_2)\sigma(\chi_2, \chi_3)},$$

and hence that the relation $\sigma(\chi', \chi'') < 1$ between characters χ', χ'' of A is ar
lence relation on $\mathbf{X}(A)$. The equivalence classes for this relation are called
parts of $\mathbf{X}(A)$.

(c) Let $u \in A$ be such that $\rho(u) \leqq 1$, and let χ_1, χ_2 be two characters of A, s
$\chi_1(u) = 0$. Show that there exists a complex number λ such that $|\lambda| < 1$ and su
if we put $v = (\lambda - u)(1 - \bar{\lambda}u)^{-1} \in A$, then $\chi_1(v) = \lambda$ and $\chi_2(v) = -\lambda$. Deduce

$$\chi_2(u) \leqq \frac{4|\chi_1(v) - \chi_2(v)|}{4 + |\chi_1(v) - \chi_2(v)|^2}.$$

(d) If χ_1, χ_2 are two characters of A, put

$$\tau(\chi_1, \chi_2) = \sup_{u \in A,\ \rho(u) \leqq 1} |\chi_1(u) - \chi_2(u)|.$$

Show that

$$\sigma(\chi_1, \chi_2) = \frac{4\tau(\chi_1, \chi_2)}{4 + (\tau(\chi_1, \chi_2))^2}$$

(use (c)). The equivalence relation defined in (b) is therefore equivalent to $\tau(\chi',$
(e) Let $u \in A$ be such that $\mathscr{R}\chi(u) \geqq 0$ for all $\chi \in \mathbf{X}(A)$. Show that, for a
characters χ_1, χ_2, we have

$$\mathscr{R}\chi_2(u) \geqq \frac{1 - \sigma(\chi_1, \chi_2)}{1 + \sigma(\chi_1, \chi_2)} \mathscr{R}\chi_1(u).$$

(Apply (*) to the element e^{-tu}, where $t > 0$, and let t tend to 0.)

19. With the hypotheses of Problem 8, let χ_1, χ_2 be two characters of B.

(a) Let μ_1, μ_2 be representative measures for χ_1, χ_2 respectively (Prob
If $\sigma(\chi_1, \chi_2) = 1$ (Problem 18), show that μ_1 and μ_2 are disjoint (13.18). (Wri
$h \cdot \mu_1 + \nu$ where ν is disjoint from μ_1, and by majorizing $|\chi_2(u) - \chi_1(u)|$ fo
and $\|u\| \leqq 1$, show that the hypothesis implies that $\int (h + 1)\, d\mu_1 \leqq \int |h -$

(b) If $\sigma(\chi_1, \chi_2) < 1$, show that there exist representative measures μ_1 for
μ_2 for χ_2 such that

$$\mu_2 \geqq \frac{1 - \sigma(\chi_1, \chi_2)}{1 + \sigma(\chi_1, \chi_2)} \mu_1 \qquad \text{and} \qquad \mu_1 \geqq \frac{1 - \sigma(\chi_1, \chi_2)}{1 + \sigma(\chi_1, \chi_2)} \mu_2.$$

(Put $c = (1 - \sigma(\chi_1, \chi_2)(1 + \sigma(\chi_1, \chi_2))^{-1}$. Use Problem 18(e) and the Hahn-

theorem to show that there exist two positive measures α, β on X such that $\chi_2(u) - c\chi_1(u) = \int u\, d\alpha$ and $\chi_1(u) - c\chi_2(u) = \int u\, d\beta$.) Deduce that if χ is a character of B, then the Gleason part (Problem 18(b)) containing χ is the set of characters for which a representative measure has base μ, for at least one representative measure μ for χ.

(c) Suppose χ is a character for which the representative measure is *unique*. If χ' belongs to the Gleason part containing χ, show that the representative measure μ' for χ' is also unique, and that

$$\frac{1-\sigma(\chi,\chi')}{1+\sigma(\chi,\chi')}\,\mu \leqq \mu' \leqq \frac{1+\sigma(\chi,\chi')}{1-\sigma(\chi,\chi')}\,\mu.$$

(d) Suppose that there exists a hyperextremal representative measure μ for χ (Problem 11). Show that if a character χ' admits a representative measure μ' with base μ, then μ' is the only representative measure for χ' with base μ. (If $\mu' = f \cdot \mu$ and $\mu'' = g \cdot \mu$ are two representative measures for χ', show that $f - g$ is zero almost everywhere with respect to μ, by using Problem 13(j).)

20. Let A and B be two commutative Banach algebras having unit elements and $h : A \to B$ a homomorphism which sends unit element to unit element.

(a) Let Γ be the graph of h in $A \times B$. For each character χ of B, show that the mapping $(x, y) \mapsto \chi(h(x)) - \chi(y)$ is continuous on $A \times B$. Deduce that if (a, b) is in the closure of Γ, then $\chi(h(a)) = \chi(b)$.

(b) Deduce from (a) that if B is without radical (Section 15.2, Problem 7) then h is necessarily continuous (use the closed-graph theorem). In particular, for a commutative C-algebra having a unit element, in which the intersection of the maximal ideals is $\{0\}$, two norms which define Banach algebra structures are necessarily equivalent.

21. Let $\mathscr{D}$ be the algebra of indefinitely differentiable complex-valued functions on [0, 1].

(a) Let A be a subalgebra of $\mathscr{D}$, containing the unit element and endowed with a norm which makes A a Banach algebra. Show that there exists a sequence $(m_n)_{n \geqq 0}$ of finite real numbers $\geqq 0$ such that, for *all* functions $x \in A$, we have

$$\sup_{0 \leqq t \leqq 1} |x^{(n)}(t)| = O(m_n)$$

as n increases indefinitely. (If $\mathscr{D}_n$ is the Banach algebra of Section 15.1, Problem 1, then the canonical injection of A into $\mathscr{D}_n$ is continuous (Problem 20).)

(b) Show that there exists no norm on $\mathscr{D}$ with respect to which $\mathscr{D}$ is a Banach algebra. (Use (a) and Problem 4 of Section 8.14.)

22. Let A be a separable Banach algebra with unit element. Show that if $\mathfrak{a} \neq A$ is a left ideal of A, there exists a maximal left ideal containing $\mathfrak{a}$. (Observe that if $\mathfrak{n}$ is a closed left ideal and (x_m) a dense sequence in A such that either $x_m \in \mathfrak{n}$ or $\mathfrak{n} + Ax_m = A$ for each m, then $\mathfrak{n}$ is maximal.) Show that the radical of A (Section 15.2, Problem 7) is the intersection of the maximal left ideals of A, and is also the intersection of the maximal right ideals of A.

4. BANACH ALGEBRAS WITH INVOLUTION. STAR ALGEBRAS

Let A be an algebra over the field **C** of complex numbers. An *involution* on A is a bijection $x \mapsto x^*$ of A onto A, satisfying the following conditions:

(15.4.1) $\quad (x^*)^* = x, \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \bar{\lambda} x^*, \quad (xy)^* = y^* x^*$

for all x, y in A and all $\lambda \in \mathbf{C}$. The element x^* is often called the *adjoint* of x, and a subset of A which is stable under the involution is said to be *self-adjoint*.

If A has a unit element e, then $x \cdot e^* = (e \cdot x^*)^* = x^{**} = x$, and similarly $e^* \cdot x = x$ for all $x \in \mathrm{A}$, by virtue of (15.4.1). Hence

(15.4.2) $\qquad e^* = e.$

If x is invertible in A, then $(x^{-1})^* x^* = (x x^{-1})^* = e^* = e$, and similarly $x^*(x^{-1})^* = e$, by (15.4.1) and (15.4.2). Hence x^* is invertible, and

(15.4.3) $\qquad (x^{-1})^* = (x^*)^{-1}.$

A *normed algebra with involution* is a normed algebra A equipped with an involution $x \mapsto x^*$ such that

(15.4.4) $\qquad \|x^*\| = \|x\|$

for all $x \in \mathrm{A}$.

A *star algebra* is a *Banach algebra* A equipped with an involution $x \mapsto x^*$ such that

(15.4.5) $\qquad \|x\|^2 = \|x^* x\|$

for all $x \in \mathrm{A}$.

A star algebra satisfies (15.4.4), for it follows from (15.4.5) and (15.1.1) that $\|x\|^2 \leqq \|x\| \cdot \|x^*\|$, whence $\|x\| \leqq \|x^*\|$ for all $x \in \mathrm{A}$; interchanging x and x^*, we obtain (15.4.4).

Examples of Banach Algebras with Involution and Star Algebras

(15.4.6) If X is a metrizable compact space, the Banach algebra $\mathscr{C}_{\mathbf{C}}(\mathrm{X})$ is a star algebra with respect to the involution $f \mapsto \bar{f}$ (the complex conjugate of f). If X consists of a single point, $\mathscr{C}_{\mathbf{C}}(\mathrm{X})$ is isomorphic to the field **C**.

(15.4.7) Let E be a complex Hilbert space. Then the Banach algebra $A = \mathscr{L}(E)$ (15.1.6) is a star algebra with respect to the involution $u \mapsto u^*$ (11.5).

Hilbert–Schmidt Operators

(15.4.8) Let E be an infinite-dimensional separable complex Hilbert space. An operator $u \in \mathscr{L}(E)$ is said to be a Hilbert–Schmidt operator if, for *one* choice of a Hilbert basis (a_n), the series with general term $\|u(a_n)\|^2$ is *convergent*. If (b_n) is another Hilbert basis of E, then $\|u(a_n)\|^2 = \sum_m |(u(a_n) \mid b_m)|^2$ by Parseval's identity (6.5.2), and therefore the series $\sum_n \|u(a_n)\|^2$ converges if and only if the double family $(|(u(a_n) \mid b_m)|^2)$ is summable (5.3.4). But $(u(a_n) \mid b_m) = (a_n \mid u^*(b_m))$, and

$$\sum_n |(a_n \mid u^*(b_m))|^2 = \|u^*(b_m)\|^2$$

by Parseval's identity. Hence an equivalent condition is that the series with general term $\|u^*(b_n)\|^2$ should be convergent. It follows that, if u is a Hilbert–Schmidt operator, the series with general term $\|u(a_n)\|^2$ converges for *all* choices of the Hilbert basis (a_n), that u^* is a Hilbert–Schmidt operator, and that the number

$$\|u\|_2 = \left(\sum_n \|u(a_n)\|^2\right)^{1/2} = \|u^*\|_2 \tag{15.4.8.1}$$

is *independent of the Hilbert basis* (a_n) chosen. We shall show that the set $\mathscr{L}_2(E)$ of Hilbert–Schmidt operators is an algebra, on which $\|u\|_2$ is a *norm*, and that, endowed with this norm and the involution $u \mapsto u^*$, the algebra $\mathscr{L}_2(E)$ is a *Banach algebra with involution*, without unit element.

The fact that u is a Hilbert–Schmidt operator also means that, if (a_n) is any Hilbert basis of E, the sequence $(\|u(a_n)\|) \in \mathbf{R}^{\mathbf{N}}$ belongs to the Hilbert space $l^2_{\mathbf{R}}$ (6.5), and the norm of this sequence in $l^2_{\mathbf{R}}$ is equal to $\|u\|_2$. If $u, v \in \mathscr{L}_2(E)$, then the inequality $\|(u+v)(a_n)\| \leqq \|u(a_n)\| + \|v(a_n)\|$ and the preceding remark show that $u + v \in \mathscr{L}_2(E)$ and that $\|u+v\|_2 \leqq \|u\|_2 + \|v\|_2$. Clearly also $\lambda u \in \mathscr{L}_2(E)$ for all $\lambda \in \mathbf{C}$, and $\|\lambda u\|_2 = |\lambda| \cdot \|u\|_2$.

For *every* continuous operator $w \in \mathscr{L}(E)$, the operator $w \circ u$ is a Hilbert–Schmidt operator, by reason of the inequality

$$\|w(u(a_n))\| \leqq \|w\| \cdot \|u(a_n)\|$$

(5.5.1), and we have

$$\|w \circ u\|_2 \leqq \|w\| \cdot \|u\|_2 . \tag{15.4.8.2}$$

Since $u^* \in \mathscr{L}_2(E)$ and $(u \circ w)^* = w^* \circ u^* \in \mathscr{L}_2(E)$, it follows that $u \circ w \in \mathscr{L}_2(E)$ and that, by (11.5.2) and (15.4.8.1),

$$\|u \circ w\|_2 \leqq \|w\| \cdot \|u\|_2 . \tag{15.4.8.3}$$

Hence $\mathscr{L}_2(E)$ is a *self-adjoint two-sided ideal* in the Banach algebra with involution $\mathscr{L}(E)$. In general $\mathscr{L}_2(E)$ is not closed in $\mathscr{L}(E)$ (Problem 12).

For each $x \in E$ we can write $x = \sum_n \xi_n a_n$, and $\|x\|^2 = \sum_n |\xi_n|^2$. The interpretation given earlier for a Hilbert–Schmidt operator u in terms of $l^2_{\mathbf{R}}$ shows that the series with general term $|\xi_n| \cdot \|u(a_n)\|$ is convergent, and that

$$\|u(x)\|^2 \leqq \|x\|^2 \cdot \|u\|_2^2 , \tag{15.4.8.4}$$

so that

$$\|u\| \leqq \|u\|_2 . \tag{15.4.8.5}$$

This, together with (15.4.8.2), shows that the norm $\|u\|_2$ on the algebra $\mathscr{L}_2(E)$ satisfies the inequality (15.1.1). Also $\mathscr{L}_2(E)$ contains all continuous endomorphisms *of finite rank*. For if u is of finite rank, then its kernel $N = u^{-1}(0)$ is closed and of finite codimension in E, hence admits a (finite-dimensional) orthogonal supplement M in E (6.3.1). If (a_n) is a Hilbert basis of E obtained by taking the union of a Hilbert basis of N and a Hilbert basis of M, then $u(a_n) = 0$ for all but a finite number of indices n; hence $u \in \mathscr{L}_2(E)$. It follows that $\mathscr{L}_2(E)$ has no unit element, for the unit element (if it existed) could only be the identity mapping of E, and this is clearly not a Hilbert–Schmidt operator because E is of infinite dimension.

If $u \in \mathscr{L}_2(E)$, if (a_n) is a Hilbert basis of E and if u_n is the continuous endomorphism of E such that $u_n(a_k) = u(a_k)$ for $k \leqq n$ and $u_n(a_k) = 0$ for $k > n$, then it is immediate that $\lim_{n \to \infty} \|u - u_n\|_2 = 0$. Hence, in view of (15.4.8.5), it follows that u is a *compact* operator (11.2.10). But there exist compact operators which are not Hilbert–Schmidt operators (Problem 12).

It remains to be shown that $\mathscr{L}_2(E)$ is *complete* with respect to the norm $\|u\|_2$. Let (u_n) be a Cauchy sequence with respect to this norm; then by (15.4.8.5) (u_n) is also a Cauchy sequence with respect to the norm $\|u\|$ on $\mathscr{L}(E)$, hence *converges* to an element $v \in \mathscr{L}(E)$ with respect to this norm. On the other hand, there exists a real number $\beta > 0$ such that

$$\|u_n\|_2^2 = \sum_{k=0}^{\infty} \|u_n(a_k)\|^2 \leqq \beta$$

for all n, where (a_k) is a Hilbert basis of E. For each integer N we have therefore $\sum_{k=0}^{N} \|u_n(a_v)\|^2 \leqq \beta$ for all n; hence, letting n tend to $+\infty$, we have

$$\sum_{k=0}^{N} \|v(a_k)\|^2 \leqq \beta$$

for all N; consequently (5.3.1), $v \in \mathscr{L}_2(E)$ and $\|v\|_2^2 \leqq \beta$. Finally, for each $\varepsilon > 0$, there exists n_0 such that $\|u_m - u_n\|_2 \leqq \varepsilon$ for all $m \geqq n_0$ and all $n \geqq n_0$, and the same argument shows that $\|v - u_n\|_2 \leqq \varepsilon$ for all $n \geqq n_0$. Hence $v = \lim_{n \to \infty} u_n$ in $\mathscr{L}_2(E)$.

It is easily verified that the Banach algebra $\mathscr{L}_2(E)$ *is not* a star algebra (Problem 13).

Remark

(15.4.8.6) Let (a_n) be a Hilbert basis of E. To each continuous operator $u \in \mathscr{L}(E)$ we may associate the double sequence $(c_{mn}) = ((u(a_n) \mid b_m))$ (sometimes called the *infinite matrix* of u with respect to (a_n)). We have seen above that, if u is a Hilbert–Schmidt operator, then $\|u\|_2^2 = \sum_{m,n} |c_{mn}|^2$. Conversely, let (c_{mn}) be any double sequence of complex numbers such that the double family $(|c_{mn}|^2)$ is summable. Then, for all n, the vector $b_n = \sum_m c_{mn} a_m$ is defined, and we have $\|b_n\|^2 = \sum_m |c_{mn}|^2$ (6.5.2). Furthermore, for each vector $x = \sum_n \xi_n a_n$ (with $\|x\|^2 = \sum_n |\xi_n|^2$), the vector $u(x) = \sum_n \xi_n b_n$ is defined, and $\|u(x)\|^2 \leqq \left(\sum_{m,n} |c_{mn}|^2\right) \cdot \|x\|^2$; for the Cauchy–Schwarz inequality shows that if $h \leqq k$ we have

$$\begin{aligned}
\left\| \sum_{n=h}^{k} \xi_n b_n \right\|^2 &\leqq \left(\sum_{n=h}^{k} |\xi_n| \cdot \|b_n\| \right)^2 \\
&\leqq \left(\sum_{n=h}^{k} |\xi_n|^2 \right) \left(\sum_{n=h}^{k} \|b_n\|^2 \right) \\
&\leqq \left(\sum_{n=h}^{k} |\xi_n|^2 \right) \left(\sum_{m,n} |c_{mn}|^2 \right)
\end{aligned}$$

from which follows the absolute convergence of the series $\sum_n \xi_n b_n$ and the inequality above for $\|u(x)\|$. We have therefore defined a continuous endomorphism u of E (5.5.1), and it is immediately verified that

$$\sum_n \|u(a_n)\|^2 = \sum_{m,n} |c_{mn}|^2,$$

so that $u \in \mathscr{L}_2(E)$.

(15.4.9) Let G be a separable, metrizable, locally compact group. Then the mapping $\mu \mapsto \overline{(\check{\mu})} = (\bar{\mu})^{\check{}}$ (denoted by $\check{\bar{\mu}}$) is an *involution* on the Banach algebra $M_c^1(G)$ (15.1.7); this follows from (14.7.3.1). Also we have $\|\mu\| = \|\bar{\mu}\| = \|\check{\bar{\mu}}\|$ (13.20.1), hence $M_c^1(G)$ is a Banach algebra with involution. It can be shown that it is not necessarily a star algebra.

(15.4.10) It is clear that every self-adjoint subalgebra (resp. self-adjoint closed subalgebra) of a normed algebra with involution (resp. of a star algebra) is a normed algebra with involution (resp. a star algebra).

(15.4.11) In an algebra A with involution, an element x is said to be *self-adjoint* (or *hermitian*) if $x^* = x$ (the terminology is inspired by the example (15.4.7)). For each $x \in A$, the elements $x_1 = \frac{1}{2}(x + x^*)$ and $x_2 = (x - x^*)/2i$ are self-adjoint; so are xx^* and x^*x, and in general these two products are distinct. The element x is said to be *normal* if $xx^* = x^*x$, or equivalently if the self-adjoint elements x_1 and x_2 commute. If A has a unit element e, then $x \in A$ is said to be *unitary* if $xx^* = x^*x = e$, that is if x is invertible and $x^{-1} = x^*$. The unitary elements of A form a multiplicative *group*; for if x is unitary, then so is x^{-1} because (15.4.3) $(x^{-1})^* = (x^*)^{-1} = x = (x^{-1})^{-1}$, and if x, y are unitary then so is xy, because $(xy)^* = y^*x^* = y^{-1}x^{-1} = (xy)^{-1}$.

(15.4.12) *Let* A *be a Banach algebra with involution, having a unit element* $e \neq 0$.

(i) *For each* $x \in A$, $\mathrm{Sp}(x^*)$ *is the image of* $\mathrm{Sp}(x)$ *under the mapping* $\zeta \mapsto \bar{\zeta}$.

(ii) *If* A *is a star algebra, then for each self-adjoint element* $x \in A$ *we have* $\mathrm{Sp}(x) \subset \mathbf{R}$, *and for each unitary element* $y \in A$ *we have* $\mathrm{Sp}(y) \subset \mathbf{U}$.

(i) If $x - \zeta e$ is invertible, then so is $(x - \zeta e)^* = x^* - \bar{\zeta} e$ ((15.4.1) and (15.4.2)). Hence the result.

(ii) Suppose that $\alpha + i\beta \in \mathrm{Sp}(x)$, where α and β are real. Then, for each *real* number λ, the complex number $\alpha + i(\beta + \lambda)$ belongs to $\mathrm{Sp}(x + i\lambda e)$ and therefore, by (15.2.4),

$$\alpha^2 + (\beta + \lambda)^2 \leqq \|x + i\lambda e\|^2 = \|x^*x + \lambda^2 e\| \leqq \|x^*x\| + \lambda^2 = \|x\|^2 + \lambda^2,$$

so that

$$2\beta\lambda \leqq \|x\|^2 - \alpha^2 - \beta^2.$$

Since this is true for *all* real numbers λ, it follows that $\beta = 0$. On the other hand, if y is unitary, we have $\|y\|^2 = \|y^*y\| = \|e\| = 1$, hence $\|y\| = 1$; similarly $\|y^{-1}\| = 1$, and therefore $\mathrm{Sp}(y) \subset \mathbf{U}$ (15.2.6).

(15.4.13) *Let* B *be a star algebra with unit element* $e \neq 0$, *and* A *a closed self-adjoint subalgebra of* B *containing* e. *Then* $\mathrm{Sp}_A(x) = \mathrm{Sp}_B(x)$ *for all* $x \in A$.

If x is self-adjoint, this follows because $\mathrm{Sp}_A(x) \subset \mathbf{R}$ and therefore all the points of $\mathrm{Sp}_A(x)$ are frontier points in $\mathbf{C}$, and we need only apply (15.2.8). On the other hand, if $x \in A$ is invertible *in* B, then so is xx^* (15.4.3), which is self-adjoint and belongs to A. From above, xx^* is invertible *in* A, hence x has a right inverse in A. The same argument, applied to x^*x, shows that x is invertible in A. If we replace x by $x - \zeta e$ in this argument, we obtain the relation $\mathrm{Sp}_A(x) = \mathrm{Sp}_B(x)$.

(15.4.14) (Gelfand–Neumark theorem) *Let* A *be a separable commutative star algebra with unit element* $e \neq 0$. *Then the Gelfand transformation* (15.3) *is an isometric isomorphism of* A *onto* $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$ (*that is to say,* $\rho(x) = \|\mathscr{G}x\| = \|x\|$) *such that* $\mathscr{G}x^* = \overline{\mathscr{G}x}$ *for all* $x \in A$.

If χ is a character of A, then $\chi(x)$ is real for all self-adjoint $x \in A$, by virtue of (15.4.12) and (15.3.1). For each element $x = x_1 + ix_2$ of A, where x_1 and x_2 are self-adjoint (15.4.11), we have therefore

$$\chi(x^*) = \chi(x_1 - ix_2) = \chi(x_1) - i\chi(x_2) = \overline{\chi(x)},$$

and therefore $\mathscr{G}x^* = \overline{\mathscr{G}x}$. Next, the elements $\mathscr{G}x \in \mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$ *separate* the points of $\mathbf{X}(A)$ (7.3), for if $\chi_1 \neq \chi_2$ are two distinct characters, then by definition there exists $x \in A$ such that $\chi_1(x) \neq \chi_2(x)$, in other words such that $(\mathscr{G}x)(\chi_1) \neq (\mathscr{G}x)(\chi_2)$. Also the constants belong to $\mathscr{G}(A)$, because $\mathscr{G}e$ is the constant 1. Hence by (7.3.2) the subalgebra $\mathscr{G}(A)$ is *dense* in $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$. To complete the proof it is enough to show that $\mathscr{G}$ is *isometric*, for then $\mathscr{G}(A)$, being isomorphic to A, will be a *complete* subspace of $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$, and therefore will be equal to $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(A))$ (3.14.4).

We shall first prove the following lemma:

(15.4.14.1) *In a* (*commutative or noncommutative*) *star algebra* A, *we have* $\rho(y) = \|y\|$ *for every self-adjoint* $y \in A$.

We have $\|y\|^2 = \|y^2\|$ by (15.4.5) and therefore $\|y\|^{2^n} = \|y^{2^n}\|$ by induction on n. Hence $\|y\| = \|y^{2^n}\|^{2^{-n}}$; letting $n \to \infty$, the right-hand side tends to $\rho(y)$ (15.2.7), which proves the result.

Now return to the proof of (15.4.14). Let x be any element of A; then $\|x\|^2 = \|x^*x\| = \|\mathscr{G}(x^*x)\|$ by the lemma and (15.3.4(i)). But $\mathscr{G}(x^*x) = \mathscr{G}x^* \cdot \mathscr{G}x = |\mathscr{G}x|^2$, and $\|\mathscr{G}(x^*x)\| = \|\mathscr{G}x\|^2$. Q.E.D.

(15.4.15) *Let* A *be a commutative star algebra with unit element* $e \neq 0$. *Suppose that there exists* $x_0 \in A$ *such that the subalgebra generated by* e, x_0 *and* x_0^* *is dense in* A. *Then the mapping* $\chi \mapsto \chi(x_0)$ *is a homeomorphism of* $\mathbf{X}(A)$ *onto* $\mathrm{Sp}_A(x_0)$.

As in (15.3.6) we see that A is separable, and that it is enough to prove that the relation $\chi_1(x_0) = \chi_2(x_0)$ for two characters χ_1, χ_2 implies that $\chi_1 = \chi_2$. Now, from (15.4.14) we deduce that $\chi_1(x_0^*) = \chi_2(x_0^*)$ and thence that $\chi_1(P(x_0, x_0^*)) = \chi_2(P(x_0, x_0^*))$ for all polynomials $P \in \mathbf{C}[X, Y]$. Since by hypothesis the $P(x_0, x_0^*)$ are dense in A, and since χ_1, χ_2 are continuous on A, the result follows.

PROBLEMS

1. Let A be a normed algebra endowed with a continuous involution $x \mapsto x^*$. If $\|x\|_1 = \sup(\|x\|, \|x^*\|)$, show that with the norm $\|x\|_1$ the algebra A becomes a normed algebra with involution, and that the norms $\|x\|$ and $\|x\|_1$ are equivalent.

2. Let A be a commutative Banach algebra without radical (Section 15.2, Problem 7). Show that every involution on A is continuous (use Problem 20 of Section 15.3).

3. Let A be a commutative Banach algebra with involution, having a unit element. A character χ of A is said to be *hermitian* if $\chi(x^*) = \overline{\chi(x)}$ for all $x \in A$. Show that every character of A is hermitian if and only if, for each $x \in A$, the element $1 + xx^*$ is invertible in A. (If every character of A is hermitian, then $\mathscr{G}(1 + xx^*)$ is a function which is everywhere >0 on $\mathbf{X}(A)$; use (15.3.4). If a character χ of A is not hermitian, then there exists a self-adjoint $y \in A$ such that $\chi(y) = i$, whence $\chi(1 + yy^*) = 0$, and $1 + yy^*$ is not invertible.) Hence give an example of a commutative Banach algebra with involution, having a unit element, for which there exist nonhermitian characters.

4. (a) If A is a star algebra with unit element, then $\|x\|^2 = \rho(x^*x)$ for all $x \in A$ (prove this first when x is self-adjoint). Deduce that A is without radical.

   (b) Let A be the algebra of Problem 2 of Section 15.1. If $x = \sum_{n=0}^{\infty} \xi_n T^n$, define x^* to be $\sum_{n=0}^{\infty} \bar{\xi}_n T^n$. This defines an involution on A for which A is a commutative Banach algebra with involution, having a unit element, and such that the only character of A (Section 15.3, Problem 2) is hermitian.

5. With respect to the involution $f^*(t) = \overline{f(-t)}$ on the Beurling algebra (Section 15.1, Problem 4 and Section 15.2, Problem 6), show that all the characters are hermitian.

6. On the algebra A of Problem 7 of Section 15.3, consider the involution induced by $f \mapsto \check{f}$, and extend this involution to $\tilde{A}$ by putting $e^* = e$. Then A is a Banach algebra with involution. Find its hermitian characters.

7. (a) In a star algebra, every hermitian quasi-nilpotent element (Section 15.2, Problem 5) is zero.
(b) Let A be a star algebra with unit element, and let x be an element of A which is not left-invertible. Then x^*x is not invertible in the star subalgebra B of A generated by 1 and x^*x, and therefore there exists a sequence (y_n) of elements of B such that $\|y_n\| = 1$ and $\lim_{n\to\infty} y_n x^* x = 0$ (use the Gelfand–Neumark theorem). Deduce that a noninvertible element of A is necessarily a (left or right) topological zero-divisor.

8. Let A be a star algebra with unit element, x a normal element of A, and $S = Sp_A(x)$. Then there exists a unique homomorphism φ of the star algebra $\mathscr{C}_C(S)$ into A, transforming unit element into unit element, and such that $\varphi(1_S) = x$. This homomorphism is an isometry, which sends $\bar{f}$ to $\varphi(f)^*$, of $\mathscr{C}_C(S)$ onto the star subalgebra of A generated by 1, x and x^*; all the elements of this subalgebra are therefore normal. Put $\varphi(f) = f(x)$, and show that if f is analytic in a neighborhood of S, then the element $f(x)$ is equal to the element so denoted in Section 15.2, Problem 11.

9. Let A be a star algebra with unit element. Let P denote the set of self-adjoint elements of A such that $Sp_A(x) \subset [0, +\infty[$.
(a) Show that if $x \in A$ is self-adjoint and $\|e - x\| \leqq 1$, then $x \in P$. If $x \in P$ and $\|x\| \leqq 1$, then $\|e - x\| \leqq 1$. (Consider the subalgebra generated by e and x.) Show that a self-adjoint element x belongs to P if and only if $\|x - \|x\|e\| \leqq \|x\|$.
(b) Deduce from (a) that P is a closed convex cone in A, such that $P \cap (-P) = \{0\}$.
(c) Show that the relation $x^*x \in -P$ implies that $x = 0$. (Observe first that also $xx^* \in -P$, and by writing $x = u + iv$, where u and v are self-adjoint, deduce that $x^*x \in P$, whence $x^*x = 0$.)
(d) Deduce from (c) that $x^*x \in P$ for all $x \in A$. (Write $x^*x = u - v$, where u, v are hermitian and belong to P, and $uv = vu = 0$ (use Problem 8). If $z = xv$, show that $z^*z \in -P$, and deduce that $v = 0$.) Deduce that $e + x^*x$ is invertible in A (use Problem 8).

10. (a) Let A be a commutative Banach algebra with unit element, and let $x \mapsto x^*$ be any involution on A. For each character χ of A, put $\chi^*(x) = \overline{\chi(x^*)}$. Show that $\chi \mapsto \chi^*$ is an involutory homeomorphism of the space $\mathbf{X}(A)$.
(b) Let X be a metrizable compact space, and φ an involutory homeomorphism of X onto X. For each $f \in \mathscr{C}_C(X)$, put $f^*(x) = \overline{f(\varphi(x))}$. Show that this defines an involution on $\mathscr{C}_C(X)$, and that every involution on $\mathscr{C}_C(X)$ may be so obtained.
(c) Let X be the compact subspace of $\mathbf{R}^2$ which is the union of the segment $y = 0$, $-1 \leqq x \leqq 2$, the segment $x = 0$, $1 \leqq y \leqq 2$, the circle $x^2 + y^2 = 1$ and the open half-disk $D : y > 0$, $x^2 + y^2 < 1$. Assume that the only involutory homeomorphism of X onto X is the identity (Chapter XXIV). Let A be the Banach subalgebra of $\mathscr{C}_C(X)$ consisting of functions which are analytic in D. Show that the spectrum of A can be canonically identified with X, and that the algebra A has *no* involution other than the identity.

11. Let A be a noncommutative star algebra with unit element.
(a) Show that there exists a hermitian element $y \in P$ (notation of Problem 9) such that y is invertible and y^2 is not in the center Z of A. (Using Problem 9, show that if the result were false, the intersection of Z with the real vector subspace H of A consisting of the hermitian elements would contain a neighborhood of e in H, which would imply that A was commutative.)

(b) Deduce from (a) that, if we put $x' = y^{-1}x^*y$ for all $x \in A$, then $x \mapsto x'$ is continuous involution on A such that $(x')^* \neq (x^*)'$ for some $x \in A$.

12. Let E be a Hilbert space. Show that a self-adjoint compact operator U is a Hilbert-Schmidt operator if and only if, (λ_n) being the sequence of eigenvalues of U eacl counted according to its multiplicity, the sum $\sum_n |\lambda_n|^2$ is finite. The sum is then equa to $\|U\|_2^2$.

13. (a) The norm $\|u\|_2$ is defined as in (15.4.8) when E is a finite-dimensional Hilber space. In a Hilbert space of dimension n, give an example of an operator u such tha

$$\|u^*u\|_2 \leqq \frac{1}{(n-1)^{1/2}} \|u\|_2^2.$$

(b) Let E be a Hilbert space of infinite dimension. Give an example of a sequence (u_n) of Hilbert–Schmidt operators on E such that the sequence of numbers $\|u_n^*u_n\|_2/\|u_n\|_2^2$ tends to 0 as n tends to ∞.

14. Let X be a (metrizable, separable) locally compact space, and let μ be a positive measure on X. Let $K(x, y)$ be a function on $X \times X$, belonging to $\mathscr{L}^2_{\mathbb{C}}(X \times X, \mu \otimes \mu)$. For each function $f \in \mathscr{L}^2_{\mathbb{C}}(X, \mu)$, the function $x \mapsto \int K(x, y)f(y)\, d\mu(y)$ is defined almost everywhere, and its class belongs to $L^2_{\mathbb{C}}(X, \mu)$ (use the Lebesgue–Fubini theorem and the Cauchy–Schwarz inequality). If this class is denoted by $U \cdot f$, show that U is a Hilbert–Schmidt operator; U is said to be associated to the *kernel* K. Show that $\|U\|_2 = N_2(K)$ and that U^* is associated to the kernel $(x, y) \mapsto \overline{K(y, x)}$. If U_1, U_2 are associated to kernels K_1, K_2 respectively, belonging to $\mathscr{L}^2_{\mathbb{C}}(X \times X, \mu \otimes \mu)$, then U_1U_2 is associated to the kernel

$$K(x, y) = \int K_1(x, t)K_2(t, y)\, d\mu(t).$$

15. Let E be an infinite-dimensional Hilbert space and U a Hilbert–Schmidt operator on E. If λ is a regular value for U, show that the operator $(U - \lambda \cdot 1_E)^{-1}$ is of the form $V - \lambda^{-1} \cdot 1_E$, where V is a Hilbert–Schmidt operator. Deduce that the spectrum of U in the algebra $\mathscr{L}(E)$ is the same as in the algebra obtained by adjoining a unit element to $\mathscr{L}_2(E)$. For every function f which is analytic in an open set containing $Sp(U)$, the operator $f(U)$ (Section 15.2, Problem 11) is a Hilbert–Schmidt operator.

16. Let $H_1, \ldots, H_n$ be self-adjoint operators, each pair of which commute, on a separable Hilbert space E, and suppose that $\|H_j\| \leqq 1$ and that $\|H_jH_k\| \leqq \varepsilon^{|j-k|}$ for all j, k, where ε is a real number such that $0 \leqq \varepsilon < 1$. Then

$$\|H_1 + \cdots + H_n\| \leqq (1 + \sqrt{\varepsilon})/(1 - \varepsilon)$$

("Cotlar's lemma").

(Using the Gelfand–Neumark theorem, reduce to proving the same property for real numbers $u_1, \ldots, u_n$. For each $t \geqq 0$, let $\nu(t)$ be the number of indices j such that $|u_j| \geqq t$. Remark that

$$\sum_{j=1}^{n} |u_j| = \int_0^1 \nu(t)\, dt$$

and prove that $\nu(t) \leqq 1 + [2 \log t/\log \varepsilon]$, by observing that if $|u_j| \geqq t$ and $|u_k| \geqq t$, then $\varepsilon^{|j-k|} \geqq |u_j u_k| \geqq t^2$.)

17. Let A be a Banach algebra with unit element e, and let $x \mapsto x^*$ be an involution on A which is *not necessarily continuous*.

(a) Show that the radical $\mathfrak{R}$ of A (Section 15.2, Problem 7) is a self-adjoint subset of A. Passing to the quotient, it follows that the involution on A defines an involution (also written $z \mapsto z^*$) on $A/\mathfrak{R}$.

(b) Let $a = a^*$ be a self-adjoint element of A, such that $\mathrm{Sp}_A(a)$ is contained in the open half-plane $\mathscr{R}\zeta > 0$. Then there exists an element $b \in A$, which is the limit of a sequence (u_n) of self-adjoint elements which are polynomials in a, and which satisfies $b^2 = a$. (Write $a = \alpha(e - (e - \alpha^{-1}a))$ for a suitable $\alpha > 0$, and use Problem 11(f) and Section 15.2, Problem 15.) Show that b and b^* commute, and that we may therefore write $b = u + iv$, where u and v are self-adjoint and commute with each other and with a. Deduce that $uv = 0$ and $u^2 - v^2 = a$.

(c) Let C be a commutative Banach subalgebra of A which contains e, a, b, b^* and is such that $\mathrm{Sp}_C(b) = \mathrm{Sp}_A(b)$ (Section 15.2, Problem 15). Let C′ be a commutative Banach subalgebra of A containing the image of C under the homomorphism $\varphi : x \mapsto x^*$ of C into A. If $\mathfrak{R}'$ is the radical of C′ and $\pi : C' \to C'/\mathfrak{R}'$ the canonical homomorphism, then the homomorphism $\pi \circ \varphi : C \to C'/\mathfrak{R}'$ is continuous (Section 15.3, Problem 20). Deduce that $v = \dfrac{1}{2i}(b^* - b) \in \mathfrak{R}'$, and thence that $\mathrm{Sp}_A(a) = \mathrm{Sp}_A(u^2)$. (Observe that $\mathrm{Sp}_A(a) = \mathrm{Sp}_{C'/\mathfrak{R}'}(\pi(u^2 - v^2)) = \mathrm{Sp}_{C'/\mathfrak{R}'}(\pi(u^2))$.) Consequently u is invertible, $v = u^{-1}uv = 0$, and finally $b = u$ is *self-adjoint*. Furthermore, if $\mathrm{Sp}(a)$ is contained in $]0, +\infty[$, then the same is true of $\mathrm{Sp}(b)$.

18. With the hypotheses of Problem 17, put $p_A(x) = (\rho(x^*x))^{1/2}$ (also denoted by $p(x)$). We have $p(x^*) = p(x)$; also if $\mathfrak{R}$ is the radical of A and if $\pi : A \to A/\mathfrak{R}$ is the canonical homomorphism, then $p_{A/\mathfrak{R}}(\pi(x)) = p_A(x)$.

The involution $x \mapsto x^*$ on A is said to be *hermitian*, and the involutive algebra A is said to be *hermitian*, if $\mathrm{Sp}(a) \subset \mathbf{R}$ for all self-adjoint elements $a \in A$. If A is commutative, an equivalent condition is that every character of A is hermitian (Problem 2).

Assume for the rest of this Problem that A is hermitian.

(a) Let $x \in A$ be such that $p(x) < 1$. Show that $(e + x^*)(e - x)$ is invertible. (Remark that $e - x^*x = w^2$, where w is self-adjoint and invertible, by virtue of Problem 17; hence show that the spectrum of $w^{-1}(x^* - x)w^{-1}$ is contained in $i\mathbf{R}$.) Deduce that

$$\rho(x) \leqq p(x)$$

for all $x \in A$ (Ptàk's inequality).

Consequently, if x is normal, we have $\rho(x) = p(x)$. (Remark that if x, y are two elements of a Banach algebra which commute, then $\rho(xy) \leqq \rho(x)\rho(y)$, by (15.3.4).) In particular, if x is unitary, then $\rho(x) = 1$ and $\mathrm{Sp}(x) \subset \mathbf{U}$.

(b) Deduce from (a) that for each pair a, b of self-adjoint elements of A we have $\rho(ab) \leqq \rho(a)\rho(b)$. (Observe that $\rho(ab) \leqq p(ab) = (\rho(a^2b^2))^{1/2} \leqq \|a^2\|^{1/2} \cdot \|b^2\|^{1/2}$.) For all x, $y \in A$ we have $p(xy) \leqq p(x)p(y)$, and the radical of A is the set of all $x \in A$ such that $p(x) = 0$.

(c) Suppose that a, b are self-adjoint elements of A such that $\mathrm{Sp}(a)$ and $\mathrm{Sp}(b)$ are contained in $[0, +\infty[$. Show that $\mathrm{Sp}(a + b)$ is contained in $[0, +\infty[$. (Write

$$e + a + b = (e + a)(e - uv)(e + b),$$

with $\rho(u) < 1$ and $\rho(v) < 1$.) Deduce that for any two self-adjoint elements a, b of A, we have $\rho(a + b) \leqq \rho(a) + \rho(b)$. Show also that $\mathrm{Sp}(x^*x + xx^*) \subset [0, +\infty[$ for all $x \in A$.

(d) If $a, b \in A$ are self-adjoint, show that $\rho(a^2) \leqq \rho(a^2 + b^2)$. (Remark that

$$\rho(a^2 + b^2)e - a^2 = (\rho(a^2 + b^2)e - a^2 - b^2) + b^2,$$

and use (c).) Deduce that $\rho(x^* + x) \leqq 2p(x)$ for all $x \in A$.
(e) Deduce from (b) and (d) that $p(x + y) \leqq p(x) + p(y)$ for all $x, y \in A$. without radical, it follows that p defines on A a *normed algebra* structure, fo the topology is coarser than that defined by the norm $\|x\|$. These two top coincide only if A is complete relative to the norm p.
(f) Show that if A is without radical, the function $x \mapsto x^*$ is continuous on the closed graph theorem, by remarking that if a sequence (x_n) is such that and $x_n^* \to y$, then $p(x_n^*)$ and $p(y - x_n^*)$ both tend to 0; then use (e).)
(g) Show that p is a *continuous* function on A. (Use (f), by remarking tha exists a constant $c > 0$ such that $\|\pi(x^*)\| \leqq c\|\pi(x)\|$, and that $p_A(x) = p_{A/\mathfrak{R}}$
(h) Show that, for each $x \in A$, $\mathrm{Sp}(x^*x)$ is contained in $[0, +\infty[$. (Argue by diction: suppose that there exists $x \in A$ such that $-1 \in \mathrm{Sp}(x^*x)$. Write $z = x$ for each positive integer n let w_n be a self-adjoint element of A which commutes and is such that $w_n^2 = z^2 + \frac{1}{n}e$ and $\mathrm{Sp}(w_n) \subset [0, +\infty[$ (Problem 17). Let $b_n =$ By considering a commutative Banach subalgebra B of A containing w_n and such that the spectra of z, w_n, and b_n are the same in B and in A, show that

$$\mathrm{Sp}(b_n) \subset [0, +\infty[.$$

Show also that $\rho(b_n) \leqq 1 + 2\rho(z) = \alpha$, independent of n (use (c)). Put $y_n =$ that $y_n^* y_n = -b_n^2 w_n + (1/n)b_n$; deduce that $\mathrm{Sp}(y_n^* y_n)$ is contained in the $]-\infty, \alpha/n]$, and then that $\mathrm{Sp}(y_n y_n^*)$ is contained in the interval $[-\alpha/n, +\infty[$ (u hence that $\rho(y_n^* y_n) \leqq \alpha/n$ (Section 15.2, Problem 2(b)), and consequent $|\chi(y_n^* y_n)| \leqq \alpha/n$ for all characters χ of B. Finally, obtain a contradiction by that $y_n^* y_n = b_n^2 z$, that $\chi(w_n) \geqq 0$ and that there exists a character χ such that $\chi(z) =$

19. Let A be a Banach algebra with unit element, and let $x \mapsto x^*$ be a (not nec continuous) involution on A. Show that the following properties of A are equi

(α) A is hermitian.
(β) $\rho(x) \leqq p(x)$ for all normal $x \in A$.
(γ) $\rho(x) = p(x)$ for all normal $x \in A$.
(δ) $\rho(x^* + x) \leqq 2p(x)$ for all $x \in A$.
(ε) $p(x + y) \leqq p(x) + p(y)$ for all $x, y \in A$.
(ζ) The set of numbers $\rho(x)$, where x runs through the unitary element is bounded.
(η) For each $x \in A$ the element $e + x^*x$ is invertible in A.

(To show that (η) implies (α), argue by contradiction by showing tl spectrum of a hermitian element cannot contain the number i. To show t implies (α), argue as in (15.4.12). To show that (δ) implies (β), remark t implies that $\rho(x)^n = \rho(x^n) \leqq 2p(x)^n$ for x normal and $n \geqq 1$. Finally, to show t implies (α), notice first that (ζ) implies that $\mathrm{Sp}(x) \subset U$ for all unitary element considering the powers x^n ($n \in \mathbf{Z}$). Next observe that if a is self-adjoint and $\rho($ there exists a self-adjoint element b, commuting with a, such that $b^2 =$ (Problem 17), hence $a + ib$ is unitary. Then consider a commutative Banach subs containing a and b and such that the spectra of a, b and $a + ib$ are the same in A

20. Let A be a Banach algebra with unit element e, and let $x \mapsto x^*$ be a (not necessarily continuous) involution on A. Show that if $\|x\| \leqq p(x)$, then A is a star algebra.

5. REPRESENTATIONS OF ALGEBRAS WITH INVOLUTION

Let A be an algebra with involution (not necessarily endowed with a norm, and not necessarily having a unit element), and let H be a *Hilbert* space. A *representation*† of A in H is a homomorphism $s \mapsto U(s)$ of A into the algebra $\mathscr{L}(H)$ of continuous endomorphisms of H, such that

$$U(s^*) = (U(s))^*. \tag{15.5.1}$$

This implies in particular that if s is self-adjoint, then so is $U(s)$. If A has a unit element e, we require in addition that

$$U(e) = 1_H. \tag{15.5.2}$$

The representation U is said to be *faithful* if the homomorphism $s \mapsto U(s)$ is *injective*, that is if the relation $U(s) \cdot x = 0$ for *all* $x \in H$ implies that $s = 0$.

Let H, H′ be two Hilbert spaces. A representation $s \mapsto U(s)$ of A in H and a representation $s \mapsto U'(s)$ of A in H′ are said to be *equivalent* if there exists a *Hilbert space isomorphism* $T : H \to H'$ such that $U'(s) = TU(s)T^{-1}$ for all $s \in A$.

When $H = H'$, the *Hilbert space automorphisms* T of H are precisely the *unitary* elements of the star algebra $\mathscr{L}(H)$. For T must be invertible and satisfy $(T \cdot x \mid T \cdot y) = (x \mid y)$ for all x, y in H, so that $(T^*T \cdot x \mid y) = (x \mid y)$ and therefore $T^*T = 1_H$, hence $T^* = T^{-1}$; the converse is immediate.

Let H_1, H_2 be two Hilbert spaces, $s \mapsto U_1(s)$ a representation of A in H_1, and $s \mapsto U_2(s)$ a representation of A in H_2. Let H be the *Hilbert sum* of H_1 and H_2, so that H_1 and H_2 are identified with supplementary subspaces of H. If $x = x_1 + x_2$ and $y = y_1 + y_2$ are two elements of H (where x_i, y_i in H_i for $i = 1, 2$), then (6.4)

$$(x \mid y) = (x_1 \mid y_1) + (x_2 \mid y_2).$$

If we put $U(s) \cdot (x_1 + x_2) = U_1(s) \cdot x_1 + U_2(s) \cdot x_2$, it is immediately verified that $U(s) \in \mathscr{L}(H)$ for each $s \in A$, and that $s \mapsto U(s)$ is a representation of A, called the *Hilbert sum* of the given representations.

† Strictly speaking, a *unitary representation*. Since we shall not consider other types of representations, we shall suppress the word "unitary," by abuse of language.

Let $s \mapsto U(s)$ be a representation of A in a Hilbert space H. A vector subspace E of H is said to be *stable* with respect to this representation if $U(s)E \subset E$ for *all* $s \in A$. If E is stable with respect to U, then so is the closure $\bar{E}$ of E (3.11.4). If E is closed and if E′ is the *orthogonal supplement* of E in H (6.3), then E′ is also *stable with respect to* U. For if $x \in E$ and $x' \in E'$, we have $(x \mid U(s) \cdot x') = ((U(s))^* x \mid x') = (U(s^*)x \mid x') = 0$ by hypothesis, hence $U(s) \cdot x'$ is orthogonal to all $x \in E$ and therefore belongs to E′. If $U_1(s)$ and $U_2(s)$ are the restrictions of $U(s)$ to E and E′, respectively, then the representation U is the Hilbert sum of U_1 and U_2.

(15.5.3) *For a closed subspace* E *of* H *to be stable with respect to* U, *it is necessary and sufficient that* $P_E U(s) = U(s)P_E$ *for all* $s \in A$, *where* P_E *is the orthogonal projection on* E (6.3).

The condition is necessary, for if $x \in E$ we have $P_E \cdot x = x$ and $P_E \cdot (U(s) \cdot x) = U(s) \cdot x$, because E is stable with respect to U; and if $x \in E'$ we have $P_E \cdot x = 0$ and $P_E \cdot (U(s) \cdot x) = 0$, because E′ is also stable with respect to U. Conversely, if the given condition is satisfied, then $U(s) \cdot x = P_E \cdot U(s) \cdot x \in E$ for all $x \in E$ and all $s \in A$.

On a Hilbert space H, an *orthogonal projector* is by definition any continuous operator on H which is an orthogonal projection onto a closed subspace of H. The importance of such projectors is due to (15.5.3) and to their characterization in terms of the structure of algebra with involution of $\mathscr{L}(H)$:

(15.5.3.1) *A continuous operator* P *on a Hilbert space* H *is an orthogonal projector if and only if it is idempotent and hermitian.*

The necessity of these conditions has already been proved (11.5). Conversely, if $P^2 = P = P^*$, then $(P \cdot x \mid y - P \cdot y) = (x \mid P \cdot y - P^2 \cdot y) = 0$ for all $x, y \in H$. Since $P(H)$ is also the kernel of $1_H - P$, it is closed, and H is the Hilbert sum of $P(H)$ and $P^{-1}(0)$. Hence the result.

Suppose that H is the *Hilbert sum* of an infinite sequence (H_n) of subspaces which are *stable* with respect to the representation U. Let $U_n(s)$ denote the restriction of $U(s)$ to H_n, so that for each n the mapping $s \mapsto U_n(s)$ is a representation of A in H_n. By abuse of language, the representation U is said to be the *Hilbert sum* of the representations U_n. For each $s \in A$ and each $x = \sum_n x_n \in H$, where $x_n \in H_n$ for each n, we have $U(s) \cdot x = \sum_n U_n(s) \cdot x_n$, and $\sum_n \|U_n(s) \cdot x_n\|^2 = \|U(s) \cdot x\|^2$.

A representation $s \mapsto U(s)$ of A in H is said to be *topologically irreducible* if there exists no *closed* vector subspace E of H, other than $\{0\}$ and H, which is stable with respect to U.

From (15.5.3) we obtain the following *irreducibility criterion:*

(15.5.4) *In order that U should be topologically irreducible, it is necessary and sufficient that the only orthogonal projections P such that $PU(s) = U(s)P$ for all $s \in A$ should be 0 and 1_H.*

For this condition expresses that $\{0\}$ and H are the only closed subspaces of H which are stable under U.

(15.5.5) *Let $s \mapsto U(s)$ be a representation of A in H, let E be the closure in H of the vector subspace generated by the elements $U(s) \cdot x$ for $s \in A$ and $x \in H$, and let E′ be the set of all $x \in H$ such that $U(s) \cdot x = 0$ for all $s \in A$. Then E and E′ are stable with respect to U and are orthogonal supplements of each other in H.*

Since $U(st) = U(s)U(t)$, it is clear that E and E′ are stable subspaces of H. Let E″ be the orthogonal supplement of E in H. We have seen earlier that E″ is stable with respect to U. But if $x \in E''$, we have $U(s) \cdot x \in E$ by definition, hence $U(s) \cdot x \in E \cap E'' = \{0\}$ for all $s \in A$, so that $E'' \subset E'$. Conversely, if $x \in E'$, $s \in A$ and $y \in H$, we have $(x \mid U(s) \cdot y) = (U(s^*) \cdot x \mid y) = 0$ by definition, so that x is orthogonal to E, and therefore $E' \subset E''$.

The subspace E is called the *essential subspace* for U. If $E' = \{0\}$, the representation U is said to be *nondegenerate*. An equivalent condition is that the elements $U(s) \cdot x$ should form a *total* subspace of H; by (15.5.2), this is always the case if A has a unit element.

A vector $x_0 \in H$ is a *totalizer* or *totalizing vector* for a representation U of A in H if the vector subspace of H generated by the transforms $U(s) \cdot x_0$ of x_0, as s runs through A, is *dense* in H. (In any case, this subspace is *stable* with respect to U.) A representation which admits a totalizer is said to be *topologically cyclic*. If U is topologically irreducible, *every* nonzero vector $x_0 \in H$ is a totalizer, and conversely.

(15.5.6) *Suppose that A has a unit element. Let $s \mapsto U(s)$ be a nondegenerate representation of A in a separable Hilbert space H. Then H is the Hilbert sum of a sequence (H_n) (finite or infinite) of closed subspaces, stable with respect to U and such that the restriction of U to each H_n is topologically cyclic.*

Let $(x_n)_{n \geqq 1}$ be a dense sequence in H. We define the H_n inductively, as follows. H_1 is the closure of the vector subspace of H generated by the $U(s) \cdot x_1$, as s runs through A; we have $x_1 \in H_1$, because A has a unit element. If $H_1, \ldots, H_{n-1}$ have been constructed, it may be the case that H is equal to the (direct) sum L of the H_i, and the construction stops. If not, let L′ be the orthogonal supplement of L in H, and let $p(n)$ be the smallest integer such that, if x'_n is the orthogonal projection of $x_{p(n)}$ on L′, the vector subspace H'_n of L′ generated by the $U(s) \cdot x'_n$ is not equal to $\{0\}$: there exists such an index because the representation U is nondegenerate, and we take H_n to be the closure of H'_n. Since $x'_n \in H_n$, the subspaces H_n satisfy the required conditions, by virtue of (6.4.2).

(15.5.7) *If* A *is a Banach algebra with involution, having a unit element, then every representation* $s \mapsto U(s)$ *of* A *in a Hilbert space satisfies* $\|U(s)\| \leqq \|s\|$ (and in particular U is a *continuous* mapping of A into $\mathscr{L}(H)$).

We have $\|U(s)\|^2 = \|U(s)^* U(s)\| = \rho(U(s)^* U(s))$ in the star algebra $\mathscr{L}(H)$ (15.4.14.1). Since $U(s)^* U(s) = U(s^* s)$, it follows from (15.2.8(i)) that

$$\rho(U(s^* s)) \leqq \rho(s^* s) \leqq \|s^* s\| \leqq \|s^*\| \cdot \|s\| = \|s\|^2.$$

In particular, we recover in this way (15.3.1(ii)).

6. POSITIVE LINEAR FORMS, POSITIVE HILBERT FORMS AND REPRESENTATIONS

Let A be an algebra with involution (not necessarily normed, and not necessarily having a unit element). A linear form $f: A \to \mathbf{C}$ on A is said to be *positive* if

(15.6.1) $$f(x^* x) \geqq 0$$

for all $x \in A$.

(15.6.2) *Let f be a positive linear form on an algebra* A *with involution.*

(i) *The mapping* $(x, y) \mapsto g(x, y) = f(y^* x)$ *is a positive hermitian form on* $A \times A$: *in other words,*

(15.6.2.1) $$f(x^* y) = \overline{f(y^* x)}$$

for all x, y in A.

(ii) *For all* x, y *in* A *we have*

(15.6.2.2) $$|f(y^*x)|^2 \leqq f(x^*x)f(y^*y).$$

(iii) *If* A *has a unit element* e, *then*

(15.6.2.3) $$f(x^*) = \overline{f(x)},$$

(15.6.2.4) $$|f(x)|^2 \leqq f(e)f(x^*x).$$

To prove (i), express that

$$g(x + y, x + y) = g(x, x) + g(x, y) + g(y, x) + g(y, y)$$

is real: by virtue of the hypothesis (15.6.1), we obtain

$$\mathscr{I}(g(y, x)) = -\mathscr{I}(g(x, y));$$

changing x into ix, this becomes $\mathscr{R}(g(x, y)) = \mathscr{R}(g(y, x))$, hence $g(y, x) = \overline{g(x, y)}$. Assertion (ii) is the Cauchy–Schwarz inequality (6.2.1) applied to g. The relations (iii) are obtained by replacing y by e in (15.6.2.1) and (15.6.2.2), and using (15.4.2).

The hermitian form g so obtained from f is not arbitrary, because it clearly satisfies the relation

(15.6.3) $$g(xy, z) = g(y, x^*z)$$

for all x, y, z in A. We therefore make the following definition:

A *positive Hilbert form* on the algebra with involution A is a *positive hermitian form* on A × A which satisfies the relation (15.6.3).

If A has a unit element, every positive Hilbert form g comes as above from a positive linear form f: we have only to define $f(x) = g(x, e)$. But we shall see later that this is no longer the case when A has no unit element (15.7.4).

We shall now show that there are remarkable relationships between *Hilbert forms* on A × A and *representations* of A.

In the first place, every representation $s \mapsto U(s)$ of A in a Hilbert space H gives rise to *positive linear forms* (and hence to positive Hilbert forms) in the following way: for each $x_0 \in$ H, if we put

(15.6.4) $$f_{x_0}(s) = (u(s) \cdot x_0 \,|\, x_0)$$

then it is clear that f_{x_0} is a linear form on A, and we have

$$f_{x_0}(s^*s) = (U(s^*)U(s) \cdot x_0 \mid x_0) = (U(s) \cdot x_0 \mid U(s) \cdot x_0) = \|U(s) \cdot x_0\|^2 > 0$$

by virtue of (15.5.1). The corresponding Hilbert form is

$$g_{x_0}(s, t) = (u(s) \cdot x_0 \mid u(t) \cdot x_0). \tag{15.6.5}$$

For example, if $A = \mathscr{L}(H)$ where H is a Hilbert space of *finite* dimension n, and $1_A : T \mapsto T$ is the *identity* representation of A in H, then the form f_{x_0} is calculated explicitly as follows: if $(e_i)_{1 \leqq i \leqq n}$ is an orthonormal basis of H, and (ζ_{ij}) the matrix of T with respect to this basis, then for $x_0 = \sum_{i=1}^{n} \lambda_i e_i$ we have

$$f_{x_0}(T) = \sum_{i,j} \lambda_i \bar{\lambda}_j \zeta_{ij}. \tag{15.6.6}$$

When studying positive linear forms of the type (15.6.4) we may always assume that x_0 is a *totalizer* for U (15.5), because f_{x_0} is unchanged when we replace U by its restriction to the stable subspace of H which is the closure of the stable subspace generated by x_0 and the $U(s) \cdot x_0$ ($s \in A$). Under this additional assumption, the form f_{x_0} determines the representation U *up to equivalence*:

(15.6.7) *Let U, U' be two representations of A in Hilbert spaces* H, H', *respectively, and suppose that U (resp. U') has a totalizer x_0* (resp. x'_0). *Then if* $(U(s) \cdot x_0 \mid x_0) = (U'(s) \cdot x'_0 \mid x'_0)$ *for all $s \in A$, the representations U and U' are equivalent.*

For all s, t in A we have

$$\begin{aligned}(U(s) \cdot x_0 \mid U(t) \cdot x_0) &= (U(t^*s) \cdot x_0 \mid x_0) \\ &= (U'(t^*s) \cdot x'_0 \mid x'_0) \\ &= (U'(s) \cdot x'_0 \mid U'(t) \cdot x'_0).\end{aligned} \tag{15.6.7.1}$$

Since the vectors $U(t) \cdot x_0$ (resp. $U'(t) \cdot x'_0$) form a dense subspace H_0 (resp. H'_0) of H (resp. H'), this proves already that $U(s) \cdot x_0 = 0$ if and only if $U'(s) \cdot x'_0 = 0$. It follows that, for each $z \in H_0$ and each $s \in A$ such that $U(s) \cdot x_0 = z$, the vector $U'(s) \cdot x'_0$ is constant and equal to say $z' = T \cdot z \in H'_0$. By (15.6.7.1) the mapping T is an isomorphism of the prehilbert space H_0 onto the prehilbert space H'_0, which extends uniquely to an isomorphism (also denoted by T) of the Hilbert space H onto the Hilbert space H' (by virtue of (5.5.4) and the principle of extension of identities). It

remains to be shown that $T \cdot (U(s) \cdot z) = U'(s) \cdot (T \cdot z)$, and by the principle of extension of identities we may assume that z is of the form $U(t) \cdot x_0$. But then $U(s) \cdot (U(t) \cdot x_0) = U(st) \cdot x_0$, and therefore

$$T \cdot (U(st) \cdot x_0) = U'(st) \cdot x_0 = U'(s) \cdot (U'(t) \cdot x_0) = U'(s) \cdot (T \cdot z).$$

This completes the proof.

Furthermore, the prehilbert space H_0 *can be defined directly in terms of the form* f_{x_0}, and even in terms of the Hilbert form g_{x_0}. We have the following general proposition:

(15.6.8) *Let g be a positive Hilbert form on* A. *Then the set* $\mathfrak{n}$ *of elements* $s \in A$ *such that* $g(s, s) = 0$ *is a left ideal of* A, *and is equal to the set of* $s \in A$ *such that* $g(s, t) = 0$ for *all* $t \in A$. *If* $\pi : A \to A/\mathfrak{n}$ *is the canonical linear mapping, then there exists a unique structure of prehilbert space on* $A/\mathfrak{n}$ *such that* $(\pi(s) \mid \pi(t)) = g(s, t)$ *for all* $s, t \in A$.

The Cauchy–Schwarz inequality $|g(s, t)|^2 \leqq g(s, s)g(t, t)$ (6.2.1) shows that $\mathfrak{n}$ is the set of all $s \in A$ such that $g(s, t) = 0$ for *all* $t \in A$. The relation (15.6.3) then proves that $\mathfrak{n}$ is a left ideal in A. By virtue of the relation $g(t, s) = \overline{g(s, t)}$, we have $g(t, s) = 0$ for $s \in \mathfrak{n}$ and $t \in A$; it follows that if $s - s' \in \mathfrak{n}$ and $t - t' \in \mathfrak{n}$ (that is, if $\pi(s) = \pi(s')$ and $\pi(t) = \pi(t')$) then $g(s, t) = g(s', t')$, because $g(s, t) - g(s, t') = g(s, t - t') + g(s - s', t')$. Since $g(s, t)$ depends only on $\pi(s)$ and $\pi(t)$, we may write $g(s, t) = (\pi(s)|\pi(t))$, and it is straightforward to check that the function $(x \mid y)$ so defined on $(A/\mathfrak{n}) \times (A/\mathfrak{n})$ is a non-degenerate positive hermitian form.

In the case where $g = g_{x_0}$, we have $g_{x_0}(s, s) = \| U(s) \cdot x_0 \|^2$, hence $\mathfrak{n}$ is the kernel of the linear mapping $u : s \mapsto U(s) \cdot x_0$ of A onto H_0. The formula (15.6.5) shows that the bijective mapping $S : A/\mathfrak{n} \to H_0$ induced by u is a *prehilbert space isomorphism*. This shows, as stated earlier, that the Hilbert space H is *determined* by g_{x_0} up to isomorphism. Further, the representation $s \mapsto U(s)$ can be reconstructed from the algebra structure of A and the form g_{x_0}, because

$$U(s) \cdot (U(t) \cdot x_0) = U(s) \cdot (S \cdot \pi(t)) = S \cdot \pi(st);$$

and once $U(s)$ is known in H_0, it extends uniquely by continuity to H (5.5.4).

Nevertheless, positive Hilbert forms of the type g_{x_0} *are not the most general ones*, for they satisfy an additional condition which expresses that each

$U(s)$ is a *continuous* operator on H_0. In view of (15.6.5) and (5.5.1), this continuity is equivalent to the following condition on the form $g = g_{x_0}$:

(U) *For each* $s \in A$ *there exists a real number* $M_s \geqq 0$ *such that*

$$g(st, st) \leqq M_s g(t, t) \tag{15.6.9}$$

for all $t \in A$.

Conversely:

(15.6.10) *Let* g *be a positive Hilbert form on* A, *satisfying the condition* (U), *and suppose that (with the notation of* (15.6.8)) *the prehilbert space* $A/\mathfrak{n}$ *is separable, so that* (6.6.2) *it can be identified with a dense vector subspace of a Hilbert space* H. *Then for each* $s \in A$ *the endomorphism* $\pi(t) \mapsto \pi(st)$ *of* $A/\mathfrak{n}$ *extends to a continuous endomorphism* $x \mapsto V(s) \cdot x$ *of* H, *and* $s \mapsto V(s)$ *is a representation of* A *in* H.

If $\pi(t) = \pi(t')$, then $\pi(st) = \pi(st)'$) because $\mathfrak{n}$ is a left ideal. Hence the endomorphism of H_0 under consideration is well-defined, and the definition of the scalar product on $A/\mathfrak{n}$, together with (15.6.9), shows that this endomorphism is *continuous* (5.5.1). Hence the existence of the continuous operator $V(s)$ (5.5.4). Since $\pi((ss')t) = \pi(s(s't))$, we have $V(ss') = V(s)V(s')$. Also, by virtue of (15.6.3)

$$\begin{aligned}(V(s^*) \cdot \pi(t) \mid \pi(t')) &= g(s^*t, t') = g(t, st') \\ &= \overline{g(st', t)} = \overline{(V(s) \cdot \pi(t') \mid \pi(t))} = (\pi(t) \mid V(s) \cdot \pi(t'))\end{aligned}$$

which shows that $V(s^*) = (V(s))^*$. Finally, if A has a unit element e, then evidently $V(e) = 1_H$, and therefore $s \mapsto V(s)$ is a representation of A in H.

It is useful to know when the representation $s \mapsto V(s)$ so defined is non-degenerate (15.5.5). This is equivalent to the following condition on g:

(N) *The elements* $\pi(st)$ *form a total set in the prehilbert space* $A/\mathfrak{n}$.

This condition is trivially satisfied when A has a unit element.

Under the conditions of (15.6.10), there does not in general exist an element $x_0 \in H$ such that $g(s, t) = (V(s) \cdot x_0 \mid V(t) \cdot x_0)$ (cf. Section 15.9, Problem 3). However, such a vector always exists when A has a unit element e: we may take $x_0 = \pi(e)$.

When A has a unit element and is a *Banach algebra* with involution, not only does the theory of positive Hilbert forms reduce to that of positive linear forms, and the condition (N) is automatically satisfied, but also the condition (U) is satisfied. Precisely,

(15.6.11) *Let* A *be a Banach algebra with involution, having a unit element* $e \neq 0$. *Let* f *be a positive linear form on* A. *Then*

(i) f *is continuous and* $\|f\| = f(e)$.

(ii) $|f(y^*xy)| \leqq \|x\| f(y^*y)$.

We shall use the following lemma:

(15.6.11.1) *If* $x \in A$ *is self-adjoint and* $\|x\| < 1$, *then there exists a self-adjoint* $y \in A$ *such that* $y^2 = e + x$.

If t is real and $|t| < 1$, Taylor's formula (8.14.3) gives

$$(1+t)^{1/2} = 1 + \tfrac{1}{2}t + \binom{\frac{1}{2}}{2}t^2 + \cdots + \binom{\frac{1}{2}}{n}t^n + r_n(t),$$

where

$$r_n(t) = \frac{(-1)^n}{2} \cdot \frac{1 \cdot 3 \cdot \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \int_0^t \left(\frac{t-s}{1+s}\right)^n \frac{ds}{(1+s)^{1/2}}.$$

This leads immediately to the estimate

$$|r_n(t)| \leqq |t|^n$$

and therefore we have

$$(15.6.11.2) \qquad (1+t)^{1/2} = 1 + \tfrac{1}{2}t + \binom{\frac{1}{2}}{2}t^2 + \cdots + \binom{\frac{1}{2}}{n}t^n + \cdots,$$

the series on the right being absolutely convergent for $|t| < 1$ (9.1.2). It follows that in the Banach algebra A the series

$$e + \tfrac{1}{2}x + \binom{\frac{1}{2}}{2}x^2 + \cdots + \binom{\frac{1}{2}}{n}x^n + \cdots$$

converges absolutely for $\|x\| < 1$, and its sum y is self-adjoint when x is self-adjoint. Furthermore, since the square of the power series on the right-hand side of (15.6.11.2) is $1 + t$, it follows that $y^2 = e + x$, by virtue of (5.5.3).

This lemma shows that, if $x \in A$ is self-adjoint and $\|x\| < 1$, there exists $y \in A$ such that $y^*y = e - x$, so that $f(e - x) \geqq 0$, or equivalently $f(x) \leqq f(e)$. If now $z \in A$ is such that $\|z\| < 1$, then also $\|z^*z\| < 1$ and therefore, using

(15.6.2.4), we have $|f(z)|^2 \leqq f(e)f(z^*z) \leqq f(e)^2$. This proves that f is continuous and that $\|f\| \leqq f(e)$ (5.7.1). Since also $\|e\| = 1$, we have $f(e) \leqq \|f\|$, and assertion (i) is proved.

To prove (ii), observe that for $y \in A$ the linear form $x \mapsto f(y^*xy)$ on A is positive, because $f(y^*x^*xy) = f((xy)^*(xy)) \geqq 0$; by virtue of (i), the norm of this linear form is $f(y^*y)$. This completes the proof.

PROBLEMS

1. Let A be an algebra with involution and let U be a representation of A on a Hilbert space H. In order that U should be irreducible, it is necessary and sufficient that the subalgebra B of $\mathscr{L}(H)$ consisting of the endomorphisms V such that $VU(s) = U(s)V$ for all $s \in A$ should be equal to $\mathbf{C} \cdot 1_H$. (To show that the condition is necessary, observe that B is an involutive closed subalgebra of $\mathscr{L}(H)$, hence is a star algebra. If S is a hermitian operator belonging to B, consider the closed commutative subalgebra C of B generated by S, which is separable and therefore isomorphic to $\mathscr{C}_C(X)$ for some compact metrizable space X. Show that C has no divisors of zero, other than 0, and conclude that X consists of a single point.)

2. Let A be a Banach algebra with unit element e, and let $x \mapsto x^*$ be a (not necessarily continuous) involution on A.

(a) Let f be a linear form on A such that $f(a^2) \geqq 0$ for all self-adjoint elements $a \in A$. Show that if also $p(a) < 1$, then $f(a)$ is real and $|f(a)| \leqq f(e)$ (use Section 15.4, Problem 17). Show that, for each self-adjoint element $a \in A$, $f(a)$ is real, $|f(a)| \leqq f(e)\rho(a)$ and $f(a)^2 \leqq f(e)f(a^2)$. (Consider $f((a + \xi e)^2)$ where $\xi \in \mathbf{R}$.) Deduce that $|f(x)| \leqq f(e)p(x)$, in the notation of Section 15.4, Problem 18, and *a fortiori* that f is *continuous* on A. Furthermore, we have $|f(y^*xy)| \leqq \rho(x)f(y^*y)$.

(b) Conversely, if f is a linear form on A such that $|f(x)| \leqq f(e)p(x)$ for all $x \in A$, show that for each self-adjoint element $a \in A$ we have $f(a)$ real and $|f(a)| \leqq f(e)\rho(a)$ (consider $f(a + i\xi e)$, where $\xi \in \mathbf{R}$).

(c) Suppose that the involution $x \mapsto x^*$ is *hermitian* (Section 15.4, Problem 18). Show that the following conditions are equivalent, for a linear form f on A:

(α) f is a positive linear form.

(β) For every self-adjoint element $a \in A$, $f(a)$ is real and $|f(a)| \leqq f(e)\rho(a)$.

(γ) For every self-adjoint element $a \in A$, $f(a^2) \geqq 0$.

(δ) For each $x \in A$, $|f(x)| \leqq f(e)\rho(x)$.

(Use (a) and (b). To show that (β) implies (α), observe that if $\mathrm{Sp}(a)$ is contained in an interval $[\lambda, \mu] \subset \mathbf{R}$ and if $f(e) \neq 0$, then $f(e) > 0$ and $f(a) \in [f(e)\lambda, f(e)\mu]$.)

3. Let A be a separable Banach algebra with unit element e, and let $x \mapsto x^*$ be a (not necessarily continuous) involution on A.

(a) Show that if A is hermitian, then for each $x \in A$ there exists a positive linear form f on A such that $f(e) = 1$ and $f(x^*x)^{1/2} = p(x)$. (Consider the commutative Banach subalgebra C generated by x^*x, and a commutative Banach subalgebra $B \supset C$ such that $\mathrm{Sp}_B(y) = \mathrm{Sp}_A(y)$ for all $y \in C$. For each character χ of B and each

element y of C, we have $\chi(y^*) = \overline{\chi(y)}$, and there exists a character χ such that $\rho(x^*x) = \chi(x^*x)$. If g is the restriction of χ to C, we have $|g(y)| \leqq p(y)$ for all $y \in C$. Now use Section 15.4, Problem 18(e); Section 12.15, Problem 4; and Problem 2(c) above).

(b) Conversely, if for each $x \in A$ there exists a positive linear form f on A such that $f(e) = 1$ and $f(x^*x)^{1/2} = p(x)$, then A is hermitian. (If $\alpha = \rho(x^*x)$ and $u = \alpha e - x^*x$, show that $f(u^2) \leqq \alpha^2$ for all positive linear forms f on A such that $f(e) = 1$, and deduce from the hypothesis that $\rho(u^2) \leqq \alpha^2$, so that $\rho(\alpha e - x^*x) \leqq \alpha$; then use Section 15.4, Problem 19.)

(c) Suppose that A is hermitian, and let S(A) be the subspace of the dual A' of the Banach space A, consisting of the positive linear forms f on A such that $f(e) = 1$. Then S(A) is compact and metrizable for the weak topology (12.15.9). Let (f_n) be a dense sequence in S(A), and let $s \mapsto U_n(s)$ be the representation of A on a separable Hilbert space H_n constructed from f_n by the procedure of (15.6.10). We obtain a representation U of A on the Hilbert space H which is the Hilbert sum of the H_n (6.4) by putting $U(s) \cdot \sum_n x_n = \sum_n U_n(s) \cdot x_n$. Show that $\|U(s)\| = p(s)$, and hence that the kernel of U is the radical $\mathfrak{R}$ of A, and U is continuous.

(d) A star algebra A with unit element is hermitian (15.4.12), and for each $x \in A$ we have $\|x\| = p(x)$ (Section 15.4, Problem 4(a)). If A is separable, deduce that there exists an isometric isomorphism U of A onto a closed involutive subalgebra of $\mathscr{L}(E)$ (where E is a separable Hilbert space), such that $U(s^*) = (U(s))^*$ for all $s \in A$.

(e) If E is a separable Hilbert space, the algebra $A = \mathbf{C} \cdot 1_E \oplus \mathscr{L}_2(E)$ is a hermitian Banach algebra relative to the norm $\|\xi \cdot 1_E + u\|_2 = |\xi| + \|u\|_2$, in the notation of (15.4.8). Compare $\|u\|_2$ and $p(u)$ for $u \in \mathscr{L}_2(E)$.

4. Let A be a Banach algebra with unit element, and let $x \mapsto x^*$ be a (not necessarily continuous) involution on A. Show that the following properties are equivalent:

- (α) There exists a norm on A, equivalent to the given norm, relative to which A is a star algebra.
- (β) There exists a number $\lambda > 0$ such that $\|x^*x\| \geqq \lambda \|x^*\| \cdot \|x\|$ for all $x \in A$.
- (γ) The set of unitary elements of A is bounded.
- (δ) The involution $x \mapsto x^*$ is hermitian, and there exists a number $\mu > 0$ such that $\|a\| \leqq \mu \cdot \rho(a)$ for all hermitian elements $a \in A$.

(To show that (δ) implies (α), show that (δ) implies that $\|x\| \leqq 2\mu p(x)$, by writing $x = a + ib$ with a, b hermitian, and using Section 15.4, Problem 18(d) and 18(g). To show that (β) implies (δ), show first that if a is hermitian, then $\lambda \|a\| \leqq \rho(a)$, by using (15.2.7); deduce that if x is normal, then $\lambda^2 \|x^n\| \cdot \|(x^*)^n\| \leqslant \rho(x^*x)^n$ for all integers $n > 1$, and use Section 15.4, Problem 19 to show that A is hermitian. Finally, to show that (γ) implies (δ), observe that for each hermitian element a such that $\rho(a) < 1$ there exists in A a hermitian element b, commuting with a, such that $b^2 = e - a^2$ (Section 15.4, Problem 17); consequently $u = a + ib$ is unitary, and $a = \frac{1}{2}(u + u^*)$.)

5. Let A be a Banach algebra with unit element e, endowed with a *hermitian* involution $x \mapsto x^*$.

(a) Let $x \in A$ be such that $p(x) < 1$. Show that $x(e - x^*x)^{1/2} = (e - xx^*)^{1/2}x$ (cf. Section 15.2, Problem 11(f)).

(b) Under the same hypotheses, show that the function

$$F(\zeta) = (e - xx^*)^{-1/2}(\zeta e + x)(e + \zeta x^*)^{-1}(e - x^*x)^{1/2}$$

is defined and holomorphic in an open disk $|\zeta| < 1 + \varepsilon$, for some $\varepsilon > 0$, and that $F(\zeta)$ is unitary when $|\zeta| = 1$. (If $|\zeta| = 1$, and if we put $a = e - x^*x$, $b = e - xx^*$, $y = \zeta e + x$, we have $e + \zeta x^* = \zeta y^*$, $yy^{*-1} = \zeta(x + by^{*-1})$, $y^{*-1}y = \zeta(x + y^{*-1}a)$, and finally $yy^{*-1}a = by^{*-1}y$.)

(c) Show that the set of elements $x \in A$ such that $p(x) \leqq 1$ is the closed convex hull (Section 12.14, Problem 13) of the set of unitary elements of A. (Use the relation

$$x = F(0) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{it})\,dt.)$$

6. Let A be a Banach algebra with unit element, endowed with a (not necessarily continuous) involution $x \mapsto x^*$. Show that the following properties are equivalent:

(α) A is a star algebra.

(β) $\|x\| \leqq p(x)$ for all $x \in A$.

(γ) $\|x^*\| \cdot \|x\| = \|x^*x\|$ for all normal $x \in A$.

(δ) $\|x\| = 1$ for all unitary $x \in A$.

(To prove that (δ) implies (β), observe first that (δ) implies that A is a star algebra relative to some norm equivalent to the given norm (Problem 4), and in particular that A is hermitian; then apply Problem 5(c).)

7. TRACES, BITRACES, AND HILBERT ALGEBRAS

Let A be an algebra with involution and f a positive linear form on A. In general $f(xy) \neq f(yx)$ for $x, y \in A$, as example (15.6.6) shows.

We say that f is a *trace* on A if f is a *positive linear form satisfying the condition*

(15.7.1) $$f(yx) = f(xy)$$

for all x, y in A.

For f to be a trace it is sufficient that $f(xx^*) = f(x^*x)$ for all $x \in A$; for, replacing x by $x + y$ and using (15.6.2.1), we obtain $\mathscr{R}(f(y^*x)) = \mathscr{R}(f(xy^*))$, and then, replacing x by ix we get $\mathscr{I}(f(y^*x)) = \mathscr{I}(f(xy^*))$, so that $f(y^*x) = f(xy^*)$ for all x, y in A.

The condition (15.7.1) is trivially satisfied if A is *commutative*.

Example

(15.7.2) Let $A = \mathscr{L}(H)$, where H is a Hilbert space of *finite* dimension n. Then the positive linear form $f(T) = \sum_{i=1}^{n} (T \cdot e_i \mid e_i)$ is a trace, for it is the trace $\mathrm{Tr}(T) = \sum_{i=1}^{n} \zeta_{ii}$ of the endomorphism T. On the other hand, it is easily seen

that there exists no trace on $\mathscr{L}(H)$ if H is an *infinite*-dimensional Hilbert space (Problem at the end of this section).

The corresponding notion for positive Hilbert forms is that of a *bitrace*. A *bitrace* on A is by definition a positive Hilbert form g on A which satisfies the additional condition

(15.7.3) $$g(y^*, x) = g(x, y).$$

This reduces to (15.7.1) when $g(x, y) = f(y^*x)$, f being a positive linear form.

Example

(15.7.4) Consider the algebra $\mathscr{L}_2(E)$ of Hilbert–Schmidt operators on an infinite-dimensional separable Hilbert space (15.4.8). For any Hilbert basis (a_n) of E and any pair (u, v) of Hilbert–Schmidt operators on E, the sum $\sum_n (u(a_n) \mid v(a_n))$ is defined, because $(u(a_n) \mid v(a_n)) = ((v^*u)(a_n) \mid a_n)$, and we have seen that the double family $((v^*u)(a_n) \mid a_m)$ is absolutely summable (15.4.8), since v^*u is a Hilbert–Schmidt operator. Moreover, for each Hilbert basis (b_n) of E, we have

$$\begin{aligned}(u(a_n) \mid v(a_n)) &= \sum_m (u(a_n) \mid b_m)(b_m \mid v(a_n)) \\ &= \sum_m (a_n \mid u^*(b_m))(v^*(b_m) \mid a_n)\end{aligned}$$

and therefore

$$\sum_n (u(a_n) \mid v(a_n)) = \sum_m (v^*(b_m) \mid u^*(b_m)).$$

This shows that the sum $\sum_n (u(a_n) \mid v(a_n))$ does not depend on the orthonormal basis (a_n) originally chosen. If we denote this sum by $g(u, v)$, then it is immediately seen that g is a positive hermitian form on $\mathscr{L}_2(E)$, such that $g(u, u) = \|u\|_2^2$. Also, if u, v, w are three Hilbert–Schmidt operators, then $(u(v(a_n)) \mid w(a_n)) = (v(a_n) \mid u^*(w(a_n)))$, hence $g(uv, w) = g(v, u^*w)$; and the calculation above shows that $g(v^*, u^*) = g(u, v)$. Hence g is a *bitrace* on $\mathscr{L}_2(E)$. It can be shown that it *does not come* from a trace on this algebra (Problem). Note also that g satisfies conditions (U) and (N) of (15.6). For (U), this follows from the inequality (15.4.8.2); for (N), observe that the endomorphisms of finite rank are dense in $\mathscr{L}_2(E)$ (15.4.8) and that if $u(E) = F$ is finite-dimensional then we may write $u = P_F \circ u$ (in the notation of (6.3.1)) where P_F is also of finite rank.

Remark

(15.7.4.1) If E is a Hilbert space of *finite* dimension n, we can define a scalar product $(u \mid v)$ on $\mathscr{L}(E) = \mathrm{End}_{\mathbf{C}}(E)$ as in (15.7.4). As in (15.4.8) we see that, if we put $\|u\|_2^2 = (u \mid u)$, then $\|u\|_2$ is a norm on $\mathscr{L}(E)$ such that $\|uv\|_2 \leqq \|u\|_2 \cdot \|v\|_2$; it is necessarily equivalent to the usual norm $\|u\|$ (5.9.1), but does not satisfy (15.1.2) unless $n = 1$, because $\|1_E\|_2 = n^{1/2}$.

(15.7.5) Let A be an *arbitrary* algebra with involution, and let g be a bitrace on A. Since $g(s^*, s^*) = g(s, s)$, the left ideal $\mathfrak{n}$ defined in (15.6.8) is *self-adjoint* and is therefore a *two-sided* ideal. The vector space $A/\mathfrak{n}$ is therefore naturally an *algebra.* Since $\mathfrak{n}$ is self-adjoint, the mapping $s \mapsto s^*$ induces an *involution* on $A/\mathfrak{n}$ such that $\pi(s^*) = (\pi(s))^*$; also the scalar product $(x \mid y)$ on $A/\mathfrak{n}$ induced by g is a *bitrace* on $A/\mathfrak{n}$; finally, if g satisfies the condition (U) (resp. (N)), then the same is true of the scalar product on $A/\mathfrak{n}$.

A *Hilbert algebra* is defined to be an algebra A with involution, endowed with the structure of a *prehilbert* space defined by a bitrace $(x \mid y)$ satisfying the conditions (U) and (N). In other words, this scalar product satisfies the following conditions:

(15.7.5.1) $$(y^* \mid x^*) = (x \mid y)$$

(15.7.5.2) $$(xy \mid z) = (y \mid x^*z)$$

(15.7.5.3) *For each* $x \in A$ *there exists a real number* $M_x \geqq 0$ *such that*

$$(xy \mid xy) \leqq M_x(y \mid y).$$

(15.7.5.4) *The elements of* A *of the form* xy, *with* $x \in A$ *and* $y \in A$, *form a total set in* A.

Note that it follows from (15.7.5.1) and (15.7.5.2) that

(15.7.5.5) $$(yx \mid z) = (y \mid zx^*).$$

For $(yx \mid z) = \overline{(x^*y^* \mid z^*)} = \overline{(y^* \mid xz^*)} = (y \mid zx^*)$.

Furthermore, we have

(15.7.5.6) $$(yx \mid yx) \leqq M_{x^*}(y \mid y)$$

because $(yx \mid yx) = (x^*y^* \mid x^*y^*) \leqq M_{x^*}(y^* \mid y^*) = M_{x^*}(y \mid y)$.

The inequalities (15.7.5.3) and (15.7.5.6) express that the linear mappings $y \mapsto xy$ and $x \mapsto xy$ are continuous on A with respect to its topology as a prehilbert space. But it should be carefully noted that in general A *is not a normable algebra* for this topology (15.1.8): that is to say, in general the mapping $(x, y) \mapsto xy$ *is not continuous.*

Finally, we have

(15.7.5.7) *The relation* $x^*x = 0$ *for* $x \in A$ *implies that* $x = 0$.

For it follows from (15.7.5.2) that $(xy \mid xy) = (x^*xy \mid y) = 0$, hence $xy = 0$ for all $y \in A$. Hence, by (15.7.5.5), $(x \mid yz) = (xz^* \mid y) = 0$ for all y and all z, and by (15.7.5.4) this implies that $x = 0$.

In particular, the right (resp. left) *annihilator* of A consists only of 0, and for each $x \neq 0$ in A the left ideal Ax is nonzero.

In the next two sections we shall determine the structure of two of the most important types of Hilbert algebras.

PROBLEM

(a) Show that on the algebra $A = M_n(\mathbf{C})$ of all $n \times n$ matrices with complex entries, the only linear forms f such that $f(XY) = f(YX)$ for all pairs of matrices X, Y are scalar multiples $X \mapsto \rho \mathrm{Tr}(X)$ of the trace.

(b) Deduce that, if H is an infinite-dimensional separable Hilbert space, there exists no trace on the algebra $\mathscr{L}(H)$ or on the algebra $\mathscr{L}_2(H)$. (Consider a compact operator U such that $U(e_n) = \lambda_n e_n$, where the e_n form an orthonormal basis of H and the λ_n are real numbers > 0 such that $\sum_n \lambda_n^2$ converges but $\sum_n \lambda_n$ does not. Use (a) by considering the restriction of a hypothetical trace f to the algebra of endomorphisms of the vector subspace of H generated by the e_k with $k \leqq n$.)

8. COMPLETE HILBERT ALGEBRAS

Throughout this section we shall assume that the Hilbert algebra A is *complete* (and thus a Hilbert space) with respect to the norm $\|x\| = (x \mid x)^{1/2}$. We shall also assume that the bilinear mapping $(x, y) \mapsto xy$ of $A \times A$ into A is *continuous* with respect to this norm (it can be shown (Section 12.16, Problem 8c)) that this is in fact a consequence of the other hypotheses). Every *closed self-adjoint* subalgebra B of A is a complete Hilbert algebra.

(15.8.1) *Let* $\mathfrak{b}$ *be a closed left ideal in* A, *and put* $U_{\mathfrak{b}}(x) \cdot y = xy$ *for all* $y \in \mathfrak{b}$ *and all* $x \in A$. *Then* $x \mapsto U_{\mathfrak{b}}(x)$ *is a representation of the algebra* A *on the Hilbert space* $\mathfrak{b}$.

First of all, $U_{\mathfrak{b}}(x)$ is a continuous operator on $\mathfrak{b}$, by (15.7.5.3). Clearly $U_{\mathfrak{b}}(xx') = U_{\mathfrak{b}}(x)U_{\mathfrak{b}}(x')$. For y, z in $\mathfrak{b}$, we have

$$(U_{\mathfrak{b}}(x)^* \cdot y \mid z) = (y \mid U_{\mathfrak{b}}(x) \cdot z) = (y \mid xz) = (x^*y \mid z) = (U_{\mathfrak{b}}(x^*) \cdot y \mid z)$$

by virtue of (15.7.5.2), hence $U_{\mathfrak{b}}(x)^* = U_{\mathfrak{b}}(x^*)$. Also if A has a unit element e, then $U_{\mathfrak{b}}(e)$ is the identity transformation.

When $\mathfrak{b} = A$, we write $U(x)$ in place of $U_A(x)$; the representation U is called the *regular representation* of A. It is *faithful* (15.5) by virtue of (15.7.5.7). Moreover, $x \mapsto U(x)$ is a *continuous* linear mapping of A into $\mathscr{L}(A)$, by the continuity of $(x, y) \mapsto xy$.

The study of complete Hilbert algebras is founded on the consideration of their *minimal left ideals* and the *idempotents* which generate them.

For each left ideal $\mathfrak{l}$ of A, we denote by $\mathfrak{l}^*$ the image of $\mathfrak{l}$ under the involution $s \mapsto s^*$; clearly $\mathfrak{l}^*$ is a right ideal.

(15.8.2) *For each left ideal* $\mathfrak{l}$ *of* A, *the orthogonal supplement* $\mathfrak{l}^\perp$ *of the closure* $\bar{\mathfrak{l}}$ *of* $\mathfrak{l}$ *is a left ideal.*

Since $\bar{\mathfrak{l}}$ is a left ideal (15.1.3), we may as well assume that $\mathfrak{l}$ is closed. If $y \in \mathfrak{l}^\perp$ and $z \in A$, then for each $x \in \mathfrak{l}$ we have $(zy \mid x) = (y \mid z^*x) = 0$ because $\mathfrak{l}$ is a left ideal, whence it follows that $zy \in \mathfrak{l}^\perp$.

(15.8.3) *Let* $e \neq 0$ *be an idempotent in* A. *Then*

(i) $\|e\| \geqq 1$;

(ii) e^* *is idempotent*;

(iii) *the left ideal* Ae *is the set of all* $x \in A$ *such that* $x = xe$, *and is closed in* A.

The first assertion follows from the inequality $\|e\| = \|e^2\| \leqq \|e\|^2$. The second is trivial. As to (iii), it is clear that if $x = xe$ then $x \in Ae$, and conversely if $x \in Ae$, then $x = ye$ for some $y \in A$, hence $xe = ye^2 = ye = x$. The fact that Ae is closed then follows from the continuity of the mapping $x \mapsto x - xe$, and (3.15.1).

Consider in particular the *self-adjoint* idempotents (cf. (15.5.3.1)):

(15.8.4) *If* e_1, e_2 *are self-adjoint idempotents in* A, *then the following properties are equivalent*: (a) $(e_1 \mid e_2) = 0$; (b) $e_1e_2 = 0$; (c) $e_2e_1 = 0$.

Since $(e_1e_2)^* = e_2^*e_1^* = e_2e_1$, it is clear that (b) and (c) are equivalent. If $(e_1 \mid e_2) = 0$ then by (15.7.5.2) and (15.7.5.5) we have $0 = (e_1^2 \mid e_2^2) = (e_1e_2 \mid e_1e_2) = \|e_1e_2\|^2$, so that (a) implies (b). Conversely, if $e_1e_2 = 0$, then $(e_1 \mid e_2) = (e_1^2 \mid e_2) = (e_1 \mid e_1e_2) = 0$, and therefore (b) implies (a).

(15.8.5) *Every left ideal* $\mathfrak{l} \neq \{0\}$ *in* A *contains a self-adjoint idempotent* $\neq 0$.

Let x be a nonzero element of $\mathfrak{l}$. Then $z = x^*x \neq 0$ (15.7.5.7), and z is a self-adjoint element of $\mathfrak{l}$. Multiplying z by a suitable nonzero real scalar, we may assume that $\|U(z)\| = 1$, where U is the regular representation of A (15.8.1). Hence, by (15.8.1) and (11.5.3), $\|U(z^2)\| = 1$, and therefore $\|U(z^{2^n})\| = 1$ by induction on n. On the other hand, we have

$$\|U(z^{k+1})\| = \|U(z)U(z^k)\| \leqq \|U(z)\| \cdot \|U(z^k)\| = \|U(z^k)\|$$

for all k, hence the sequence $(\|U(z^k)\|)$ is decreasing. Since it has infinitely many terms equal to 1, it follows that $\|U(z^k)\| = 1$ for all integers k, and hence that $\|z^k\| \geqq 1/\|U\|$ for all k. We shall show that the sequence (z^{2^k}) is a *Cauchy sequence* in the Hilbert space A. Let n, p be two integers >0, and put $m = n + p$. Then

$$(z^{2m} \mid z^{2n}) = (z^{2p}z^{2n} \mid z^{2n}) = (z^{p+2n} \mid z^{p+2n}) = \|U(z^p) \cdot z^{2n}\|^2 \leqq (z^{2n} \mid z^{2n})$$

and

$$(z^{2m} \mid z^{2m}) = \|U(z^p) \cdot z^{p+2n}\|^2 \leqq (z^{p+2n} \mid z^{p+2n}) = (z^{2m} \mid z^{2n})$$

so that, for all $m > n$,

$$1/\|U\|^2 \leqq (z^{2m} \mid z^{2m}) \leqq (z^{2m} \mid z^{2n}) \leqq (z^{2n} \mid z^{2n}).$$

This shows that the sequence $(\|z^{2^n}\|^2)$ is decreasing and has a limit $a > 0$; moreover, we have

$$\|z^{2m} - z^{2n}\|^2 = (z^{2m} \mid z^{2m}) - 2(z^{2m} \mid z^{2n}) + (z^{2n} \mid z^{2n}) \leqq \|z^{2n}\|^2 - a$$

which proves that (z^{2^k}) is a Cauchy sequence, as asserted. Hence the sequence (z^{2^k}) has a limit e. By continuity, $e^2 = \lim_{k\to\infty}(z^{4k}) = e$, and $e^* = \lim_{k\to\infty}(z^*)^{2k} = e$ because z is self-adjoint, and finally $ez^2 = \lim_{k\to\infty} z^{2k+2} = e$, so that $e \in \mathfrak{l}$. Finally, since $\|z^k\| \geqq 1/\|U\|$ for all $k \geqq 1$, we have $\|e\| > 0$, and the proof is complete.

A self-adjoint idempotent $e \neq 0$ is said to be *reducible* if there exist two *orthogonal* self-adjoint idempotents e_1, e_2, each nonzero, and such that $e = e_1 + e_2$. In this case, by (15.8.4), we have $ee_1 = e_1e = e_1$ and $ee_2 = e_2e = e_2$. If $e \neq 0$ is not reducible, it is said to be *irreducible*.

(15.8.6) (i) *Every self-adjoint idempotent $e \neq 0$ is the sum of a finite number of irreducible self-adjoint idempotents belonging to* Ae.

(ii) *Every left ideal* $\mathrm{I} \neq \{0\}$ *contains an irreducible self-adjoint idempotent.*

Clearly (ii) is a consequence of (i) and (15.8.5). Hence it is enough to prove (i). If $\|e\|^2 < 2$, then e must be irreducible, for otherwise we should have $\|e\|^2 = \|e_1\|^2 + \|e_2\|^2$, where e_1 and e_2 are self-adjoint idempotents $\neq 0$, hence $\|e\|^2 \geqq 2$ by (15.8.3). We shall proceed by induction on the smallest integer n such that $\|e\|^2 < n$. If $e = e_1 + e_2$ is reducible, where e_1 and e_2 are orthogonal self-adjoint idempotents $\neq 0$, then $e_1 = e_1 e$ and $e_2 = e_2 e$, so that e_1 and e_2 lie in Ae; moreover, we have

$$\|e_1\|^2 = \|e\|^2 - \|e_2\|^2 \leqq \|e\|^2 - 1 < n - 1$$

and similarly $\|e_2\|^2 < n - 1$, so that the inductive hypothesis can be applied to e_1 and e_2. This completes the proof.

A left ideal I in A is said to be *minimal* if $\mathrm{I} \neq \{0\}$ and if there exists no nonzero left ideal $\mathrm{I}' \neq \mathrm{I}$ contained in I. Similarly for minimal right ideals.

(15.8.7) *A left ideal* I *in* A *is minimal if and only if it is of the form* $\mathrm{I} = \mathrm{A}e$, *where e is an irreducible self-adjoint idempotent* $\neq 0$.

If I is minimal, it contains an irreducible self-adjoint idempotent $e \neq 0$ (15.8.6). We have $\mathrm{A}e \subset \mathrm{I}$, and $e = e^2 \in \mathrm{A}e$, hence $\mathrm{A}e = \mathrm{I}$. Conversely, let e be an irreducible self-adjoint idempotent $\neq 0$, and let $\mathrm{I} = \mathrm{A}e$. Suppose that I contains a left ideal I′ distinct from $\{0\}$ and I. Let e' be a self-adjoint idempotent $\neq 0$ contained in I′ (15.8.5). Then we can write $e = e_1 + e_2$, where $e_2 = ee'$ and $e_1 = e - ee'$. We shall show that e_1 and e_2 are *orthogonal self-adjoint idempotents*. Since $e' \in \mathrm{A}e$, we have $e' = e'e$ (15.8.3), hence

$$e_2^2 = ee'ee' = ee'^2 = ee' = e_2, \qquad ee_2 = e_2, \qquad e_2 e = ee'e = ee' = e_2,$$

and therefore

$$e_1 e_2 = (e - e_2)e_2 = 0, \qquad e_2 e_1 = e_2(e - e_2) = 0,$$
$$e_1^2 = (e - e_2)^2 = e - e_2 = e_1;$$

finally $e_2^* = (ee'e)^* = ee'e = e_2$, and hence $e_1 = e - e_2$ is also self-adjoint. We shall obtain a contradiction if we can show that e_1 and e_2 are both nonzero. If $e_1 = 0$, it follows that $e = ee' \in \mathrm{I}'$, hence $\mathrm{I} = \mathrm{I}'$, contrary to hypothesis. On the other hand, $e'e_2 = e'ee' = e'^2 = e' \neq 0$, and therefore $e_2 \neq 0$. This completes the proof.

view of (15.8.6) and (15.8.3), we obtain the following corollary:

3) *Every left ideal in* A *contains a minimal left ideal. Every minimal left s closed.*

9) (i) *If e, e' are two orthogonal self-adjoint idempotents, then the left* Ae *and* Ae' *are orthogonal.*

Let $(e_i)_{1 \leqq i \leqq n}$ *be a finite family of pairwise orthogonal, self-adjoint otents. Then for each* $x \in A$, *the element* $x - \sum_{i=1}^{n} xe_i$ *is orthogonal to each* $\leqq j \leqq n$).

We have $(xe \mid ye') = (xe \mid ye'^2) = (xee' \mid ye') = 0$, by (15.7.5.5) and 4).

Likewise,

$$\left(x - \sum_{i=1}^{n} xe_i \mid ye_j\right) = \left(xe_j - \sum_{i=1}^{n} xe_i e_j \mid ye_j\right) = 0$$

j.

10) *For each* $x \in A$, *there exists a finite or infinite sequence* (e_n) *of pair-rthogonal, irreducible self-adoint idempotents belonging to the closure* I *of eal* Ax, *such that* $x = \sum_n xe_n$ (*this series being convergent in* A), *and* $- \sum_n \|xe_n\|^2$.

e may assume that $x \neq 0$. Then there exists at least one self-adjoint otent $e \in I$ such that $xe \neq 0$, for the construction in (15.8.5) produces adjoint idempotent $e \neq 0$ in I such that $e(x^*x)^2 = e$, and therefore 0, so that $xe = (ex^*)^* \neq 0$. Since e is the sum of a finite number of cible self-adjoint idempotents belonging to Ae (15.8.6), there exists at ne of these idempotents whose product with x is $\neq 0$. We next remark $(e_i)_{1 \leqq i \leqq n}$ is a finite sequence of pairwise orthogonal, self-adjoint idem-s such that $\|xe_i\|^2 \geqq \alpha > 0$ for all i, then by virtue of (15.8.9) we have

$$\|x\|^2 = \left\|x - \sum_{i=1}^{n} xe_i\right\|^2 + \sum_{i=1}^{n} \|xe_i\|^2 \geqq \sum_{i=1}^{n} \|xe_i\|^2,$$

herefore $n \leqq \|x\|^2/\alpha$. Now define inductively an increasing sequence $\geqq 0$ of integers $\geqq 0$; a (finite or infinite) sequence (e_k) of pairwise onal, irreducible self-adjoint idempotents belonging to I; and a

sequence $(x_n)_{n\geqq 0}$ of elements of A, as follows: $x_0 = x$, $\varphi(0) = 0$; suppose that $\varphi(n)$ has been defined and that the e_k have been defined for $1 \leqq k \leqq \varphi(n)$, and let $x_n = x - \sum_{k=1}^{\varphi(n)} xe_k$. If $x_n = 0$, the sequence (e_k) is finite and has $\varphi(n)$ elements; we take $x_m = x_n = 0$ for $m \geqq n$, and $\varphi(m) = \varphi(n)$ for $m \geqq n$. If $x_n \neq 0$, take a finite (possibly empty) sequence $(e'_i)_{1\leqq i\leqq r}$ of irreducible self-adjoint idempotents belonging to I, orthogonal in pairs and orthogonal to each e_k for $1 \leqq k \leqq \varphi(n)$, such that $\|xe'_i\|^2 \geqq \|x\|^2/2^{n+1}$, and that the number r is *as large as possible* among all finite sequences having the above properties. (We have seen above that this number is $\leqq 2^{n+1}$.) Then put $\varphi(n+1) = \varphi(n) + r$; $e_k = e'_i$ for $k = \varphi(n) + i$, $1 \leqq i \leqq r$; and $x_{n+1} = x_n - \sum_{i=1}^{r} xe'_i$. If no x_n vanishes, the sequence $(\varphi(n))$ tends to $+\infty$; for if it were bounded, we should have $\varphi(m) = \varphi(n)$ for some n and all $m \geqq n$, and by definition this would mean that for each irreducible self-adjoint idempotent e' in the orthogonal supplement F of $\sum_{k=1}^{\varphi(n)} Ae_k$, we should have $xe' = x_n e' = 0$; but $x_n \in F$, $x_n \neq 0$, and F is a closed left ideal (15.8.2), so that this would contradict what was established at the beginning of the proof.

This being so, it is clear from the construction that $\sum_n \|xe_n\|^2 \leqq \|x\|^2$ by virtue of (15.8.9); hence if the series $\sum_n xe_n$ is not a finite sum, it converges in any case to an element $y \in I$ which is the orthogonal projection of x on the closure of the left ideal $\mathfrak{a}$ which is the sum of the Ae_n (for $x - y$, being the limit of $x - \sum_{k=1}^{m} xe_k$, is orthogonal to all the e_n) (6.5.2). If $x - y \neq 0$, there would exist in the left ideal $\mathfrak{a}^\perp$ an irreducible self-adjoint idempotent $e'' \in I$ such that $xe'' = (x-y)e'' \neq 0$. If n is the smallest integer such that $\|xe''\|^2 \geqq \|x\|^2/2^{n+1}$, the existence of e'' would contradict the maximality of the family of e_i such that $\varphi(n) < i \leqq \varphi(n+1)$. Q.E.D.

(15.8.11) *Suppose that the algebra* A *is separable. Then every closed left ideal* $\mathfrak{b}$ *is the Hilbert sum of a (finite or infinite) sequence of minimal left ideals* $I_n = Ae_n$, *where* e_n *is an irreducible self-adjoint idempotent. For each* $x \in \mathfrak{b}$ *we have* $x = \sum_n xe_n$, *and for all* x, y *in* $\mathfrak{b}$ *we have* $(x \mid y) = \sum_n (xe_n \mid ye_n)$.

The second and third assertions are consequences of the first and of the definition of a Hilbert sum (6.4), since xe_n is the orthogonal projection of x on Ae_n (15.8.9). To prove the first assertion, let $(x_n)_{n\geqq 1}$ be a dense sequence in $\mathfrak{b}$ (3.10.9). We define inductively, for each n, a (finite or infinite) sequence $(e_{n,i})_{i\in I_n}$ of irreducible self-adjoint idempotents, as follows. For $(e_{1,i})_{i\in I_n}$

ke any sequence of pairwise orthogonal, irreducible self-adjoint idempo- belonging to $\mathfrak{b}$ and such that $x_1 = \sum_i x_1 e_{1,i}$ (15.8.10). Now suppose that $_{,i}$ have been defined for $m \leqq n$ in such a way that they are pairwise gonal and belong to $\mathfrak{b}$, and are such that the x_m with $m \leqq n$ belong to the e $\mathfrak{a}_n \subset \mathfrak{b}$ of the left ideal which is the sum of the ideals $Ae_{m,i}$ for all and $i \in I_m$ for each m. Let x'_{n+1} be the orthogonal projection of x_{n+1} $\cap \mathfrak{b}$; then take for $(e_{n+1,i})_{i \in I_{n+1}}$ a sequence of pairwise orthogonal cible self-adjoint idempotents which belong to $\mathfrak{a}^\perp \cap \mathfrak{b}$ and are such $_{n+1} = \sum_i x'_{n+1} e_{n+1,i}$ (15.8.10). It is clear that the double family $(e_{n,i})$, ged in a single sequence, has the required properties, by virtue of 9) and (6.4.2).

is theorem applies in particular when $\mathfrak{b} = A$. In this case we get a position of A as a Hilbert sum of minimal left ideals. It should be ked that in general there will exist *infinitely many* such decompositions ter (15.8.14)). More precisely:

1.1) *Suppose that* A *is separable, and let* I *be a minimal left ideal of* A. *here exists a decomposition of* A *as a Hilbert sum of minimal left ideals h that* $I_1 = I$.

pply (15.8.11) to $\mathfrak{b} = I^\perp$.

2) *Let* e, e' *be two irreducible self-adjoint idempotents, and let* $I = Ae$, e' *be the corresponding minimal left ideals.*

) *Every homomorphism of the* A*-module* I *into the* A*-module* I′ *is of the* $_a : x \mapsto xa$, *where* $a \in eAe' = eA \cap Ae'$; *it is either zero or bijective, and pping* $a \mapsto f_a$ *is an isomorphism of the complex vector space* eAe' *onto* (I, I′) *such that* $f_{ab} = f_b \circ f_a$.

) *The* **C***-algebra* eAe, *isomorphic to* $\mathrm{End}_A(I)$, *is a field, equal to* $\mathbf{C}e$ (*and re isomorphic to* **C**).

) *If* I *and* I′ *are not isomorphic as* A*-modules, then* e *and* e' (*and uently* I *and* I′) *are orthogonal, and* $II' = I'I = \{0\}$. *If* I *and* I′ *are iso- ic as* A*-modules, then* eAe' *is a complex vector space of dimension* 1, $= I'$.

If $x \in A$, *then* Ix *is a left ideal which is either zero or isomorphic* (*as ule*) *to* I.

If $g : I \to I'$ is an A-module homomorphism and $a = g(e)$, then for $\in I$ we have $g(x) = g(xe) = xg(e) = xa$ (15.8.3(iii)). Since $a \in I'$ we

have $a = ae'$ (15.8.3(iii)), and on the other hand $a = g(e^2) = eg(e) = ea$, so that $a \in eAe'$, and $g = f_a$. It is clear that $eAe' \subset eA \cap Ae'$; conversely, if $y \in eA \cap Ae'$, then $y = ye'$ and $y = ey$ (15.8.3(iii)), so that $y = eye' \in eAe'$. The image $g(I)$ is a left ideal contained in I', so it is either $\{0\}$ or I'; likewise, the kernel $g^{-1}(0)$ is a left ideal contained in I, hence is either $\{0\}$ or I. If $g^{-1}(0) = I$, then $g(I) = \{0\}$; if $g^{-1}(0) = \{0\}$, then we must have $g(I) \neq \{0\}$, hence $g(I) = I'$ and g is bijective. Finally, if $f_a = 0$, we have $f_a(e) = ea = 0$; but $a \in eAe'$, so that $ea = a$, and consequently $a = 0$.

(ii) The C-algebra eAe is a closed subalgebra of A (15.8.3(iii)). Since we have seen above that every element of $\mathrm{End}_A(I)$ is either zero or invertible, it follows that $\mathrm{End}_A(I)$ is a (possibly noncommutative) *field*, and hence the same is true for eAe. Clearly e is the unit element of eAe, and because A is a *normable* algebra (15.1.8) it follows from the Gelfand–Mazur theorem that $eAe = \mathbf{C}e$.

(iii) If I and I' are not isomorphic, we have $eAe' = \{0\}$ by (i) above, and in particular $ee' = 0$; hence e and e' are orthogonal (15.8.4). The same is true of I and I' (15.8.9), and $II' = \{0\}$. If I and I' are isomorphic, and if g is an isomorphism of I onto I', then every homomorphism of I into I' is of the form $g \circ u$, where u is an endomorphism of I. Hence eAe' is a complex vector space of dimension 1, by virtue of (i) and (ii) above. Clearly II' is a left ideal contained in I'; since it contains $eAe' \neq \{0\}$, it must be equal to I'.

(iv) Since Ix is the image of I under the homomorphism $y \mapsto yx$ of I into A, it is a left ideal isomorphic to I/I', where I' is the kernel of the above homomorphism. But I' must be equal to either $\{0\}$ or I, and so $y \mapsto yx$ is either zero or injective.

(15.8.13) *Suppose that* A *is separable. Then*:

(i) *There exists a finite or infinite sequence* $(I_k)_{k \in J}$ *of minimal left ideals, no pair of which are isomorphic, such that every minimal left ideal of* A *is isomorphic (as an* A*-module) to some* I_k.

(ii) *For each index* $k \in J$, *the closure of the sum of all the minimal left ideals of* A *which are isomorphic to* I_k *is a self-adjoint two-sided ideal* $\mathfrak{a}_k$. *Every minimal left ideal of the Hilbert algebra* $\mathfrak{a}_k$ *is a minimal left ideal of* A, *isomorphic to* I_k, *and the algebra* $\mathfrak{a}_k$ *contains no closed two-sided ideals other than* $\{0\}$ *and* $\mathfrak{a}_k$.

(iii) *Each of the algebras* $\mathfrak{a}_k$ *is the Hilbert sum of a (finite or infinite) sequence of minimal left ideals isomorphic to* I_k. *The algebra* A *is the Hilbert sum of the sequence* $(\mathfrak{a}_k)_{k \in J}$, *and* $\mathfrak{a}_h \mathfrak{a}_k = \{0\}$ *whenever* $h \neq k$.

Start with a decomposition of A as a Hilbert sum of minimal left ideals I'_n (15.8.11). We take $I_1 = I'_1$ and define inductively I_{k+1} to be equal to I'_m,

where m is the smallest integer such that $\mathfrak{l}'_m$ is not isomorphic to any of the ideals $\mathfrak{l}_1, \ldots, \mathfrak{l}_k$. (If all the $\mathfrak{l}'_n$ are isomorphic to one or other of $\mathfrak{l}_1, \ldots, \mathfrak{l}_k$, the induction stops at $\mathfrak{l}_k$.) Let J be the sequence of indices k so obtained, and for each $k \in \mathrm{J}$ let I_j be the sequence of integers n such that $\mathfrak{l}'_n$ is isomorphic to $\mathfrak{l}_k$. We define $\mathfrak{a}_k$ to be the Hilbert sum of the $\mathfrak{l}'_n$ for $n \in \mathrm{I}_k$. Clearly A is the Hilbert sum of the left ideals $\mathfrak{a}_k$ (6.4.2).

Let I be any minimal left ideal in A. Then I must be isomorphic to one of the $\mathfrak{l}_k$, for otherwise it would be orthogonal to all the $\mathfrak{l}'_n$ (15.8.12(iii)) and hence orthogonal to A itself, which is absurd. The same argument shows that I is orthogonal to all the $\mathfrak{a}_h$ with $h \neq k$. Hence, as $\mathfrak{a}_k$ is the orthogonal supplement of the Hilbert sum of the $\mathfrak{a}_h$ such that $h \neq k$, we must have $\mathrm{I} \subset \mathfrak{a}_k$. From this it follows already that $\mathfrak{a}_k$ is the closure of the sum of *all* the minimal left ideals of A which are isomorphic to $\mathfrak{l}_k$, and therefore $\mathfrak{a}_k$ is independent of the decomposition of A as the Hilbert sum of the $\mathfrak{l}'_n$, from which we started. Moreover, for each $x \in \mathrm{A}$ and each $n \in \mathrm{I}_k$, $\mathfrak{l}'_n x$ is a left ideal which is either $\{0\}$ or isomorphic to $\mathfrak{l}'_n$ (15.8.12(iv)), hence is contained in $\mathfrak{a}_k$. This proves that $\mathfrak{a}_k$ is a two-sided ideal. If $\mathfrak{l}'_n = \mathrm{A}e'_n$, where e'_n is an irreducible self-adjoint idempotent, then $\mathfrak{l}'^*_n = e'_n \mathrm{A}$, hence $\mathfrak{a}^*_k = \mathfrak{a}_k$.

Let $\mathfrak{l}''$ be a minimal left ideal *of the Hilbert algebra* $\mathfrak{a}_k$. We have $\mathfrak{l}'' = \mathfrak{a}_k e''$, where e'' is a self-adjoint idempotent (15.8.7), and $e'_n e''$ cannot vanish for all $n \in \mathrm{I}_k$, otherwise $\mathfrak{l}''$ would be orthogonal to all the $\mathfrak{l}'_n$ ($n \in \mathrm{I}_k$) and therefore to the closure of their sum, namely to $\mathfrak{a}_k$: which is absurd because $\mathfrak{l}'' \neq \{0\}$. Hence there exists at least one index $n \in \mathrm{I}_k$ such that $\mathfrak{l}'_n \mathfrak{l}'' \neq \{0\}$; since $\mathfrak{l}'_n \mathfrak{l}''$ is a left ideal in $\mathfrak{a}_k$, we must have $\mathfrak{l}'_n \mathfrak{l}'' = \mathfrak{l}''$, which shows that $\mathfrak{l}''$ is a minimal left ideal of A, necessarily isomorphic to $\mathfrak{l}'_n$ and therefore to $\mathfrak{l}_k$ (15.8.12(iii)). If now $\mathfrak{b}$ is a nonzero two-sided ideal *of the algebra* $\mathfrak{a}_k$, it contains at least one minimal left ideal $\mathfrak{l}''$ of this algebra (15.8.8), hence also contains all the $\mathfrak{l}''\mathfrak{l}'_n$ ($n \in \mathrm{I}_k$). But $\mathfrak{l}''\mathfrak{l}'_n = \mathfrak{l}'_n$ (15.8.12(iii)), and therefore $\mathfrak{b}$ contains the sum of all the $\mathfrak{l}'_n$ ($n \in \mathrm{I}_k$). If $\mathfrak{b}$ is closed, it follows that $\mathfrak{b} = \mathfrak{a}_k$. Finally, $\mathfrak{a}_h \mathfrak{a}_k \subset \mathfrak{a}_h \cap \mathfrak{a}_k = \{0\}$ if $h \neq k$, because $\mathfrak{a}_h$ and $\mathfrak{a}_k$ are two-sided ideals.

A complete Hilbert algebra A is said to be *topologically simple* if it contains no *closed* two-sided ideal other than A and $\{0\}$. It follows from (15.8.13) that the study of the structure of a separable complete Hilbert algebra A reduces completely to the study of the $\mathfrak{a}_k$, that is to the case where A is *topologically simple*.

(15.8.14) *Let* A *be a topologically simple, complete, separable Hilbert algebra. Then for each minimal left ideal* I *of* A, *the representation* $x \mapsto U_{\mathrm{I}}(x)$ *of* A *in the Hilbert space* I *is faithful.*

If A *is infinite-dimensional, then so is* I. *The image of* A *under* U_{I} *is the algebra* $\mathscr{L}_2(\mathrm{I})$ *of Hilbert–Schmidt operators on* I (15.4.8), *and there exists a*

constant $\gamma > 0$ *such that*

$$\gamma(x \mid y) = (U_{\mathrm{I}}(x) \mid U_{\mathrm{I}}(y)) \tag{15.8.14.1}$$

(*for the scalar product defined in* (15.7.4)).

If A *is finite-dimensional, then the image of* A *under* U_{I} *is the algebra* $\mathrm{End}_{\mathbf{C}}(\mathrm{I})$ *of all endomorphisms of the vector space* I, *and the relation* (15.8.14.1) *remains valid, for the scalar product defined in* (15.7.4.1) (*from the scalar product on* I *which is the restriction of that on* A).

We can assume that A is the Hilbert sum of a (finite or infinite) sequence of minimal left ideals $\mathrm{I}_n = \mathrm{A}e_n$, where $\mathrm{I} = \mathrm{I}_1$ (15.8.11.1) and all the I_n are isomorphic (15.8.13). If $x \neq 0$ and $U_{\mathrm{I}}(x) = 0$, that is, if $x\mathrm{I} = \{0\}$, we should have $(\mathrm{A}x)\mathrm{I} = \{0\}$, and since the ideal $\mathrm{A}x$ is nonzero, it contains a minimal left ideal I′ (15.8.8), which must be isomorphic to I (15.8.13); hence $\mathrm{I}'\mathrm{I} = \{0\}$, contrary to (15.8.12(iii)). Hence the representation U_{I} is faithful. Put $P_n = U_{\mathrm{I}}(e_n)$, which is the *orthogonal projection* of $\mathrm{I} = \mathrm{A}e_1$ onto the *one-dimensional* subspace $e_n\mathrm{A}e_1$ (15.8.12), because $(xe_1 - e_n xe_1 \mid e_n ye_1) = (e_n xe_1 - e_n^2 xe_1 \mid ye_1) = 0$. Since $e_m e_n = 0$ if $m \neq n$, we have $P_m P_n = 0$ and therefore the subspaces $e_n\mathrm{A}e_1$ are *orthogonal in pairs.* Moreover, I is the *Hilbert sum* of the subspaces $e_n\mathrm{A}e_1$, because if xe_1 is orthogonal to all these subspaces, we have $P_n(xe_1) = 0$ for all n, so that $e_n xe_1 = 0$ for all n, and therefore xe_1 belongs to the right annihilator of A, which is zero (15.7.5.7). This shows that the sequence (I_n) is *finite* if and only if I (and therefore each I_n) is *finite*-dimensional over $\mathbf{C}$, or equivalently if and only if A is *finite-dimensional.*

Let (a_n) be an orthonormal basis of I such that $a_n \in e_n\mathrm{A}e_1$ for each n. Then $a_n a_n^* \in e_n\mathrm{A}e_n$, hence $a_n a_n^* = \lambda_n e_n$ for some $\lambda_n \in \mathbf{C}^*$ (15.8.12). Likewise $a_n^* a_n = \lambda_n' e_1$. We have $\lambda_n' = \lambda_n$, because on the one hand

$$a_n a_n^* a_n a_n^* = \lambda_n^2 e_n$$

and on the other hand

$$a_n a_n^* a_n a_n^* = \lambda_n' a_n e_1 a_n^* = \lambda_n' a_n a_n^* = \lambda_n' \lambda_n e_n,$$

since $a_n e_1 = a_n$. Also

$$1 = (a_n \mid a_n) = (a_n \mid e_n a_n) = (a_n a_n^* \mid e_n) = \lambda_n (e_n \mid e_n)$$

and

$$1 = (a_n \mid a_n) = (a_n \mid a_n e_1) = (a_n^* a_n \mid e_1) = \lambda_n (e_1 \mid e_1);$$

hence

$$\bullet\ (e_n \mid e_n) = (e_1 \mid e_1) \tag{15.8.14.2}$$

ll n, and all the λ_n have the same value $\gamma = (e_1 \mid e_1)^{-1}$. Hence, for all x, y in e have

$$(xa_n \mid ya_n) = (y^*x \mid a_n a_n^*) = (y^*x \mid \gamma e_n) = \gamma(xe_n \mid ye_n);$$

the series with general term $(xe_n \mid ye_n)$ is absolutely convergent, with $(x \mid y)$ (15.8.11), it follows that if A is infinite-dimensional then $U_{\mathrm{I}}(x)$ is a ert–Schmidt operator, and the relation (15.8.14.1) is valid. Since A is a *ert* space, so is its image under U_{I}, and to show that this image is the e of the Hilbert space $\mathscr{L}_2(\mathrm{I})$ (15.4.8), it is enough to show that $U_{\mathrm{I}}(\mathrm{A})$ is in $\mathscr{L}_2(\mathrm{I})$. Now, for each pair, m, n with $m \neq n$, we have

$$e_m \mathrm{A} e_n \cdot e_n \mathrm{A} e_1 = e_m(\mathrm{A}e_n)(\mathrm{A}e_1) = e_m \mathrm{A} e_1$$

.12(iii)), and since $e_n \mathrm{A} e_1 = \mathbf{C}a_n$, it follows that there exists $e_{mn} \in e_m \mathrm{A} e_n$ that $e_{mn} a_n = a_m$ (which implies that $e_{mn} = \gamma^{-1} a_m a_n^*$), and clearly $_p = 0$ if $p \neq n$. We conclude from this that $E_{mn} = U_{\mathrm{I}}(e_{mn})$ is the con- ous endomorphism of the Hilbert space I such that $E_{mn} \cdot a_n = a_m$ and $a_p = 0$ if $p \neq n$. Our assertion now follows from the fact that the finite r combinations of the E_{mn} are dense in $\mathscr{L}_2(\mathrm{I})$ (15.4.8). The proof is ogous but simpler when A is finite-dimensional.

Ve remark that, in all cases, the $U_{\mathrm{I}}(\mathrm{I}_n)$ consist of endomorphisms of the $U(x) \circ P_n$ and therefore consist of endomorphisms *of rank* 1 of I.

.15) *Under the hypotheses of* (15.8.14), *if there exists an element* $\neq 0$ *in enter of* A, *then* A *is finite-dimensional. In that case the center of* A *is* $\mathbf{C}u$, *e* u *is the unit element of* A.

$c \in \mathrm{A}$ belongs to the center of A, then $U_{\mathrm{I}}(c)$ is an endomorphism of the odule I, hence is a homothety $x \mapsto \lambda x$, where $\lambda \in \mathbf{C}$ (15.8.12). But clearly a othety cannot be a Hilbert–Schmidt operator on an infinite-dimensional ert space, unless it is zero.

.16) *Let* A *be a separable complete Hilbert algebra and let* $\mathfrak{a}_k$ $(k \in \mathrm{J})$ *e topologically simple Hilbert algebras which are the Hilbert summands* (15.8.13). *For each* $k \in \mathrm{J}$, *let* I_k *be a minimal left ideal of* $\mathfrak{a}_k$. *Let* V *be a non- nerate representation* (15.5.5) *of* A *in a separable Hilbert space* H, *such* $V : \mathrm{A} \mapsto \mathscr{L}(\mathrm{H})$ *is continuous.*

i) H *is the Hilbert sum of subspaces* H_k $(k \in \mathrm{J})$ *stable with respect to* V, *that if* V_k *is the restriction of* V *to* H_k *we have* $V_k(s) = 0$ *for all* $s \in \mathfrak{a}_h$ *and* $\neq k$: *so that* V_k *may be considered as a representation of* $\mathfrak{a}_k$ *on* H_k.

i) *If* $\mathfrak{a}_k$ *is finite-dimensional over* $\mathbf{C}$, *the representation* V_k *is the Hilbert of a (finite or infinite) sequence of irreducible representations, each equiv- to the representation* U_{I_k} *of* $\mathfrak{a}_k$ (15.8.14).

Let H_k be the closure of the vector subspace of H generated by the vectors $V(s_k) \cdot x$, where $s_k \in \mathfrak{a}_k$ and $x \in H$. Since each $s \in A$ can be written in the form $s = \sum_k s_k$ with $s_k \in \mathfrak{a}_k$, and since V is continuous, we have

$$V(s) \cdot x = \sum_k V(s_k) \cdot x$$

(5.5.2), and therefore H is the closure of the sum of the H_k. Also, if $h \neq k$ and $s_h \in \mathfrak{a}_h$, $s_k \in \mathfrak{a}_k$, we have

$$(V(s_h) \cdot x \mid V(s_k) \cdot y) = (V(s_k^* s_h) \cdot x \mid y) = 0$$

because $\mathfrak{a}_k$ is self-adjoint and $\mathfrak{a}_k \mathfrak{a}_h = \{0\}$. This proves (i). Now assume that A is topologically simple and finite-dimensional over **C**, and therefore has a unit element; we can then restrict ourselves to the case where there exists a *totalizer* x_0 for V (15.5.6). The vector subspace of H generated by the $V(s) \cdot x_0$ is finite-dimensional, hence closed (5.9.2) and so equal to H. We may therefore argue by induction on the dimension of H. Since A is the sum of a finite number of minimal left ideals, there is at least one minimal left ideal, say I, such that the subspace $E = V(I) \cdot x_0$ is nonzero. The surjection $s \mapsto V(s) \cdot x_0$ of I onto E is then an A-module homomorphism, and since its kernel is a left ideal I′ contained in I and distinct from I, we have $I' = \{0\}$. Hence E is a stable subspace of H with respect to V, such that the restriction of V to E is equivalent to U_I. Since the orthogonal supplement H′ of E in H is stable with respect to V and of dimension strictly less than the dimension of H, we have only to apply the inductive hypothesis to complete the proof.

The result of (15.8.16(ii)) can be shown to be valid without assuming that $\mathfrak{a}_k$ is finite-dimensional (Problem 1).

PROBLEMS

1. Let A be a topologically simple, separable, complete Hilbert algebra.

Let V be a nondegenerate representation of A in a Hilbert space H. With the notation of the proof of (15.8.14), put $E_n = V(e_n)$ and $A_n = V(a_n)$. The E_n are orthogonal projectors on subspaces H_n of H, such that H is the Hilbert sum of the H_n, and we have $A_n(H_1) = H_n$ and $A_n^*(H_n) = H_1$. If $(b_{k1})_{k \in I}$ is a (finite or infinite) orthonormal basis of H_1, then for each n the vectors $b_{kn} = \gamma^{-1/2} A_n(b_{k1})$ form an orthonormal basis for H_n as k runs through the index set I. Deduce that, if H_k' is the subspace of H generated by the b_{kn} ($n \geqq 1$), then H_k' is stable with respect to V, and V is the Hilbert sum of the restrictions V_k of V to H_k'. Each of those representations is topologically cyclic, b_{k1} being a totalizer because $E_1 \cdot b_{k1} = b_{k1}$. Show that this representation is equivalent to the representation U_I.

2. Let H be an infinite-dimensional separable Hilbert space, $A = \mathscr{L}_2(H)$ the Banach algebra of Hilbert–Schmidt operators on H, and B a closed self-adjoint subalgebra of

A. Show that there exists a decomposition of H as a Hilbert sum of a subspace H_0 and a (finite or infinite) sequence of closed subspaces H_k stable with respect to B, with the following properties: (1) the restrictions to H_0 of the operators belonging to B are all zero; (2) each H_k is the Hilbert sum of a *finite* sequence $(H_{kp})_{1 \leqslant p \leqslant r_k}$ of subspaces of the *same* dimension (finite or not) which are stable with respect to B; the restrictions to H_{k1} of the operators belonging to B form the algebra of Hilbert-Schmidt operators on H_{k1}; moreover, for $2 \leqq p \leqq r_k$, there exists an isometric isomorphism T_p of H_{k1} onto H_{kp} such that, if U_1 is the restriction to H_{k1} of an operator $U \in B$, then the restriction of U to H_{kp} is $T_p U_1 T_p^{-1}$.

9. THE PLANCHEREL-GODEMENT THEOREM

In this section we shall study *commutative* (but not necessarily *complete*) Hilbert algebras. More generally it will be convenient, in view of applications, to consider a *commutative algebra* A *with involution, endowed with a bitrace g* (15.7) *satisfying conditions* (U) *and* (N) *of* (15.6). We shall denote by $\mathfrak{n}_g$ the two-sided ideal of elements $s \in A$ such that $g(s, s) = 0$, and by $\pi_g : A \to A/\mathfrak{n}_g$ the canonical mapping. We recall that $A/\mathfrak{n}_g$ is canonically endowed with a structure of Hilbert algebra (15.7). We shall assume throughout that the prehilbert space $A/\mathfrak{n}_g$ is *separable*, and hence is a dense subspace of a *separable Hilbert space*, which we denote by H_g. Thus, starting from g, we obtain canonically a representation of A in H_g (15.6.10), which we denote by U_g. The image of A under U_g is a commutative subalgebra with involution (15.6.1) of $\mathscr{L}(H_g)$. Let $\mathscr{A}_g$ denote its *closure* in $\mathscr{L}(H_g)$, so that $\mathscr{A}_g$ is a commutative star algebra (and therefore consists of *normal* operators (15.4.11)). We shall assume that this algebra $\mathscr{A}_g$ is *separable* (this is not a consequence of the separability of $A/\mathfrak{n}_g$, cf. Problem 1).

A particular example of a *trace* on A (which therefore gives rise to a bitrace, by the canonical procedure (15.6.2)) is provided by the *hermitian characters* of A, i.e., the characters χ of A which satisfy

$$\chi(x^*) = \overline{\chi(x)} \tag{15.9.1}$$

for all $x \in A$. It follows that $\chi(x^*x) = |\chi(x)|^2 \geqq 0$, and if we put

$$g(x, y) = \chi(y^*x) = \overline{\chi(y)}\chi(x),$$

the condition (U) of (15.6) is trivially verified, because

$$g(st, st) = |\chi(s)|^2 |\chi(t)|^2 = |\chi(s)|^2 g(t, t).$$

On the other hand, the ideal $\mathfrak{n}_g$, which is here the *kernel* of χ, is a hyperplane in A, so that the algebra $A/\mathfrak{n}_g$ can be identified with $\mathbf{C}$, and the condition (N)

follows immediately from the fact that $\chi(x) \neq 0$ implies $\chi(x^2) \neq 0$. The corresponding representation U_χ is evidently irreducible.

We denote by $\mathbf{H}(A)$ the set of hermitian characters of A; it is a subset of the product space $\mathbf{C}^A$ and is *closed* in the product topology (3.15.1). We give $\mathbf{H}(A)$ the topology induced by the product topology (i.e., the *weak* topology (12.15)). When A is a separable commutative Banach algebra with involution, having a unit element $e \neq 0$, the space $\mathbf{H}(A)$ is a compact metrizable subspace of $\mathbf{X}(A)$ (15.3.2). In this situation it can happen that $\mathbf{H}(A) \neq \mathbf{X}(A)$ (Section 15.4, Problem 3), but $\mathbf{H}(A)$ is equal to $\mathbf{X}(A)$ if A is a separable *star algebra* (15.4.14).

We shall show that the *bitraces g* satisfying the conditions at the beginning of this section can *all* be obtained, by a canonical process of "integration", from *hermitian characters*:

(15.9.2) (Plancherel–Godement theorem) *Let g be a bitrace on a commutative algebra* A *with involution, satisfying* (U) *and* (N), *and such that the prehilbert space* $A/\mathfrak{n}_g$ *and the star algebra* $\mathscr{A}_g \subset \mathscr{L}(H_g)$ *are separable.*

(I) *We can define canonically*: (1) *a subspace* S_g *of* $\mathbf{H}(A)$, *whose closure in* $\mathbf{C}^A$ *is either* S_g *or* $S_g \cup \{0\}$ *and is metrizable and compact* (*so that* S_g *is locally compact, metrizable and separable*); (2) *a positive measure* m_g *on* S_g, *with the following properties*:

(i) *For each* $x \in A$, *the function* $\chi \mapsto \hat{x}(\chi) = \chi(x)$ *belongs to* $\mathscr{L}^2_{\mathbf{C}}(S_g, m_g)$ *and we have*

$$g(x, y) = \int_{S_g} \chi(xy^*)\, dm_g(\chi) = \int_{S_g} \hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm_g(\chi) \tag{15.9.2.1}$$

for all x, y in A.

(ii) *As x runs through* A, *the set of functions* $\hat{x}$ *is contained in* $\mathscr{C}^0_{\mathbf{C}}(S_g)$ (13.20.6) *and is dense in this Banach space.*

(iii) *The support of the measure* m_g *is the whole of* S_g.

(iv) *The mapping* $x \mapsto \hat{x}$ *factorizes into* $x \mapsto \pi_g(x) \overset{T_0}{\mapsto} \hat{x}$, *and the mapping* T_0 *of* $A/\mathfrak{n}_g$ *into* $\mathscr{C}^0_{\mathbf{C}}(S_g)$ *extends to an isomorphism T of the Hilbert space* H_g *onto* $L^2_{\mathbf{C}}(S_g, m_g)$ *such that for all* $x \in A$ *we have* $U_g(x) = T^{-1}M(\hat{x})T$, *where* $M(\hat{x})$ *is multiplication by the class of* $\hat{x}$ *in* $L^2_{\mathbf{C}}(S_g, m_g)$ (cf. (13.21.5)). *Also*

$$\|\lambda \cdot 1_{H_g} + U_g(x)\| = \|\lambda + \hat{x}\|$$

for all $\lambda \in \mathbf{C}$ *and* $x \in A$.

(II) *Conversely, let* S *be a subspace of* $\mathbf{H}(A)$ *such that* $S \cup \{0\}$ *is compact and metrizable, and let m be a positive measure on* S, *with support equal to* S,

at for all $x \in A$ *the function* $\chi \mapsto \hat{x}(\chi) = \chi(x)$ *belongs to*

$$\mathscr{L}^2_{\mathbf{C}}(S, m) \cap \mathscr{C}^0_{\mathbf{C}}(S).$$

$'(x, y) = \int_S \hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm(\chi)$ *is a bitrace on* A *satisfying* (U) *and* (N), *at* $A/\mathfrak{n}_{g'}$ *and* $\mathscr{A}_{g'}$ *are separable; and we have* $S_{g'} = S$ *and* $m_{g'} = m$.

proof of this theorem is in several steps.

2) *Construction of* S_g *and proof of* (ii).

subalgebra $\mathbf{C} \cdot 1_{H_g} + \mathscr{A}_g = \mathscr{A}'_g$ is closed in $\mathscr{L}(H_g)$ (5.9.2) and hence is mutative star algebra with unit element. For each character $\xi' \in \mathbf{X}(\mathscr{A}'_g)$, ws from (15.4.14) that $\xi' \circ U_g$ is identically zero on A or else is a *an* character of A: in other words, $\omega : \xi' \mapsto \xi' \circ U_g$ is a mapping of into $\mathbf{H}(A) \cup \{0\}$. This mapping ω is *injective*, because we have $= 1$ for all $\xi' \in \mathbf{X}(\mathscr{A}'_g)$, and since ξ' is continuous on $\mathscr{A}'_g$, the values of $_g(A)$, which is dense in $\mathscr{A}_g$, completely determine the character ξ'. On er hand, it follows from the definitions that ω is *continuous* with respect weak topologies on $\mathbf{X}(\mathscr{A}'_g)$ and $\mathbf{C}^A$; since $\mathbf{X}(\mathscr{A}'_g)$ is metrizable and t, the same is true of its image

$$\omega(\mathbf{X}(\mathscr{A}'_g)) = S'_g \subset \mathbf{H}(A) \cup \{0\},$$

is a *homeomorphism* of $\mathbf{X}(\mathscr{A}'_g)$ onto S'_g (12.3.6). If $1_{H_g} \in \mathscr{A}_g$, then $'_g$, and S'_g does not contain the element 0 of $\mathbf{C}^A$. In this case we put . If on the other hand $1_{H_g} \notin \mathscr{A}_g$, then $\mathscr{A}_g$ is a closed hyperplane and in $\mathscr{A}'_g$, hence is a maximal ideal, and there exists a character ξ'_0 of se kernel is $\mathscr{A}_g$ (15.3.1); its image under ω is therefore the element 0 and we put $S_g = S'_g - \{0\}$ in this case. In each case, S_g is *separable, le and locally compact*, and the complements in $S_g \cup \{0\}$ of the com- sets of S_g are the open neighborhoods of 0 in $S_g \cup \{0\}$. For each he function $\chi' \mapsto (\omega^{-1}(\chi'))(U_g(x))$, which is the composition of the transformation $\mathscr{G}U_g(x)$ and ω^{-1}, is continuous on S'_g, and its restric- $_g$ is $\hat{x}$. We denote this function also by $\hat{x}$, by abuse of notation. When e have $\omega^{-1}(0) = \xi'_0$ and hence $\hat{x}(0) = 0$, which shows that $\hat{x} \in \mathscr{C}^0_{\mathbf{C}}(S_g)$ case (13.20.6). To prove the density assertion in (ii), note that the transformation is an isometry of $\mathscr{A}'_g$ onto $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(\mathscr{A}'_g))$ (15.4.14); the s $\lambda + \hat{x}$ (where $\lambda \in \mathbf{C}$) therefore form a dense subset of $\mathscr{C}_{\mathbf{C}}(S'_g)$, the assertion follows in all cases.

) *Preliminaries to the construction of* m_g.

For all x, y in A, the mapping $V \mapsto (V \cdot \pi_g(x) \mid \pi_g(y))$ is a continuous linear form on $\mathscr{A}'_g$ and so, by virtue of the isomorphism of $\mathscr{C}_{\mathbf{C}}(\mathrm{S}'_g)$ and $\mathscr{A}'_g$, there exists (13.1) a (complex) *measure* $\mu_{x,y}$ on S'_g such that

$$g(zx, y) = (U_g(z) \cdot \pi_g(z) \mid \pi_g(y)) = \int_{\mathrm{S}'_g} \chi'(z)\, d\mu_{x,y}(\chi') \tag{15.9.2.4}$$

for all $z \in \mathrm{A}$.

When $0 \in \mathrm{S}'_g$, we have seen (15.9.2.2) that the function $\hat{z} : \chi' \mapsto \chi'(z)$ is zero at the point 0; consequently, if $m_{x,y}$ is the measure *induced on* S_g by $\mu_{x,y}$ (13.1.8), we can write in all cases

$$g(zx, y) = \int_{\mathrm{S}_g} \hat{z}(\chi)\, dm_{x,y}(\chi). \tag{15.9.2.5}$$

It is clear that the measures $m_{x,y}$ are *bounded*. Moreover, for given x and y in A, the measure $m_{x,y}$ is the *only* bounded measure m' on S_g such that $g(zx, y) = \int_{\mathrm{S}_g} \hat{z}(\chi)\, dm'(\chi)$ for all $z \in \mathrm{A}$, because the functions $\hat{z}$ are *dense* in $\mathscr{C}^0_{\mathbf{C}}(\mathrm{S}_g)$ (13.20.6).

If the formula (15.9.2.1) is true, then for all x, y, z in A we must have $g(zx, y) = \int_{\mathrm{S}_g} \hat{z}(\chi)\hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm_g(\chi)$. Comparing this with (15.9.2.5), what we shall in fact prove is the existence of a measure m_g on S_g such that

$$m_{x,y} = \hat{x}\bar{\hat{y}} \cdot m_g \tag{15.9.2.6}$$

for all x, y in A (cf. (13.1.5)).

If we have constructed such a measure, then for all functions $\mathrm{F} \in \mathscr{K}_{\mathbf{C}}(\mathrm{S}_g)$ we shall have

$$\mathrm{F} \cdot m_{x,y} = (\mathrm{F}\hat{x}\bar{\hat{y}}) \cdot m_g = (\hat{x}\bar{\hat{y}}) \cdot m_{\mathrm{F}}, \tag{15.9.2.7}$$

where we have put $m_{\mathrm{F}} = \mathrm{F} \cdot m_g$, which is a *bounded* measure (13.14.4). We shall begin by constructing a *linear mapping* $\mathrm{F} \mapsto m_{\mathrm{F}}$ of $\mathscr{K}_{\mathbf{C}}(\mathrm{S}_g)$ into $\mathrm{M}^1_{\mathbf{C}}(\mathrm{S}_g)$, satisfying the equality

$$\mathrm{F} \cdot m_{x,y} = (\hat{x}\bar{\hat{y}}) \cdot m_{\mathrm{F}} \tag{15.9.2.8}$$

for all x, y in A, and such that $m_{\mathrm{F}} \geqq 0$ whenever $\mathrm{F} \geqq 0$. Having achieved this, the measure m_g will be simply the linear form $\mathrm{F} \mapsto m_{\mathrm{F}}(1)$ (13.3.1).

(15.9.2.9) *Definition of* m_F.

Let us denote by $\Phi \subset \mathscr{C}^0_c(S_g)$ *the set of all continuous functions* F *which tend to* 0 *at infinity and for which there exists a bounded measure* m_F *on* S_g *satisfying* (15.9.2.8) *for all* x, y *in* A. We shall prove that Φ *contains* $\mathscr{K}_c(S_g)$, by means of the following three lemmas.

(15.9.2.10) Φ *is an ideal in the algebra* $\mathscr{C}^0_c(S_g)$.

It is clear that Φ is a vector space. Also if $F \in \Phi$ and $G \in \mathscr{C}^0_c(S_g)$, it follows from (15.9.2.8) that

$$(GF) \cdot m_{x,y} = (G\hat{x}\bar{\hat{y}}) \cdot m_F = (\hat{x}\bar{\hat{y}}) \cdot (G \cdot m_F),$$

so that $GF \in \Phi$, and we may take $m_{GF} = G \cdot m_F$ (because G is m_F-integrable (13.20.5)).

(15.9.2.11) *For each function* $F \in \Phi$, *there exists only one bounded measure* m_F *satisfying* (15.9.2.8) *for all* x, y *in* A. *The mapping* $F \mapsto m_F$ *is linear, and* $m_F \geqq 0$ *whenever* $F \geqq 0$.

To prove the uniqueness of m_F, it is enough to show that the functions of the form $\hat{x}\bar{\hat{y}}$ (where $x, y \in A$) form a *total* set in the Banach space $\mathscr{C}^0_c(S_g)$, or again that the functions of the form $\lambda + \hat{x}\bar{\hat{y}}$ ($\lambda \in \mathbf{C}$) form a total set in the Banach space $\mathscr{C}_c(S'_g)$. Now, from the fact that A is an algebra and $x \mapsto \hat{x}$ is an algebra homomorphism, it follows that the set B of these functions is a subalgebra of $\mathscr{C}_c(S'_g)$, containing the constants. This subalgebra *separates the points* of S'_g. For if χ_1, χ_2 are two distinct points of S_g, then by virtue of (ii) there exists $x \in A$ such that $\hat{x}(\chi_1) \neq \hat{x}(\chi_2)$. Hence either $\hat{x}^2(\chi_1) \neq \hat{x}^2(\chi_2)$ or $\hat{x}^3(\chi_1) \neq \hat{x}^3(\chi_2)$. On the other hand, if $S'_g = S_g \cup \{0\}$, then for each $\chi \neq 0$ there exists $x \in A$ such that $\hat{x}(\chi) \neq 0$, and therefore $\hat{x}^2(\chi) \neq 0$, which proves our assertion. Finally, since $\bar{\hat{x}} = (x^*)\hat{}$ (15.9.1), the *conjugate* of each function in B belongs to B. Hence we may apply the Stone–Weierstrass theorem (7.3.2), which establishes the uniqueness of m_F. The linearity of the mapping $F \mapsto m_F$ follows immediately from the uniqueness just established. It remains now to show that the relation $F \geqq 0$ implies that $m_F \geqq 0$. Let $G \geqq 0$ be a function belonging to $\mathscr{K}_R(S_g)$. By virtue of (ii), there exists a sequence (x_n) of elements of A such that the functions $\hat{x}_n$ converge uniformly on S_g to the function $G^{1/2}$. The functions $\hat{x}_n\bar{\hat{x}}_n$ therefore converge uniformly on S_g to G, and hence, by virtue of (15.9.2.8)

$$\int G(\chi)\, dm_F(\chi) = \lim_{n\to\infty} \int |\hat{x}_n(\chi)|^2\, dm_F(\chi) = \lim_{n\to\infty} \int F(\chi)\, dm_{x_n, x_n}(\chi).$$

Hence it is enough to show that, for each $x \in A$, the measure $m_{x,x}$ is *positive*, or equivalently that $\int G(\chi)\, dm_{x,x}(\chi) \geqq 0$ for all functions $G \geqq 0$ belonging to $\mathscr{K}_{\mathbf{R}}(S_g)$. By the same reasoning, we reduce to showing that, for each $y \in A$, we have $\int |\hat{y}(\chi)|^2\, dm_{x,x}(\chi) \geqq 0$; but by (15.9.2.5), this is equivalent to $g(y^*yx, x) \geqq 0$, which can also be written $g(yx, yx) \geqq 0$ (15.6.3). This completes the proof of (15.9.2.11).

(15.9.2.12) *Every function of the form* $F = \hat{x}\bar{\hat{y}}$, *where* $x, y \in A$, *belongs to* Φ, *and we have* $m_F = m_{x,y}$.

We have to show that, for all u, v in A,

$$(\hat{x}\bar{\hat{y}}) \cdot m_{u,v} = (\hat{u}\bar{\hat{v}}) \cdot m_{x,y},$$

or equivalently (in view of (ii)) that, for all $z \in A$,

$$\int \hat{z}(\chi)\hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm_{u,v}(\chi) = \int \hat{z}(\chi)\hat{u}(\chi)\overline{\hat{v}(\chi)}\, dm_{x,y}(\chi).$$

But the left-hand side of this relation is equal to $g(zxy^*u, v)$, and the right-hand side to $g(zuv^*x, y)$. From (15.6.3) and the commutativity of A, we have

$$g(zxy^*u, v) = g(y^*zxu, v) = g(zxu, yv)$$

and

$$g(zuv^*x, y) = g(v^*zux, y) = g(zux, vy) = g(zxu, yv).$$

Hence (15.9.2.12).

We can now prove that *every function* $G \in \mathscr{K}_{\mathbf{C}}(S_g)$ *belongs to* Φ. Let $K = \mathrm{Supp}(G)$; then it is enough to show that there exists a function $F \in \Phi$ which does not vanish on K. For then we can write $G = G'F$, with $G' \in \mathscr{C}^0_{\mathbf{C}}(S_g)$, and the lemma (15.9.2.10) will show that $G \in \Phi$. For each $\chi \in K$, there exists by definition an element $x \in A$ such that $\hat{x}(\chi) \neq 0$, and hence a neighborhood $V(\chi)$ of χ in S_g such that $\hat{x}(\chi') \neq 0$ for all $\chi' \in V(\chi)$. Cover K by a finite number of such neighborhoods $V(\chi_i)$. If x_i are the corresponding elements of A, then the function $F = \sum_i \hat{x}_i \bar{\hat{x}}_i$ does what is required, by virtue of (15.9.2.12).

(15.9.2.13) *Definition of* m_g *and proof of* (i).

From (15.9.2.11), the linear form $F \mapsto m_F(1)$ is a positive measure m_g on S_g. Also, for each function $F \in \Phi$, we have

(15.9.2.14) $$m_F = F \cdot m_g.$$

To prove this, observe that for each $G \in \mathscr{K}_C(S_g)$ we have $m_g(GF) = m_{GF}(1)$ and, by virtue of (15.9.2.10), $m_{GF} = G \cdot m_F$; hence $m_{GF}(1) = \int G(\chi)\, dm_F(\chi)$; but since $m_{GF}(1) = \int G(\chi)F(\chi)\, dm_g(\chi)$ by definition, this implies the relation (15.9.2.14). In particular, bearing in mind (15.9.2.12), we have

$$m_{x,y} = \hat{x}\bar{\hat{y}} \cdot m_g \tag{15.9.2.15}$$

for all x, y in A. Since $m_{x,y}$ is a bounded measure, this relation implies that $\hat{x}\hat{y}$ is m_g-*integrable* (13.14.4). The functions $\hat{x}$ therefore all belong to $\mathscr{L}^2_C(S_g, m_g)$; also, it follows from (15.9.2.15) and (15.9.2.5) that, for all $z \in A$,

$$g(zx, y) = \int \hat{z}(\chi)\, dm_{x,y}(\chi) = \int \hat{z}(\chi)\hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm_g(\chi). \tag{15.9.2.16}$$

In other words, we have proved (15.9.2.1) in the particular case where x is replaced by a *product* zx. It remains to show that (15.9.2.1) is true in general. The remark at the beginning of (15.9.2.3) shows that, for all operators $V \in \mathscr{A}'_g$,

$$(V \cdot \pi_g(x) \,|\, \pi_g(y)) = \int_{S'_g} (\mathscr{G}V)(\omega^{-1}(\chi'))\, d\mu_{x,y}(\chi'); \tag{15.9.2.17}$$

in particular, taking $V = 1_{H_g}$, we have

$$g(x, y) = \int_{S'_g} d\mu_{x,y}(\chi'). \tag{15.9.2.18}$$

But, from (15.9.2.15), we have

$$\int_{S_g} \hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm_g(\chi) = \int_{S_g} dm_{x,y}(\chi).$$

Since the measure $m_{x,y}$ is by definition induced by $\mu_{x,y}$ on S_g, it follows that what we have to prove is that, when $0 \in S'_g$ (and therefore $S_g = S'_g - \{0\}$)

$$\mu_{x,y}(\{0\}) = 0. \tag{15.9.2.19}$$

Now it follows immediately from the definition of $\mu_{x,y}$ that the mapping $(x, y) \mapsto \mu_{x,y}$ is *sesquilinear*, and hence (13.16.1) the complex-valued function $(x, y) \mapsto \mu_{x,y}(\{0\})$ is a *sesquilinear form* on $A \times A$. Moreover, it follows directly from (15.9.2.17) and the Gelfand–Neumark theorem (15.4.14) that, for each continuous function F on the compact space $S'_g = S_g \cup \{0\}$,

$$\left| \int F(\chi')\, d\mu_{x,y}(\chi') \right| \leqq \|F\| \cdot \|\pi_g(x)\| \cdot \|\pi_g(y)\|,$$

hence $\|\mu_{x,y}\| \leqq \|\pi_g(x)\| \cdot \|\pi_g(y)\|$, and therefore

$$|\mu_{x,y}(\{0\})| \leqq \|\pi_g(x)\| \cdot \|\pi_g(y)\|.$$

Since $A/\mathfrak{n}_g$ is dense in H_g, we infer that there exists a vector $E \cdot x \in H_g$ such that $|\mu_{x,y}(\{0\})| = (E \cdot x \mid \pi_g(y))$ for all $y \in A$ (6.3.2), and we have

$$\|E \cdot x\| \leqq \|\pi_g(x)\|.$$

This shows first that $E \cdot x$ depends only on $\pi_g(x)$ and can therefore be written in the form $W_0 \cdot \pi_g(x)$, where W_0 is a linear mapping of $A/\mathfrak{n}_g$ into H_g; also, we have $\|W_0 \cdot \pi_g(x)\| \leqq \|\pi_g(x)\|$, so that W_0 is continuous, and extends to a continuous operator $W \in \mathscr{L}(H_g)$. Next, replacing x by zx in (15.9.2.17) and noting that $V \cdot \pi_g(zx) = (VU_g(z)) \cdot \pi_g(x)$ for $V \in \mathscr{A}'_g$, we obtain

$$\mu_{zx,y} = ((\mathscr{G}U_g(z)) \circ \omega^{-1}) \cdot \mu_{x,y}.$$

Since the function $\chi' \mapsto (\mathscr{G}U_g(z))(\omega^{-1}(\chi'))$ is zero at the point 0, we see that $\mu_{zx,y}(\{0\}) = 0$ for all $z \in A$, or equivalently $(W \cdot \pi_g(zx) \mid \pi_g(y)) = 0$. Since the elements $\pi_g(y)$ are dense in H_g, it follows that $W \cdot \pi_g(zx) = 0$ for all z and all x in A; but *by virtue of the condition* (N) (which is used only at this point of the proof) the $\pi_g(zx)$ form a *total* set in H_g. Hence we have $W = 0$, and the proof of (i) is complete.

(15.9.2.20) *Proof of* (iii).

To show that the support of m_g is the whole of S_g, it is enough to prove that, for each $\chi \in S_g$ and each open neighborhood V of χ in S_g, there exists at least one continuous function $F \geqq 0$, with support contained in V, and such that $m_g(F) \neq 0$. Suppose that this is not the case. Then for every F as above we have

$$\int F(\chi)\, dm_{x,y}(\chi) = \int F(\chi)\hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm_g(\chi) = 0$$

for all x, y in A. But by virtue of the Gelfand–Neumark theorem (15.4.14), F is the restriction to S_g of a function of the form $\mathscr{G}V \circ \omega^{-1}$, where $V \in \mathscr{A}'_g$, and by (15.9.2.17) we have $(V \cdot \pi_g(x) \mid \pi_g(y)) = 0$ for all x, y in A. Since $A/\mathfrak{n}_g$ is dense in H_g, it follows that $V = 0$ and therefore $F = 0$, which is absurd (4.5.2).

(15.9.2.21) *Proof of* (iv).

By virtue of (iii), the vector space $\mathscr{C}^0_{\mathbf{c}}(S_g) \cap \mathscr{L}^2_{\mathbf{c}}(S_g, m_g)$ (and therefore also its subspace $\mathscr{K}_{\mathbf{c}}(S_g)$) can be identified (algebraically) with a dense subspace of $L^2_{\mathbf{c}}(S_g, m_g)$ (13.11.6). Since on the other hand the relation (15.9.2.1) can be put in the form $(\pi_g(x) \mid \pi_g(y)) = \int \hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm_g(\chi)$ and therefore defines

an isomorphism T_0 of the prehilbert space $A/\mathfrak{n}_g$ onto its image in $\mathscr{C}^0_{\mathbf{c}}(S_g)$ such that $T_0 \cdot \pi_g(x) = \hat{x}$, it is enough to show that this image is dense in $\mathscr{C}^0_{\mathbf{c}}(S_g) \cap \mathscr{L}^2_{\mathbf{c}}(S_g, m_g)$, *with respect to the topology of* $\mathscr{L}^2_{\mathbf{c}}(S_g, m_g)$. Given $\varepsilon > 0$ and any function $F \in \mathscr{K}_{\mathbf{c}}(S_g)$, it is sufficient to prove the existence of $x \in A$ such that

$$\int |F(\chi) - \hat{x}(\chi)|^2 \, dm_g(\chi) \leqq \varepsilon^2.$$

Now, we have seen earlier that there exists a function $\hat{u} = \sum_i \hat{x}_i \overline{\hat{x}_i}$ $(x_i \in A)$ which does not vanish on Supp(F), so that we may write $F = G\hat{u}$, where $G \in \mathscr{K}_{\mathbf{c}}(S_g)$. By virtue of (ii), there exists $y \in A$ such that

$$|G(\chi) - \hat{y}(\chi)| \leqq \varepsilon / N_2(\hat{u})$$

for all $\chi \in S_g$, and it follows that

$$\int |F(\chi) - \hat{u}(\chi)\hat{y}(\chi)|^2 \, dm_g(\chi) \leqq (\varepsilon / N_2(\hat{u}))^2 \int |\hat{u}(\chi)|^2 \, dm_g(\chi) = \varepsilon^2.$$

Finally, for x and y in A, we have by definition

$$T_0 \cdot (U_g(x) \cdot \pi_g(y)) = T_0 \cdot \pi_g(xy) = \hat{x}\hat{y} = M(\hat{x}) \cdot (T_0 \cdot \pi_g(y)),$$

and by extending T_0 by continuity to an isomorphism T of H_g onto $L^2_{\mathbf{c}}(S_g, m_g)$ we have $TU_g(x) = M(\hat{x})T$. The equality of the norm of $\lambda \cdot 1_{H_g} + U_g(x)$ in $\mathscr{L}(H_g)$ and the norm of $\lambda + \hat{x}$ in $\mathscr{C}_{\mathbf{c}}(S'_g)$ is a consequence of the Gelfand–Neumark theorem (15.4.14).

(15.9.2.22) *Uniqueness of* S_g *and* m_g.

It is immediately checked that the hermitian form g' satisfies (15.6.3) and (15.7.3), hence is a bitrace on A. Since the functions $\hat{x}$ are bounded, we have

$$g'(xy, xy) = \int |\hat{x}(\chi)|^2 |\hat{y}(\chi)|^2 \, dm(\chi) \leqq \|\hat{x}\|^2 g'(y, y),$$

so that g' satisfies the condition (U). To prove that g' satisfies (N), let S′ denote the closure of S in $\mathbf{H}(A) \cup \{0\}$, so that $S' = S$ if S is compact, and $S' = S \cup \{0\}$ otherwise. By virtue of (13.11.6), it is enough to show that the functions $\lambda + \hat{x}\hat{y}$, where $\lambda \in \mathbf{C}$ and $x, y \in A$, form a *total* set in the space $\mathscr{C}_{\mathbf{c}}(S')$. The proof is the same as in (15.9.2.11), using the Stone–Weierstrass theorem and the fact that by definition the functions $\hat{x}$ separate the points of $S \cup \{0\}$. The ideal $\mathfrak{n}_{g'}$ is the set of all $x \in A$ for which $\hat{x}$ is m-negligible; but since the support of m is S, and $\hat{x}$ is continuous, $\hat{x}$ can be m-negligible only if $\hat{x} = 0$, and therefore $\pi_{g'}(x) \mapsto \hat{x}$ is an isomorphism of $A/\mathfrak{n}_{g'}$ onto the algebra of the functions $\hat{x}$ such that $(\pi_{g'}(x) \mid \pi_{g'}(y)) = \int \hat{x}(\chi)\overline{\hat{y}(\chi)} \, dm(\chi)$. The argument of (15.9.2.21) then shows that the set of functions $\hat{x}$ is dense in $\mathscr{L}^2_{\mathbf{c}}(S, m)$, and

therefore the isomorphism above extends to an isomorphism of Hilbert spaces $T' : H_{g'} \to L^2_{\mathbf{C}}(S, m)$, such that $U_{g'}(x) = T'^{-1} M(\hat{x}) T'$. Let us now show that

$$\|\lambda \cdot 1_{H_{g'}} + U_{g'}(x)\| = \|\lambda + \hat{x}\|,$$

or equivalently that, if $F \in \mathscr{C}_{\mathbf{C}}(S')$, the continuous operator $M(F)$ ("multiplication by F") on the Hilbert space $L^2_{\mathbf{C}}(S, m)$ has norm equal to $\|F\|$. It follows immediately from (13.12.5) that $\|M(F)\| \leqq \|F\|$. On the other hand, there exists $\chi_1 \in S'$ such that $|F(\chi_1)| = \|F\|$ (3.17.10); hence, for all $\varepsilon > 0$, there exists a compact neighborhood V of χ_1 in S' such that $|F(\chi)| \geqq \|F\| - \varepsilon$ for all $\chi \in V$. Since by hypothesis the support of m is the whole of S, and since in all cases χ_1 lies in the closure of S in S', we have $m(V \cap S) = \int \varphi_{V \cap S}\, dm = (N_2(\varphi_{V \cap S}))^2 > 0$; also it is clear that $|F\varphi_{V \cap S}| \geqq (\|F\| - \varepsilon)\varphi_{V \cap S}$, so that

$$N_2(F\varphi_{V \cap S}) \geqq (\|F\| - \varepsilon) N_2(\varphi_{V \cap S})$$

and consequently $\|M(F)\| \geqq \|F\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves our assertion. Hence the isomorphism $\lambda \cdot 1_{H_{g'}} + U_{g'}(x) \mapsto \lambda + \hat{x}$ extends by continuity to an *isometry* of $\mathscr{A}'_{g'}$ onto $\mathscr{C}_{\mathbf{C}}(S')$. Having regard to the fact that every character of $\mathscr{C}_{\mathbf{C}}(S')$ is of the form $F \mapsto F(\chi')$ for some $\chi' \in S'$ (15.3.7), we conclude that $S'_{g'} = S'$ and hence that $S_{g'} = S$. It remains to show that $m_{g'} = m$, or equivalently that $\int F(\chi)\, dm_{g'}(\chi) = \int F(\chi)\, dm(\chi)$ for all functions $F \in \mathscr{K}_{\mathbf{C}}(S)$. But by virtue of (i) we have

$$\int \hat{z}(\chi)\hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm_{g'}(\chi) = \int \hat{z}(\chi)\hat{x}(\chi)\overline{\hat{y}(\chi)}\, dm(\chi),$$

and since the functions $\hat{z}$, as z runs through A, form a total set in $\mathscr{C}^0_{\mathbf{C}}(S)$, this establishes the equality of the bounded measures $\hat{x}\bar{\hat{y}} \cdot m_{g'}$ and $\hat{x}\bar{\hat{y}} \cdot m$ for all x, y in A. On the other hand, we have seen earlier that there exists a function $\hat{u} = \sum_i \hat{x}_i \bar{\hat{x}}_i$ (where the x_i belong to A) which does not vanish on Supp(F), so that $F = G\hat{u}$ with $G \in \mathscr{K}_{\mathbf{C}}(S)$. Consequently we have from above

$$\begin{aligned}
\int F(\chi)\, dm_{g'}(\chi) &= \int G(\chi)\hat{u}(\chi)\, dm_{g'}(\chi) \\
&= \sum_i \int G(\chi)\hat{x}_i(\chi)\overline{\hat{x}_i(\chi)}\, dm_{g'}(\chi) \\
&= \sum_i \int G(\chi)\hat{x}_i(\chi)\overline{\hat{x}_i(\chi)}\, dm(\chi) \\
&= \int G(\chi)\hat{u}(\chi)\, dm(\chi) = \int F(\chi)\, dm(\chi).
\end{aligned}$$

This completes the proof of the Plancherel–Godement theorem.

The Plancherel–Godement theorem applies in particular when the bitrace g is of the form $(x, y) \mapsto f(y^*x)$, where f is a positive linear form on A (and therefore a *trace* since A is commutative). When g comes from a trace f, the formula (15.9.2.1) leads us to ask whether we also have

$$f(x) = \int_{S_g} \hat{x}(\chi)\, dm_g(\chi). \tag{15.9.3}$$

A partial answer to this question is provided by the following theorem:

(15.9.4) (Bochner–Godement theorem) (i) *Let f be a positive linear form on a commutative algebra* A *with involution, such that the bitrace $g(x, y) = f(xy^*)$ satisfies the hypotheses of* (15.9.2). *Then, if the formula* (15.9.3) *is true and if the measure m_g is bounded, f satisfies the condition*

(B) *There exists a real number $M > 0$ such that $|f(x)|^2 \leqq M \cdot f(xx^*)$ for all $x \in A$.*

(ii) *Conversely, let f be a positive linear form on* A *which satisfies the condition* (B), *and suppose that the corresponding bitrace $g(x, y) = f(xy^*)$ satisfies* (U) *and is such that the prehilbert space $A/\mathfrak{n}_g$ and the star algebra $\mathscr{A}_g$ are separable. Then g also satisfies* (N), *the measure m_g is bounded, and the formula* (15.9.3) *is valid.*

(i) If m_g is bounded and if (15.9.3) holds, then the Cauchy–Schwarz inequality (13.11.2.2) gives us

$$\left| \int \hat{x}\, dm_g \right|^2 \leqq m_g(S_g) \cdot \int |\hat{x}|^2\, dm_g$$

and hence the inequality of (B), with $M = m_g(S_g)$.

(ii) Recall that the definition of the Hilbert space H_g does not presuppose that the condition (N) is satisfied. The inequality in (B) can be put in the form $|f(x)|^2 \leqq M\|\pi_g(x)\|^2$, and shows that f vanishes on $\mathfrak{n}_g$; hence we may write $f = f' \circ \pi_g$, where f' is a linear form on $A/\mathfrak{n}_g$. Also $|f'(\pi g(x))|^2 \leqq M\,\|\pi_g(x)\|^2$, so that f' is *continuous* on the prehilbert space $A/\mathfrak{n}_g$ (5.5.1) and therefore extends to a continuous linear form on the Hilbert space H_g (5.5.4). Hence, by (6.4.2), there exists a well-determined vector $a \in H_g$ such that

$$f(x) = (\pi_g(x) \mid a). \tag{15.9.4.1}$$

From this it follows that, for all $x \in A$, we have

$$U_g(x) \cdot a = \pi_g(x). \tag{15.9.4.2}$$

For if $y \in A$, then

$$(\pi_g(y) \mid U_g(x) \cdot a) = (U_g(x^*) \cdot \pi_g(y) \mid a) = (\pi_g(x^*y) \mid a) = f(x^*y) = (\pi_g(y) \mid \pi_g(x)),$$

and (15.9.4.2) follows because $A/\mathfrak{n}_g$ is dense in H_g.

We are now in a position to show that g satisfies the condition (N). Let $b \in H_g$ be a vector belonging to the orthogonal supplement of the subspace generated by the elements $\pi_g(xy)$ for all x, y in A; then we have $(\pi_g(xy) \mid b) = 0$, that is $(U_g(x) \cdot \pi_g(y) \mid b) = 0$, or again $(\pi_g(y) \mid U_g(x^*) \cdot b) = 0$ for all x and y in A. Since the elements $\pi_g(y)$ form a dense subspace of H_g, it follows that $(a \mid U_g(x^*) \cdot b) = 0$, hence $(U_g(x) \cdot a \mid b) = 0$, hence finally $(\pi_g(x) \mid b) = 0$ and therefore $b = 0$, because $A/\mathfrak{n}_g$ is dense in H_g.

Now consider the positive linear form $f''(V) = (V \cdot a \mid a)$ on the algebra $\mathscr{A}'_g$. We have

$$|f''(V)| \leqq \|V \cdot a\| \cdot \|a\| \leqq \|V\| \cdot \|a\|^2;$$

by virtue of the Gelfand–Neumark theorem (15.4.14), we may write $f''(V) = h(\mathscr{G}V)$ where h is a linear form on $\mathscr{C}_{\mathbf{C}}(\mathbf{X}(\mathscr{A}'_g))$, and since $\|\mathscr{G}V\| = \|V\|$, it follows that h is an (*a priori* complex) measure on the compact space $\mathbf{X}(\mathscr{A}'_g)$. Since any continuous function $G \geqq 0$ on $\mathbf{X}(\mathscr{A}'_g)$ is of the form $F \cdot \overline{F}$, hence of the form $\mathscr{G}V \cdot \mathscr{G}V^* = \mathscr{G}(VV^*)$, we have $h(G) = \|V \cdot a\|^2 \geqq 0$, so that h is a *positive* measure. Taking into account (15.9.4.2), (15.9.4.1) and the canonical homeomorphism ω of $\mathbf{X}(\mathscr{A}'_g)$ onto S'_g, we see therefore that there exists a positive measure v on S_g (induced by the measure $\omega(h)$ on S'_g, and therefore *bounded*) such that

$$f(x) = (U_g(x) \cdot a \mid a) = \int \hat{x}(\chi)\, dv(\chi)$$

for all $x \in A$.

Thus it remains to show that $v = m_g$. Since

$$g(x, y) = \int \hat{x}(\chi)\overline{\hat{y}(\chi)}\, dv(\chi)$$

it is sufficient, by virtue of (15.9.2(II)) to show that (1) the functions $\hat{x}$ (where $x \in A$) belong to $\mathscr{L}^2_{\mathbf{C}}(S_g, v)$; (2) the support of v is S_g. The first assertion is trivial, because the functions $\hat{x}$ are bounded and continuous and the measure v is bounded. To prove (2), suppose that there exists a function $F \geqq 0$ belonging to $\mathscr{K}_{\mathbf{C}}(S_g)$ such that $\int F(\chi)\, dv(\chi) = 0$. Then $\int F(\chi)\hat{x}(\chi)\overline{\hat{y}(\chi)}\, dv(\chi) = 0$ for all $x, y \in A$. Since $F \circ \omega$ is a continuous function on $\mathbf{X}(\mathscr{A}'_g)$, it is of the form $\mathscr{G}V$ where $V \in \mathscr{A}'_g$, and the relation above now takes the form

$$(U_g(y^*)VU_g(x) \cdot a \mid a) = 0,$$

or

$$(VU_g(x) \cdot a \mid u_g(y) \cdot a) = 0.$$

But, by virtue of (15.9.4.2), $(V \cdot \pi_g(x) \mid \pi_g(y)) = 0$ for all x, y in A. Since $A/\mathfrak{n}_g$ is dense in H_g, it follows that $V = 0$ and hence $F = 0$. Q.E.D.

Examples

(15.9.5) We have seen (15.6.2.4) that a positive linear form f on an algebra with involution A *having a unit element* always satisfies the condition (B). We recall that if A is in addition a *Banach algebra*, then the corresponding Hilbert form g also satisfies (U) (15.6.11). If A is a *separable Banach algebra* with unit element e, then the prehilbert space $A/\mathfrak{n}_g$ and the star algebra $\mathscr{A}_g$ are also separable. The first assertion follows from the *continuity* of f (15.6.11), which implies that $\|\pi_g(x)\|^2 = f(x^*x) \leqq \|f\| \cdot \|x^*x\| \leqq \|f\| \cdot \|x\|^2$ and shows therefore that the image under π_g of a denumerable dense set in A is dense in the prehilbert space $A/\mathfrak{n}_g$. The separability of $\mathscr{A}_g$ follows from the fact that the representation U_g is *continuous* (15.5.7) and therefore transforms a denumerable dense set in A into a denumerable dense set in $\mathscr{A}_g$. Hence the Bochner–Godement theorem can be applied to *every* positive linear form on A.

(15.9.6) If $s \mapsto U(s)$ is a representation of A on a Hilbert space H, then for each $x_0 \in H$ the form $f_{x_0}(s) = (U(s) \cdot x_0 \mid x_0)$ satisfies the condition (B), because by (6.2.4)

$$|f_{x_0}(s)|^2 \leqq \|U(s) \cdot x_0\|^2 \cdot \|x_0\|^2 = \|x_0\|^2 f_{x_0}(s^*s).$$

By (15.5.6) this shows that knowledge of the *hermitian characters* of a commutative algebra A with involution which has a *unit element* determines *all the representations of* A. In addition, we have seen in the course of the proof of (15.9.4) that, if f is a positive linear form satisfying the conditions of (15.9.4(ii)), then the corresponding representation $x \mapsto U_g(x)$ admits a *totalizer a* (15.9.4.2).

Remarks

(15.9.7) It can happen that the formula (15.9.3) (and the conditions of (15.9.2)) are satisfied but that the measure m_g is not bounded (and therefore f does not satisfy (B)) (Problem 2). On the other hand, there are examples in which the conditions (U) and (N) are satisfied but (B) is not (Problem 4), and examples in which (B) and (N) are satisfied but (U) is not (Problem 5).

PROBLEMS

1. Let A be the algebra with involution $\mathscr{C}_{\mathbf{C}}^{\infty}(\mathbf{R})$ of continuous bounded complex-valued functions on $\mathbf{R}$, and let μ be a *bounded* positive measure on $\mathbf{R}$, with support equal to $\mathbf{R}$. Show that

$$g(x, y) = \int x(t)\overline{y(t)}\, d\mu(t)$$

is a bitrace on A for which the prehilbert space $A/\mathfrak{n}_g = \mathrm{A}$ is separable, but that the star algebra $\mathscr{A}_g$ is not separable (Section 7.4, Problem 4).

2. Let λ be Lebesgue measure on $\mathbf{R}$ and let A be the subalgebra with involution of $\mathscr{C}_{\mathbf{C}}^{0}(\mathbf{R})$ consisting of square-λ-integrable functions. Then $g(x, y) = \int x(t)\overline{y(t)}\, d\lambda(t)$ is a bitrace on A satisfying (U) and (N), but the measure m_g is not bounded, and the right-hand side of (15.9.3) is not defined for all $x \in \mathrm{A}$. If $\mathrm{B} \subset \mathrm{A}$ is the subalgebra of $\mathscr{C}_{\mathbf{C}}^{0}(\mathbf{R})$ consisting of λ-integrable functions, then

$$f(x) = \int x(t)\, d\lambda(t)$$

is a positive linear form on B satisfying (15.9.3) but the measure m_g is not bounded.

3. Let A be the subalgebra with involution of $\mathscr{C}_{\mathbf{C}}(\mathrm{I})$, where $\mathrm{I} = [0, 1]$, consisting of functions x with continuous second-order derivative on I and such that $x(0) = 0$. Show that

$$f(x) = \int_0^1 x(t)\, dt + x''(0)$$

is a positive linear form on A, such that the corresponding bitrace g satisfies the condition (U) but not the condition (N). (If (f_n) is a sequence of functions belonging to $\mathscr{C}_{\mathbf{C}}(\mathrm{I})$, with values in $[0, 1]$ and continuous second derivatives, and such that $f_n(t) = 1$ in a neighborhood of 0 and $f_n(t) = 0$ for $t \geqq 1/n$, consider in the Hilbert space H_g the sequence of functions $x_n \in \mathrm{A}$ such that $x_n(t) = t f_n(t)$.)

4. Let A be the subalgebra with involution of $\mathscr{C}_{\mathbf{C}}(\mathrm{I})$ consisting of continuously differentiable functions on I which vanish at 0. Show that the form $f(x) = x'(0)$ on I is a positive linear form for which the corresponding bitrace g is zero (and therefore satisfies (U) and (N)) but does not satisfy (B).

5. Let A be the algebra with involution (the involution being $x \mapsto \bar{x}$) of complex-valued functions on $[0, 1]$ of the form $\mathrm{P}(t, \log t)$, where P is a polynomial in two variables with complex coefficients. Then the linear form $f(x) = \int_0^1 x(t)\, dt$ is positive, and satisfies (B) because A has a unit element; the corresponding bitrace g satisfies (N) but not (U).

6. Let Γ be a set endowed with an associative law of composition $(x, y) \mapsto xy$ and a neutral element e, and let $x \mapsto x^*$ be an *involution* on Γ (i.e., a bijection of Γ onto Γ such that $e^* = e$, $(x^*)^* = x$ and $(xy)^* = y^*x^*$). A *representation* of Γ in a Hilbert

is a mapping $x \mapsto U(x)$ of Γ into $\mathscr{L}(H)$ such that $U(e) = 1_H$, $U(xy) = U(x)U(y)$ $x^*) = U(x)^*$.

E be a Hilbert space and $x \mapsto T(x)$ a mapping of Γ into $\mathscr{L}(E)$. In order that ould exist a Hilbert space H which is the Hilbert sum of E and another Hilbert , and a representation $x \mapsto U(x)$ of Γ on H such that $T(x) = PU(x) \mid E$ for all where P is the orthogonal projection of H on E, it is necessary and sufficient that d satisfy the following three conditions:

$T(e) = 1_E$ and $T(x^*) = T(x)^*$ for all $x \in E$.

If $g : \Gamma \to E$ is any mapping such that $g(x) = 0$ for all but a finite set of ele- of Γ, then

$$\sum_{(x,\, y) \in \Gamma \times \Gamma} (T(x^*y) \cdot g(y) \mid g(x)) \geqq 0.$$

If $g : \Gamma \to E$ is any mapping such that $g(x) = 0$ for all but a finite set of elements hen for each $z \in \Gamma$ there exists a constant $M_z > 0$ such that

$$\sum_{(x,\, y) \in \Gamma \times \Gamma} (T(x^*z^*zy) \cdot g(y) \mid g(x)) \leqq M_z \cdot \sum_{(x,\, y) \in \Gamma \times \Gamma} (T(x^*y) \cdot g(y) \mid g(x)).$$

reover, if U and H satisfy these conditions and are such that the elements $(x \in \Gamma$ and $f \in H)$ form a total set in H, then the representation U is deter- p to equivalence (equivalence of representations being defined as in (15.5)).

show that these conditions are sufficient, let G be the subspace of E^Γ consisting pings $g : \Gamma \to E$ such that $g(x) = 0$ for all but a finite set of elements $x \in \Gamma$, nsider the form on $G \times G$

$$(g, h) \mapsto B(g, h) = \sum_{(x,\, y) \in \Gamma \times \Gamma} (T(x^*y) \cdot g(y) \mid h(x))$$

s a positive hermitian form. If N is the set of all $g \in G$ such that $B(g, g) = 0$, induces a nondegenerate positive hermitian form $(\dot{g}, \dot{h}) \mapsto (\dot{g} \mid \dot{h}) = B(\dot{g}, \dot{h})$ on nich makes G/N a prehilbert space. Assume that G/N is a dense subspace of a space H_0. Define an injection $j : G/N \to E^\Gamma$ as follows: for each $g \in G$, the $\dot{g})$ of $\dot{g}$ is the mapping

$$x \mapsto \sum_{y \in \Gamma} T(x^*y) \cdot g(y).$$

sporting the scalar product on G/N by means of j, we have a structure of pre- space on $j(G/N)$ and hence, extending by continuity, an isomorphism of H_0 Hilbert space H. Then define $U(x)$ by the condition that $U(x) \cdot j(\dot{g})$ is the mapping $T(z^*xy) \cdot g(y)$.)

be a separable commutative Banach algebra with involution, having a unit e. Let P be the set of positive linear forms on A (or of *traces* on A), which is a of the dual A′ of the Banach space A (15.6.11). If $f_1, f_2 \in P$, we write $f_1 \leqq f_2$ to at $f_2 - f_1 \in P$.

f, f_0 are traces on A such that $f_0 \leqq f$, show that there exists a sequence (y_n) of of A such that $f_0(x) = \lim_{n \to \infty} f(y_n^* x)$ for all $x \in A$. (If g is the bitrace correspond- , note that f_0 may be written in the form $f_0 = u \circ \pi_g$, where u is a continuous orm on H_g.)

P_1 be the set of traces $f \in P$ such that $\|f\| = f(e) \leqq 1$; then P_1 is convex and compact. Show that the set of extremal points of P_1 (Section 12.15, Problem 5) to $(P \cap \mathbf{X}(A)) \cup \{0\}$. (Use (a) to show that if a character is a trace it must

be an extremal point of P_1. To prove the converse, show that if $f \in P_1$ is such that $1 = \|f\| = |f(e)|$ then there exists $z \in A$ such that $\|z\| < 1$ and $f(z^*z) \neq 0$. Using (15.6.11.1), show that the linear forms $f_1(x) = f(z^*zx)/f(z^*z)$ and

$$f_2(x) = f((e - z^*z)x)/(f(e) - f(z^*z))$$

belong to P_1, and that if f is an extremal point then $f = f_1 = f_2$. Hence we have $f(z^*zx) = f(x)f(z^*z)$; replacing z by $z(e + y)$ where y is small, deduce that f is a character.)

(c) Show that for each $f \in P_1$ there exists a unique positive measure μ_f on P_1, with mass 1, such that for each $x \in A$ we have

$$f(x) = \int_{P \cap X(A)} \hat{x}(\chi)\, d\mu_f(\chi).$$

(Use Section 13.10, Problem 2(b). For the uniqueness of μ_f, use the Stone-Weierstrass theorem.)

10. REPRESENTATIONS OF ALGEBRAS OF CONTINUOUS FUNCTIONS

Let K be a *compact metrizable* space. The application of the results of Section 15.9 to the case where $A = \mathscr{C}_{\mathbf{C}}(K)$ allows us to describe very simply *all* the representations of this algebra. We consider first the *topologically cyclic* representations $u \mapsto T(u)$.

(15.10.1) *Let* K *be a compact metrizable space and* $A = \mathscr{C}_{\mathbf{C}}(K)$. *Then every topologically cyclic representation of the commutative algebra with involution* A, *in a separable Hilbert space* E, *is equivalent to one of the representations* $u \mapsto M_\mu(u)$ *defined as follows: let* μ *be a positive measure on* K, *and for each* $u \in A$, *let* $M_\mu(u)$ *denote the continuous operator on* $L^2_{\mathbf{C}}(K, \mu)$ *induced by multiplication by* u: *that is, for each* $f \in \mathscr{L}^2_{\mathbf{C}}(K, \mu)$, $M_\mu(u) \cdot \tilde{f}$ *is the class of* uf *in* $L^2_{\mathbf{C}}(K, \mu)$.

Let a be a totalizing vector for a representation $u \mapsto T(u)$ of A in a separable Hilbert space E. The representation T is determined up to equivalence by the positive linear form $f_a(u) = (T(u) \cdot a \mid a)$ on A (15.6). Since A is separable (7.4.4), the Bochner–Godement theorem applies; all the characters of A are hermitian, and the spectrum $\mathbf{X}(A)$ can be canonically identified with K (15.3.7). Hence the proposition is an immediate consequence of (15.9.2(iv)).

It follows from the definition (15.10.1) of M_μ that

$$\|M_\mu(u)\| = \operatorname*{ess\,sup}_{t \in K} |u(t)| = N_\infty(u) \tag{15.10.2}$$

(with respect to the measure μ). For it follows from (13.12.2) that $\|M_\mu(u)\| \leqq N_\infty(u)$; also, for every positive real number $\alpha < N_\infty(u)$, there exists a nonnegligible integrable subset P of K such that $|u(t)| \geqq \alpha$ for all $t \in P$; hence $N_2(u\varphi_P) \geqq \alpha \cdot N_2(\varphi_P)$, from which (15.10.2) follows.

The measure μ is not uniquely determined. We shall come back to this point a little later (15.10.7).

The definition of $M_\mu(u)$ given in (15.10.1), and the formula (15.10.2), still make sense if u is not necessarily continuous on K, but simply *μ-measurable* and *bounded in measure* (by virtue of (13.12.5)). The mapping $u \mapsto M_\mu(u)$ extended in this manner is a *representation* of the algebra with involution $\mathscr{L}^\infty_{\mathbf{C}}(K, \mu)$ on the Hilbert space $L^2_{\mathbf{C}}(K, \mu)$. Clearly, if u_1 and u_2 are equal *almost everywhere* with respect to μ, we have $M_\mu(u_1) = M_\mu(u_2)$. Usually we shall restrict the representation M_μ to a self-adjoint subalgebra of $\mathscr{L}^\infty_{\mathbf{C}}(K, \mu)$ which does not depend on the choice of μ, namely the algebra $\mathscr{U}_{\mathbf{C}}(K)$ of *universally measurable, bounded complex-valued functions* on K (13.9). By (13.9.8.1) this is indeed a C-algebra with involution, and it is a *Banach subalgebra* of $\mathscr{B}_{\mathbf{C}}(K)$ by virtue of Egoroff's theorem (13.9.10).

(15.10.3) Again let $u \mapsto T(u)$ be a representation of $\mathscr{C}_{\mathbf{C}}(K)$ in a separable Hilbert space E, admitting a totalizing vector a. (These hypotheses will be in force up to and including (15.10.7).) The preceding remarks show that the representation T may be *extended* to the algebra with involution $\mathscr{U}_{\mathbf{C}}(K)$. The extended representation (also denoted by T) *does not depend* on the choice of totalizing vector a.

To prove this assertion, let x and y be two vectors belonging to E. For each $u \in \mathscr{C}_{\mathbf{C}}(K)$, we have $\|T(u)\| \leqq \|u\|$ (15.5.7) and therefore

$$|(T(u) \cdot x \mid y)| \leqq \|u\| \cdot \|x\| \cdot \|y\|; \tag{15.10.3.1}$$

consequently the linear form $u \mapsto (T(u) \cdot x \mid y)$ is a *measure* $\mu_{x,y}$ on K such that $\|\mu_{x,y}\| \leqq \|x\| \cdot \|y\|$.

From this definition it follows immediately that

$$\mu_{y,x} = \bar{\mu}_{x,y}. \tag{15.10.3.2}$$

Also, if $x = T(v) \cdot a$ and $y = T(w) \cdot a$, where v and w are in $\mathscr{U}_{\mathbf{C}}(K)$, then for each function $u \in \mathscr{U}_{\mathbf{C}}(K)$ we have

$$(T(u) \cdot x \mid y) = (T(uv) \cdot a \mid T(w) \cdot a) = (T(\bar{w}uv) \cdot a \mid a) = \int \bar{w}uv \, d\mu$$

and therefore

(15.10.3.3) $$\mu_{T(v)\cdot a,\, T(w)\cdot a} = (v\overline{w})\cdot\mu.$$

The independence asserted above will be a consequence of the following more precise result:

(15.10.4) (i) *For all* $x, y \in \mathrm{E}$ *and all* $u \in \mathscr{U}_{\mathrm{C}}(\mathrm{K})$ *we have*

(15.10.4.1) $$(T(u)\cdot x \mid y) = \int u(t)\, d\mu_{x,y}(t).$$

(ii) *If* (u_n) *is a uniformly bounded sequence of functions in* $\mathscr{U}_{\mathrm{C}}(\mathrm{K})$ *which converges to* u, *then*

(15.10.4.2) $$(T(u)\cdot x \mid y) = \lim_{n\to\infty} (T(u_n)\cdot x \mid y),$$

for all x, y *in* E.

It is enough to prove (15.10.4.2) for x, y lying in a *total* subspace of E: the sesquilinear functions $(x, y) \mapsto (T(u_n)\cdot x \mid y)$ form an *equicontinuous* set, because

$$\begin{aligned}&|(T(u_n)\cdot x \mid y) - (T(u_n)\cdot x_0 \mid y_0)|\\ &\quad \leqq \|u_n\|(\|x - x_0\|\cdot\|y_0\| + \|x_0\|\cdot\|y - y_0\| + \|x - x_0\|\cdot\|y - y_0\|)\end{aligned}$$

(7.5.5). Take $x = T(s_1)\cdot a$ and $y = T(s_2)\cdot a$, with s_1 and s_2 in $\mathscr{C}_{\mathrm{C}}(\mathrm{K})$; then, by definition,

$$(T(u_n)\cdot x \mid y) = (T(\bar{s}_2\, u_n s_1)\cdot a \mid a) = \int s_2\, u_n s_1\, d\mu,$$

and it is enough to apply the dominated convergence theorem (13.8.4) to the measure μ. As to (15.10.4.1), it is valid by definition when $u \in \mathscr{C}_{\mathrm{C}}(\mathrm{K})$, and in general it follows by applying the dominated convergence theorem twice to the measure $\mu_{x,y}$, and using (13.7.1).

(15.10.5) *For the operator* $T(u)$ *to be hermitian* (resp. *positive hermitian* (11.5), resp. *zero*, resp. *unitary*) *it is necessary and sufficient that* $u(x)$ *should be real* (resp. $u(x) \geqq 0$, resp. $u(x) = 0$, resp. $|u(x)| = 1$) *almost everywhere with respect to the measure* μ.

It is immediately verified that if U is a hermitian (resp. positive hermitian, resp. unitary) operator on a Hilbert space E, and if S is an isomorphism of E onto a Hilbert space E', then the operator SUS^{-1} on E' has the same property (since these properties involve only the Hilbert structure). Hence we may take $T(u) = M_\mu(u)$, and in this case the sufficiency of the conditions stated is clear. On the other hand, if for example there exists a measurable subset X of K such that $\mu(X) = \alpha > 0$ and $\mathscr{F}(u(x)) \geqq \beta > 0$ for all $x \in X$, then we have $\mathscr{F}((T(u) \cdot \tilde{\varphi}_X \mid \tilde{\varphi}_X)) = \mathscr{F}\left(\int_X u \, d\mu\right) \geqq \alpha\beta$ and therefore $T(u)$ is not hermitian. The other cases are dealt with similarly.

(15.10.6) (i) *The orthogonal projectors belonging to the algebra* $T(\mathscr{U}_c(K))$ *are the operators of the form* $T(\varphi_X)$, *where* X *is a universally measurable subset of* K.

(ii) $T(\mathscr{U}_c(K))$ *is a maximal commutative subalgebra of* $\mathscr{L}(E)$.

(iii) *A closed vector subspace* F *of* E *is stable under* T *if and only if it is of the form* $T(\varphi_X)(E)$, *where* X *is a universally measurable subset of* K.

(i) By virtue of the characterization of orthogonal projectors on E (15.5.3.1) and by (15.10.5), it follows that $T(u)$ is an orthogonal projector if and only if u is almost everywhere equal to a μ-measurable bounded real-valued function v such that $v^2 = v$. Hence $v = \varphi_X$, where X is a μ-measurable subset of K, and the result follows.

(ii) It is clearly enough to prove that a continuous operator $V \in \mathscr{L}(E)$ which commutes with $T(u)$ for all $u \in \mathscr{U}_c(K)$ is of the form $T(v)$. If a is a totalizing vector for T, then for each universally measurable subset X of K we have

$$\left|\int \varphi_X \, d\mu_{V\cdot a,a}\right| = |(T(\varphi_X)V \cdot a \mid a)| = |(VT(\varphi_X) \cdot a | T(\varphi_X) \cdot a)|$$

$$\leqq \|V\| \cdot \|T(\varphi_X) \cdot a\|^2 = \|V\| \cdot \int \varphi_X \, d\mu_{a,a} = \|V\| \cdot \int \varphi_X \, d\mu$$

because $T(\varphi_X)$ is an orthogonal projector which commutes with V. The Lebesgue–Nikodym theorem (13.15.5(c')) therefore shows that $\mu_{V\cdot a,a} = h \cdot \mu$ for some μ-integrable function h. Since also it follows from the inequality above that

$$\left|\int w \, d\mu_{V\cdot a,a}\right| \leqq \|V\| \cdot N_1(w)$$

for every *step* function $w \in \mathscr{U}_c(K)$, we conclude that the linear form $w \mapsto \int hw \, d\mu$ defined on the vector space of these step functions extends by

continuity to $\mathscr{L}^1_{\mathbf{C}}(\mathrm{K}, \mu)$ ((13.9.12) and (13.9.13)). Hence, by (13.17.1), h is bounded in measure with respect to μ, and we may assume that $h \in \mathscr{U}_{\mathbf{C}}(\mathrm{K})$. For all s, $t \in \mathscr{U}_{\mathbf{C}}(\mathrm{K})$ we have

$$(VT(s) \cdot a \mid T(t) \cdot a) = (T(s)V \cdot a \mid T(t) \cdot a) = \int s\bar{t}\, d\mu_{V \cdot a, a}$$

$$= \int s\bar{t}h\, d\mu = (T(h)T(s) \cdot a \mid T(t) \cdot a);$$

since the vectors $T(s) \cdot a$ form a total set in E, it follows that $V = T(h)$.

(iii) To say that F is stable with respect to T means that the orthogonal projector P_{F} commutes with $T(u)$ for all $u \in \mathscr{U}_{\mathbf{C}}(\mathrm{K})$ (15.5.3); hence $P_{\mathrm{F}} \in T(\mathscr{U}_{\mathbf{C}}(\mathrm{K}))$ by virtue of (ii), and is of the form $T(\varphi_{\mathrm{X}})$ by virtue of (i).

We can also characterize *all* the totalizing vectors of the given representation:

(15.10.7) *For a vector $\tilde{g} \in \mathrm{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$ to be totalizing for the representation M_μ of $\mathscr{C}_{\mathbf{C}}(\mathrm{K})$, it is necessary and sufficient that $g(t) \neq 0$ almost everywhere with respect to μ.*

In order that the classes $(fg)^{\sim}$, where f runs through $\mathscr{C}_{\mathbf{C}}(\mathrm{K})$, should generate a subspace which is not dense in the Hilbert space $\mathrm{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$, it is necessary and sufficient that there should exist a nonnegligible $h \in \mathscr{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$ such that $\int fgh\, d\mu = 0$ for all $f \in \mathscr{C}_{\mathbf{C}}(\mathrm{K})$ (6.3.1). Also we have $gh \in \mathscr{L}^1_{\mathbf{C}}(\mathrm{K}, \mu)$ (13.11.7); hence the measure $(gh) \cdot \mu$ is zero, which implies that $g(t)h(t) = 0$ almost everywhere with respect to μ (13.14.4). But since by hypothesis the integrable set A of points $t \in \mathrm{K}$ at which $h(t) \neq 0$ is not negligible, we must have $g(t) = 0$ almost everywhere in A. This proves the proposition.

To such a totalizing vector $\tilde{g}$ for M_μ corresponds by definition (15.10.1) the positive measure $|g|^2 \cdot \mu$ on K. Since $|g|^2$ is μ-integrable and $\neq 0$ almost everywhere, we obtain in this way *all* the positive measures on K which are equivalent to μ (13.15.6). In other words, the measure μ of the statement of (15.10.1) is determined *only up to equivalence* by the representation T.

(15.10.8) Now let us consider an *arbitrary* nondegenerate representation T of $\mathscr{C}_{\mathbf{C}}(\mathrm{K})$ in a separable Hilbert space E. From (15.5.6) we know that E is the Hilbert sum of a sequence (E_n) of closed subspaces stable with respect to T, and such that the restriction T_n of T to E_n admits a totalizing vector a_n, for each n. The definition of the measures $\mu_{x,y}$ given in (15.10.3) applies without

change. Also, for each function $u \in \mathscr{U}_{\mathbf{C}}(\mathrm{K})$, we define $T_n(u)$ on E_n (15.10.1), and by virtue of (15.10.2) we have $\|T_n(u)\| \leqq \|u\|$ for each n.

Now we have the following lemma:

(15.10.8.1) *Let* E *be a Hilbert space which is the Hilbert sum of a sequence* (E_n) *of closed subspaces. For each* n *let* U_n *be a continuous operator on* E_n, *and suppose that the sequence of norms* $(\|U_n\|)$ *is bounded. Then there exists a unique continuous operator* U *on* E *whose restriction to* E_n *is* U_n *for each* n. *Also, the restriction of* U^* *to* E_n *is* U_n^*.

Suppose that $\|U_n\| \leqq a$ for all n. For each $x = \sum_n x_n \in \mathrm{E}$, where $x_n \in \mathrm{E}_n$ for all n and $\|x\|^2 = \sum_n \|x_n\|^2$ (6.4), we have

$$\sum_n \|U_n \cdot x_n\|^2 \leqq a^2 \sum_n \|x_n\|^2 = a^2 \|x\|^2,$$

which shows that the series $\sum_n U_n \cdot x_n$ converges in E. If $U \cdot x$ denotes its sum, it is clear that U is linear and that, from above, $\|U \cdot x\| \leqq a\|x\|$, so that U is a continuous operator (5.5.1). The uniqueness of U follows from the fact that the union of the subspaces E_n is a total set in E (6.4). Finally, we have $\|U_n^*\| = \|U_n\| \leqq a$ for all n, and therefore there exists a continuous operator V on E whose restriction to E_n is U_n^* for all n. If $y = \sum_n y_n$, with $y_n \in \mathrm{E}_n$ for all n and $\sum_n \|y_n\|^2 = \|y\|^2$, then (6.4) we have

$$(U \cdot x \mid y) = \sum_n (U_n \cdot x_n \mid y_n) = \sum_n (x_n \mid U_n^* \cdot y_n) = (x \mid V \cdot y),$$

which proves that $V = U^*$.

Applying (15.10.8.1), we see that there exists a unique *normal* continuous operator $T(u)$ on E whose restriction to E_n is $T_n(u)$, for each n. It is immediate that $u \mapsto T(u)$ is a representation of $\mathscr{U}_{\mathbf{C}}(\mathrm{K})$ in E which extends the representation T of $\mathscr{C}_{\mathbf{C}}(\mathrm{K})$. Next, the proposition (15.10.4) generalizes without any change in the proof: we have only to observe that we can take as a total subset of E the set of all $T_n(s) \cdot a_n$, where $s \in \mathscr{C}_{\mathbf{C}}(\mathrm{K})$ and n is arbitrary.

In general there exist *infinitely many* decompositions of E as a Hilbert sum of subspaces with the properties of (15.10.8). However, there is the following result:

(15.10.9) *There exists a decomposition of* E *as a Hilbert sum of a (finite or infinite) sequence* (E_n) *of closed subspaces, stable with respect to* T, *such that the*

restriction of T to E_n admits a totalizing vector a_n and such that, if μ_n is the positive measure on K corresponding to a_n (15.10.1), *then μ_{n+1} is a measure with base μ_n* (13.13) *for each n.*

We begin with a decomposition of E as the Hilbert sum of any sequence (F_n) as described above. Let b_n be a totalizing vector for the restriction of T to F_n, and let ν_n be the corresponding positive measure on K. Since the measure ν_n is determined only up to equivalence (15.10.7), we may multiply it by a strictly positive constant so as to ensure that the series of norms $\|\nu_n\|$ converges.

By induction on $n \geqq 2$ we define two sequences (ν'_n), (ν''_n) of positive measures on K, as follows. The measures ν'_2 and ν''_2 are those which appear in the Lebesgue canonical decomposition of ν_2 relative to ν_1, ν'_2 being a measure with base ν_1, and ν''_2 disjoint from ν_1 (13.8.4). For $k > 2$, the measures ν'_k and ν''_k are likewise such that $\nu_k = \nu'_k + \nu''_k$, where ν'_k is a measure with base $(\nu_1 + \nu''_2 + \cdots + \nu''_{k-1})$ and ν''_k is disjoint from $\nu_1 + \nu''_2 + \cdots + \nu''_{k-1}$. To each of these decompositions there corresponds a partition of K consisting of two universally measurable sets B'_k and B''_k such that ν'_k is concentrated on B'_k and ν''_k is concentrated on B''_k. Let $F_k = F'_k \oplus F''_k$ be the corresponding decomposition of F_k as a Hilbert sum of mutually orthogonal subspaces. If F_k is identified with $L^2_{\mathbf{C}}(K, \nu_k)$ (15.10.1) then F'_k and F''_k admit $b'_k = \tilde{\varphi}_{B'_k}$ and $b''_k = \tilde{\varphi}_{B''_k}$ as totalizing vectors, respectively (15.10.6). The sequence of measures $\nu_1 + \nu''_2 + \cdots + \nu''_k$ is increasing, and the norms of these measures are bounded above by $\sum_{n=1}^{\infty} \|\nu_n\|$. Hence (13.4.4) this sequence has a least upper bound μ_1 in $M_{\mathbf{R}}(K)$, and μ_1 is also the vague limit of the sequence. The preceding construction allows us to assume that, if ν_1 is concentrated on B_1, then the sets $B_1, B''_2, \ldots, B''_k, \ldots$ are pairwise disjoint, and ν''_k is identified with $\varphi_{B''_k} \cdot \mu_1$. If E_1 is the Hilbert sum of F_1 and the F''_k for $k \geqq 2$, then E is the Hilbert sum of E_1 and the F'_k $(k \geqq 2)$. The subspace E_1 is identified with $L^2_{\mathbf{C}}(K, \mu_1)$; since $\|b''_k\|^2 = \nu_k(B''_k) \leqq \|\nu_k\|$, the series $b_1 + b''_2 + \cdots + b''_k + \cdots$ converges in E_1, and its sum a_1 is identified with $\tilde{\varphi}_{A_1}$, where A_1 is the union of B_1 and the B''_k $(k \geqq 2)$. Clearly a_1 is a totalizing vector for the restriction of T to E_1.

Thus, starting with the given decomposition (F_n) of E, we have constructed a decomposition of E into E_1 and the F'_k, where for each $k \geqq 2$ the measure ν'_k corresponding to F'_k is a measure *with base* μ_1. Repeating the construction, we define for each n a decomposition of E as the Hilbert sum of subspaces $E_1, \ldots, E_n, F^{(n)}_{n+1}, \ldots, F^{(n)}_{n+k}, \ldots$, all stable under T, such that the restriction of T to each subspace admits a totalizing vector, and such that if the corresponding measure is μ_k for E_k $(1 \leqq k \leqq n)$ and $\nu^{(n)}_{n+k}$ for $F^{(n)}_{n+k}$, then μ_{k+1} is a measure with base μ_k for $1 \leqq k \leqq n-1$, and $\nu^{(n)}_{n+k}$ is a measure with base μ_n for all $k \geqq 1$. To achieve this we have only to apply the previous reasoning to

the Hilbert sum $E_n \oplus F_{n+1}^{(n)} \oplus \cdots \oplus F_{n+k}^{(n)} \oplus \cdots$. It is clear from the construction that

$$F_1 \oplus F_2 \oplus \cdots \oplus F_n \subset E_1 \oplus E_2 \oplus \cdots \oplus E_n$$

for each n; since E is the Hilbert sum of the F_k, it follows that E is also the Hilbert sum of the E_k (6.4.2), and the subspaces E_k and measures μ_k therefore satisfy the required conditions.

Put $\mu_k = g_k \cdot \mu_1$, and let S_k be the set of points $t \in K$ such that $g_k(t) > 0$. Evidently we may assume that the sequence (S_k) is decreasing and that each S_k is universally measurable (13.9.3). Put $M_1 = K - S_2$ and $M_k = S_k - S_{k+1}$ for $k \geqq 2$; also put $M_\infty = \bigcap_{k \geqq 1} S_k$. For $1 \leqq i \leqq k$, let $H_{ik} = T(\varphi_{M_k})(E_i)$. Then it follows from (15.10.6) that the restrictions of T to the k subspaces H_{ik} $(1 \leqq i \leqq k)$ are *equivalent*, and we put $G_k = H_{1k} \oplus \cdots \oplus H_{kk}$. Clearly G_k is the Hilbert sum of the subspaces H_{ik} $(1 \leqq i \leqq k)$. Similarly, put $H_{i\infty} = T(\varphi_{M_\infty})(E_i)$ for each $i \geqq 1$. The restrictions of T to all the subspaces $H_{i\infty}$ $(i \geqq 1)$ are equivalent, and we denote by G_∞ the Hilbert sum of the $H_{i\infty}$ for all $i \geqq 1$. Then it is clear that E is the Hilbert sum of the G_k $(k \geqq 1)$ and G_∞. It can be shown (Problem 5) that the measures μ_k are determined *up to equivalence*, and hence that the S_k (and consequently the M_k) are determined up to a μ_1-negligible set. The subspaces $G_k = T(\varphi_{M_k})(E)$ are *uniquely determined* by T. The restriction of T to G_k (resp. to G_∞) is said to have *multiplicity* k (resp. *infinite multiplicity*). A representation of multiplicity 1 is therefore *topologically cyclic*.

(15.10.10) Since the measures μ_k are determined only up to equivalence (15.10.7), we may suppose that $\mu_k = \varphi_{S_k} \cdot \mu_1$. It follows that, up to equivalence, the *most general* representation $u \mapsto T(u)$ of $\mathscr{U}_C(K)$ on a separable Hilbert space E may be described as follows:

Consider a positive measure ν on K, and a decreasing sequence $(S_j)_{1 \leqq j < \omega}$ of universally measurable subsets of K, where ω is either a positive integer or $+\infty$. The space E is the *Hilbert sum* of the Hilbert spaces $L^2_C(K, \varphi_{S_k} \cdot \nu)$; each of these spaces is stable under T, and the restriction of $T(u)$ to $L^2_C(K, \varphi_{S_k} \cdot \nu)$ is *multiplication by* $\tilde{u}$.

PROBLEMS

1. Let K be a compact space, E a Hilbert space, and $(x, y) \mapsto \mu_{x,y}$ a continuous sesquilinear mapping of $E \times E$ into the Banach space $M_C(K)$ of complex measures on K. Show that,

for each function $u \in \mathscr{U}_C(K)$, there exists a unique continuous operator $T(u)$ on E such that

$$(T(u) \cdot x \mid y) = \int u(t)\, d\mu_{x,y}(t)$$

for all x, y in E. In order that $u \mapsto T(u)$ should be a representation of $\mathscr{U}_C(K)$ on E, it is necessary and sufficient that

$$(x \mid y) = \int d\mu_{x,y}, \qquad \mu_{y,x} = \overline{\mu}_{x,y}, \qquad u \cdot \mu_{x,y} = \mu_{T(u)\cdot x,\, y}$$

for all x, y in E and all $u \in \mathscr{U}_C(K)$. In the last relation, the condition $u \in \mathscr{U}_C(K)$ may be replaced by $u \in \mathscr{C}_C(K)$.

2. (a) Deduce from Problem 1 a new proof of the fact that a representation $u \mapsto T(u)$ of $\mathscr{C}_C(K)$ in a Hilbert space E can be uniquely extended to a representation $u \mapsto T(u)$ of $\mathscr{U}_C(K)$ on E, satisfying the relation (15.10.4.1). (To prove that $T(uv) = T(u)T(v)$ for $u, v \in \mathscr{U}_C(K)$, proceed in two steps: first assume that one of the two functions is continuous and use the relation $\mu_{T(u)\cdot x,\, y} = \mu_{x,\, T(\check{u})\cdot y}$ when u is continuous; then prove that this relation remains valid for all $u \in \mathscr{U}_C(K)$.)

(b) If a continuous operator $V \in \mathscr{L}(E)$ commutes with $T(u)$ for all $u \in \mathscr{C}_C(K)$, then it commutes with $T(v)$ for all $v \in \mathscr{U}_C(K)$.

(c) If E is separable, show that each operator $T(u)$, where $u \in \mathscr{U}_C(K)$, is the limit, with respect to the weak topology on $\mathscr{L}(E)$ (Section 12.15, Problem 9) of a sequence $(T(v_n))$, where $v_n \in \mathscr{C}_C(K)$; and conversely. (For the converse, use (15.10.6(ii)) and (b) above.)

3. With the notation of Problem 1, show that $T(u) \cdot x = 0$ for $x \in E$ and $u \in \mathscr{U}_C(K)$ if and only if u is $\mu_{x,y}$-negligible for all y in a total subset of E. The mapping $u \mapsto T(u)$ of $\mathscr{C}_C(K)$ into $\mathscr{L}(E)$ is injective if and only if the union of the supports of the measures $\mu_{x,y}$ is dense in K.

4. With the notation of Problem 1, suppose that E is separable; let (a_j) be a dense sequence in E, and put $\nu_{ij} = |\mu_{a_i, a_j}|$. Then there exists a positive measure ν on K such that the relation $\nu(N) = 0$ is equivalent to the relation "$\nu_{ij}(N) = 0$ for all i and j" (13.15.8). All the functions in $\mathscr{U}_C(K)$ which are equivalent with respect to ν have the same image under T, and we may therefore regard T as an *injective* homomorphism of $L^\infty_C(\nu)$ into $\mathscr{L}(E)$. Prove that $N_\infty(u) \leqq \|T(u)\|$ for all functions $u \in \mathscr{L}^\infty_C(\nu)$ (consider first the case where u is a step-function). Consequently T is a *bicontinuous* mapping of $L^\infty_C(\nu)$ onto its image in $\mathscr{L}(E)$.

5. Suppose that, under the hypotheses of (15.10.8) there exist two sequences (E_n, μ_n) and (E'_n, μ'_n) satisfying the conditions of (15.10.9).

(a) Show that if $\mu_n = 0$ for $n \geqq 2$, then $\mu'_n = 0$ for $n \geqq 2$. (In other words, a topologically cyclic representation does not admit a canonical decomposition satisfying (15.10.9) as the sum of at least two topologically cyclic representations.) To prove this, observe that if a is a totalizing vector for T, and a'_n is its projection on E'_n, then $a'_n \neq 0$ if $E'_n \neq 0$ (i.e., if $\mu'_n \neq 0$). If $a'_n \neq 0$ for some $n \geqq 2$, show that there would exist a sequence (g_m) of functions belonging to $\mathscr{U}_C(K)$, such that $T(g_m) \cdot a'_1$ tends to 0 and $T(g_m) \cdot a'_n$ tends to a'_n, and show that this contradicts the fact that μ'_n is a measure with base μ'_1.

(b) Show that the measures μ_1, μ'_1 are equivalent, by proving that the relation $\mu_1(N) = 0$ is equivalent to $T(\varphi_N) = 0$.
(c) If F_1, F'_1 are the orthogonal supplements of E_1, E'_1, respectively, in E, show that F_1 and F'_1 have the same dimension (finite or infinite). (Assume that $\dim(F_1)$ is finite; then for $n \geqq 2$ the measures μ_n are carried by a *finite* subset M of K and are zero for all large n. Also, if ν_1 is the restriction of μ_1 to M, then μ_n ($n \geqq 2$) are measures with base ν_1. Deduce that $T(\varphi_M)(E)$ is of *finite* dimension and that the restriction of T to the orthogonal supplement $T(1 - \varphi_M)(E)$ of $T(\varphi_M)(E)$ is topologically cyclic. Use (a) to deduce that the measures μ'_n, for $n \geqq 2$, are concentrated on M, and hence that

$$T(1 - \varphi_M)(E) = T(1 - \varphi_M)(E_1) = T(1 - \varphi_M)(E'_1).$$

Also, by virtue of (b), the restriction of μ'_1 to M must be equivalent to μ_1; hence $T(\varphi_M)(E_1)$ and $T(\varphi_M)(E'_1)$ have the same finite dimension, and the result follows.)
(d) Use (c) to show that there exists a unitary transformation U_1 of E such that $U_1(E_1) = E'_1$.
(e) Prove by induction that there exists a sequence (U_n) of unitary transformations of E such that $U_n(E_k) = E'_k$ for $1 \leqq k \leqq n$ and U_{n+1} agrees with U_n on E_k for $1 \leqq k \leqq n$. Show that this sequence tends strongly (Section 12.15, Problem 8) to a unitary transformation U such that $U(E_n) = E'_n$ for all n. Show also that μ'_n is equivalent to μ_n for all n.

6. With the notation of (15.10.9), let V be an operator belonging to $\mathscr{L}(E)$ which commutes with $T(u)$ for all $u \in \mathscr{U}_C(K)$.
(a) Show that each of the subspaces G_k is stable under V, and that $V \mid G_k$ commutes with all the operators $T(\varphi_{M_k} u)$.
(b) Up to equivalence, each of the H_{ik} ($1 \leqq i \leqq k$) can be identified with $L^2_C(\mu_k)$, and the restriction $T_{ik}(u)$ of $T(u)$ to H_{ik} can be identified with multiplication $M(u)$ by the class of u, where u is any bounded μ_k-measurable function. Every continuous operator V on G_k may be written in the form of a matrix

$$V = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1k} \\ V_{21} & V_{22} & \cdots & V_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ V_{k1} & V_{k2} & \cdots & V_{kk} \end{pmatrix}$$

operating on the vectors $x = (x_i)_{1 \leqslant i \leqslant k} \in (L^2_C(\mu_k))^k$ considered as column matrices, the V_{ij} being continuous operators on $L^2_C(\mu_k)$. With this notation, the restriction of $T(u)$ to G_k is identified with the diagonal matrix all of whose diagonal elements are equal to $M(u)$.

Deduce that V commutes with the restrictions of all the $T(u)$ to G_k if and only if

$$V_{ij} = M(v_{ij})$$

with $v_{ij} \in \mathscr{U}_C(K)$.
(c) Deduce that every subspace F of E which is stable with respect to T is the Hilbert sum of subspaces $F_k \subset G_k$ stable under T, and that each F_k may be described as follows. Consider a partition of M_k consisting of k μ_k-measurable sets L_{ik} ($1 \leqq i \leqq k$), and an *invertible* matrix (w_{ij}) with k rows and k columns, whose entries w_{ij} belong to $\mathscr{U}_C(K)$.

Let F'_{ik} denote the sum of the subspaces $T(\varphi_{L_{ik}})(H_{jl})$ for $1 \leqq j \leqq i$, and let F_{ik} denote the image of F'_{ik} under the invertible operator

$$\begin{pmatrix} M(w_{11}) & \cdots & M(w_{1k}) \\ M(w_{21}) & \cdots & M(w_{2k}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ M(w_{k1}) & \cdots & M(w_{kk}) \end{pmatrix}$$

on G_k. Then F_k is the sum of the F_{ik} $(1 \leqq i \leqq k)$. (Consider the projector P_F.)

11. THE SPECTRAL THEORY OF HILBERT

(15.11.1) Let E be a separable Hilbert space and N a *normal* continuous operator on E (15.4.11). If $\mathscr{A}$ is the closure of the (commutative) subalgebra of $\mathscr{L}(E)$ generated by 1_E, N and N^*, then ((15.4.13) and (15.4.15)) $\mathscr{A}$ is a separable star algebra, and the mapping $\omega : \chi \mapsto \chi(N)$ is a homeomorphism of $\mathbf{X}(\mathscr{A})$ onto $\mathrm{Sp}(N) \subset \mathbf{C}$ (the spectrum of N with respect to $\mathscr{L}(E)$, that is to say the spectrum of N as defined in (11.1)). Hence it follows from the Gelfand–Neumark theorem (15.4.14) that the mapping $f \mapsto \mathscr{G}^{-1}(f \circ \omega)$ is a faithful representation of $\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(N))$ on E, which can therefore be extended to a faithful representation of $\mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$ on E (15.10.8). We denote this representation by $f \mapsto f(N)$. This notation is justified by the fact that, if f and g are any two functions belonging to $\mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$, then in the algebra $\mathscr{L}(E)$ we have

$$\text{(15.11.1.1)} \qquad \begin{gathered}(f+g)(N) = f(N) + g(N), \qquad (fg)(N) = f(N)g(N), \\ \bar{f}(N) = (f(N))^*,\end{gathered}$$

$$\text{(15.11.1.2)} \qquad 1_{\mathrm{Sp}(N)}(N) = N$$

(where $1_{\mathrm{Sp}(N)}$ is the identity mapping $\zeta \mapsto \zeta$ of $\mathrm{Sp}(N)$). The Gelfand–Neumark theorem also shows that, for every *continuous* function f on $\mathrm{Sp}(N)$, we have

$$\text{(15.11.1.3)} \qquad \|f(N)\| = \rho(f(N)) = \sup_{\zeta \in \mathrm{Sp}(N)} |f(\zeta)|$$

(cf. (15.11.8.1)).

Moreover ((15.10.3) and (15.10.8)), there exists a continuous sesquilinear mapping $(x, y) \mapsto m_{x,y}$ of $E \times E$ into the Banach space $M_{\mathbf{C}}(\mathrm{Sp}(N))$ of complex measures on $\mathrm{Sp}(N)$ such that $m_{y,x} = \overline{m}_{x,y}$ and

$$\text{(15.11.2)} \qquad (f(N) \cdot x \mid y) = \int f(\zeta)\, dm_{x,y}(\zeta)$$

l functions $f \in \mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$. Also it follows from (15.11.1.3) that $\cdot x \mid y)| \leqq \|f\| \cdot \|x\| \cdot \|y\|$ for all continuous functions f on $\mathrm{Sp}(N)$, and ore

2.1) $$\|m_{x,y}\| \leqq \|x\| \cdot \|y\|$$

x, y in E.
$f \in \mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$ is the restriction to $\mathrm{Sp}(N)$ of a complex-valued function ied on a subset of C containing $\mathrm{Sp}(N)$, we shall write $g(N)$ in place).

3) We shall say that N is a *simple* normal operator if the representa- $\to f(N)$ of $\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(N))$ on E is *topologically cyclic*. By virtue of (15.5.6), xists a decomposition of E as a Hilbert sum of closed subspaces E_n are stable under $\mathscr{A}$ (and therefore under N *and* N^* (cf. Problem 3) iat, for each n, the restriction N_n of N to E_n is a *simple* normal operator. ace E_n may then be identified with $\mathrm{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \mu_n)$, where μ_n is a positive e, and N_n is then identified with the operator $M_{\mu_n}(1_{\mathbf{C}})$ of multiplication class of the function $1_{\mathbf{C}}$ in $\mathrm{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \mu_n)$. If a_n is a totalizing vector for resentation $f \mapsto f(N_n)$, then (15.10.3.3) we may take

3.1) $$\mu_n = m_{a_n, a_n}.$$

h this notation:

4) *The support of μ_n is the whole of* $\mathrm{Sp}(N_n)$.

$\alpha \notin \mathrm{Supp}(\mu_n)$, then the function $\zeta \mapsto (\alpha - \zeta)^{-1}$ is continuous and ed on $\mathrm{Supp}(\mu_n)$; it extends to a bounded continuous function g on (4.5.1), and since $g(\zeta)(\alpha - \zeta) = 1$ for all $\zeta \in \mathrm{Supp}(\mu_n)$, it follows that the ious operator on $\mathrm{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \mu_n)$, defined by multiplication by the class function $\zeta \mapsto \alpha - \zeta$, is *invertible*. Hence (as this operator is identified $_{\mathrm{E}_n} - N_n$) we have $\alpha \notin \mathrm{Sp}(N_n)$.

5) $\mathrm{Sp}(N)$ *is the closure in* C *of* $\bigcup_n \mathrm{Sp}(N_n)$.

arly $\mathrm{Sp}(N_n) \subset \mathrm{Sp}(N)$; since $\mathrm{Sp}(N)$ is closed, it therefore contains the of $\bigcup_n \mathrm{Sp}(N_n)$. If α does not lie in this closure, then there exists $r > 0$ hat $|\alpha - \zeta| \geqq r$ for all $\zeta \in \mathrm{Sp}(N_n)$ and all n. Hence (15.11.1.3) we $(\alpha 1_{\mathrm{E}_n} - N_n)^{-1}\| \leqq 1/r$, from which we conclude (15.10.8.1) that the $N_n)^{-1}$ are the restrictions of a continuous operator on E, the inverse $- N$.

For each function $f \in \mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$, we have $f(N_n) = f(N) \mid \mathrm{E}_n$. For this is immediately seen to be true when f is a polynomial in ζ and $\bar{\zeta}$ (15.11.1.1); for $f \in \mathscr{C}_{\mathbf{C}}(\mathrm{Sp}(N))$, the result then follows by the Stone–Weierstrass theorem (7.3.1) and (15.11.1.3); and the general case then follows by (15.11.2).

We shall see that the study of N reduces to that of the N_n. Consider for example the *eigenvalues* of N: it is clear that x is an eigenvector of N if and only if each of its projections x_n on E_n is either zero or an eigenvector of N_n, corresponding to an eigenvalue independent of n. We have then the following characterization:

(15.11.6) *For $\alpha \in \mathrm{Sp}(N_n)$ to be an eigenvalue of N_n, it is necessary and sufficient that $\mu_n(\{\alpha\}) \neq 0$. The corresponding eigenvectors are those belonging to the image of E_n under the projector $\varphi_{\{\alpha\}}(N_n)$, and this image is a one-dimensional subspace.*

Let α be an eigenvalue of N_n, and let F be the kernel of $\alpha 1_{\mathrm{E}_n} - N_n$. Then by hypothesis F is closed and nonzero. Also it is stable under N_n^*, because $N_n \cdot (N_n^* \cdot x) = N_n^* \cdot (N_n \cdot x) = \alpha N_n^* \cdot x$. Hence F is stable with respect to the algebra generated by N_n and N_n^*. If we identify N_n with multiplication by the class of $1_{\mathbf{C}}$ in $\mathrm{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \mu_n)$, then F is identified with a subspace of the type $\mathrm{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \varphi_{\mathrm{M}} \cdot \mu_n)$, where M is a universally measurable set such that $\mu_n(\mathrm{M}) \neq 0$ (15.10.6). We assert that the support of the measure $\nu = \varphi_{\mathrm{M}} \cdot \mu_n$ *consists of a single point*. If not, there would exist two disjoint closed subsets B, C of $\mathrm{Sp}(N_n)$, such that $\nu(\mathrm{B} \cap \mathrm{M}) \neq 0$ and $\nu(\mathrm{C} \cap \mathrm{M}) \neq 0$. The function $\zeta\varphi_{\mathrm{B} \cap \mathrm{M}}(\zeta)$ (resp. $\zeta\varphi_{\mathrm{C} \cap \mathrm{M}}(\zeta)$) would be equal almost everywhere (with respect to ν) to $\alpha\varphi_{\mathrm{B} \cap \mathrm{M}}(\zeta)$ (resp. $\alpha\varphi_{\mathrm{C} \cap \mathrm{M}}(\zeta)$). In other words, we should have $\zeta = \alpha$ almost everywhere in $\mathrm{B} \cap \mathrm{M}$ and almost everywhere in $\mathrm{C} \cap \mathrm{M}$, which is absurd because $\mathrm{B} \cap \mathrm{C} = \varnothing$. Hence we may assume that $\mathrm{M} = \{\beta\}$, and then it is clear that $\mathrm{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \varphi_{\mathrm{M}} \cdot \mu_n)$ is of dimension 1, and that the restriction of N_n to this space is the homothety with ratio β.

If $\alpha \in \mathrm{Sp}(N)$ is an eigenvalue of N, then the set J of integers n such that α is an eigenvalue of N_n is not empty. If D_n is the one-dimensional subspace of E_n generated by the eigenvectors of N_n corresponding to the eigenvalue α, then the *eigenspace* $\mathrm{E}(\alpha; N)$ of N corresponding to α (11.1) is the *Hilbert sum* of the D_n for $n \in \mathrm{J}$. If J is finite, the number of elements of J (which is equal to the dimension of $\mathrm{E}(\alpha; N)$) is called the *multiplicity* of α. If J is infinite, then α is said to have *infinite multiplicity*. It is easily checked that this notion of multiplicity agrees with the notion of the multiplicity of a representation introduced in (15.10.9). Also it follows from the remarks above that if α, β are *distinct* eigenvalues of N, then the eigenspaces $\mathrm{E}(\alpha; N)$ and $\mathrm{E}(\beta; N)$ are *orthogonal*.

(15.11.7) *For N to be self-adjoint* (resp. *unitary*) *it is necessary and sufficient that* $\mathrm{Sp}(N) \subset \mathbf{R}$ (resp. $\mathrm{Sp}(N) \subset \mathbf{U}$). *If H is self-adjoint, then*

$$\begin{aligned} \inf(\mathrm{Sp}(H)) &= \inf_{\|x\|=1} (H \cdot x \,|\, x), \\ \sup(\mathrm{Sp}(H)) &= \sup_{\|x\|=1} (H \cdot x \,|\, x), \end{aligned} \tag{15.11.7.1}$$

$$\|H\| = \sup_{\|x\|=1} |(H \cdot x \,|\, x)|. \tag{15.11.7.2}$$

In the first assertion, we have already seen (15.4.12) that the conditions are necessary. To show that they are sufficient, it is enough (by virtue of (15.11.5)) to prove them for the N_n; and this is immediately done, because, when N_n is identified with multiplication by the class of $1_{\mathbf{C}}$ in $\mathbf{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \mu_n)$, the operator N_n^* is identified with multiplication by the class of the function $\zeta \mapsto \bar{\zeta}$. To prove (15.11.7.1), it is enough to show that, for a self-adjoint operator H to be such that $(H \cdot x \,|\, x) \geqq 0$ for all $x \in \mathrm{E}$ (in which case we say that H is *positive* and we write $H \geqq 0$), it is necessary and sufficient that $\mathrm{Sp}(H) \subset \mathbf{R}_+$. In view of (15.11.5) and the relation

$$(H \cdot x \,|\, x) = \sum_n (H_n \cdot x_n \,|\, x_n) \tag{15.11.7.3}$$

(with notation analogous to that of (15.11.3)) we are reduced to proving the assertion for *simple* self-adjoint operators H_n. If we identify H_n with multiplication by the class of $1_{\mathbf{C}}$ in $\mathbf{L}^2_{\mathbf{C}}(\mathrm{Sp}(H_n), \mu_n)$, what we have to prove is that $\mathrm{Sp}(H_n) \subset \mathbf{R}_+$ if and only if $\int \zeta f(\zeta)\, d\mu_n(\zeta) \geqq 0$ for every function $f \geqq 0$ in $\mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(H_n))$. Now, if M is the intersection of $\mathrm{Sp}(H_n)$ with the complement $]-\infty, 0[$ of $\mathbf{R}_+$ in $\mathbf{R}$, then the relation $\mathrm{M} \neq \varnothing$ would imply $\mu_n(\mathrm{M}) > 0$ (15.1.14). Since $]-\infty, 0[$ is the union of the intervals $]-\infty, -1/n]$, there would exist $m > 0$ such that

$$\mu_n(\mathrm{M} \cap]-\infty, -(1/m)]) = \alpha > 0,$$

and consequently $\int \zeta \varphi_{\mathrm{M}}(\zeta)\, d\mu_n(\zeta) \leqq -\alpha/m < 0$, contrary to hypothesis. Finally, the relation (15.11.7.2) follows from (15.11.7.1) and (15.4.14.1), because the spectral radius of H is equal to the larger of $|\inf(\mathrm{Sp}(H))|$, $|\sup(\mathrm{Sp}(H))|$.

(15.11.8) (i) *For each function $f \in \mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$, the spectrum of $f(N)$ is contained in* $\overline{f(\mathrm{Sp}(N))}$ (closure in $\mathbf{C}$), *and*

$$\|f(N)\| \leqq \sup_{\zeta \in \mathrm{Sp}(N)} |f(\zeta)|. \tag{15.11.8.1}$$

For every eigenvalue α of N, $f(\alpha)$ is an eigenvalue of $f(N)$, and the eigenspace $\mathrm{E}(\alpha; N)$ *is contained in* $\mathrm{E}(f(\alpha); f(N))$.

If f is continuous, then $\mathrm{Sp}(f(N)) = f(\mathrm{Sp}(N))$.

(ii) *More precisely, with the notation of* (15.11.3), *if* $f \in \mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N_n))$, *then the spectrum of* $f(N_n)$ *consists of the complex numbers* β *such that*

$$\operatorname*{ess\,inf}_{\zeta \in \mathrm{Sp}(N_n)} |\beta - f(\zeta)| = 0$$

(*relative to the measure* μ_n).

(iii) *If* $f \in \mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$ *and if g is a continuous mapping of* $\overline{f(\mathrm{Sp}(N))}$ *into* $\mathbf{C}$, *then* $g(f(N)) = (g \circ f)(N)$.

(iv) *If a sequence* (f_k) *of functions belonging to* $\mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$ *is uniformly bounded and converges simply to f, then for every* $x \in \mathrm{E}$ *the sequence* $(f_k(N) \cdot x)$ *converges in* E *to* $f(N) \cdot x$.

We shall start by proving (ii). We have seen that $f(N_n)$ may be identified with multiplication by the class of the function f in $\mathrm{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \mu_n)$. Hence $\beta \notin \mathrm{Sp}(f(N_n))$ if and only if there exists a real number $a > 0$ such that $\mathrm{N}_2((\beta - f)u) \geqq a \cdot \mathrm{N}_2(u)$ for all functions $u \in \mathscr{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \mu_n)$ (5.5.1). If $\operatorname{ess\,inf}_{\zeta \in \mathrm{Sp}(N_n)} |\beta - f(\zeta)| > 0$, then we may take a to be equal to this number, by virtue of (13.12.2), and therefore $\beta \notin \mathrm{Sp}(f(N_n))$. Conversely, if

$$\operatorname*{ess\,inf}_{\zeta \in \mathrm{Sp}(N_n)} |\beta - f(\zeta)| = 0,$$

then for each $\varepsilon > 0$ the set M of complex numbers $\zeta \in \mathrm{Sp}(N_n)$ such that $|\beta - f(\zeta)| \leqq \varepsilon$ is not μ_n-negligible, and we have $\mathrm{N}_2((\beta - f)\varphi_{\mathrm{M}}) \leqq \varepsilon \mathrm{N}_2(\varphi_{\mathrm{M}})$, hence $\beta \in \mathrm{Sp}(f(N_n))$.

It follows from (ii) that $\mathrm{Sp}(f(N_n)) \subset \overline{f(\mathrm{Sp}(N_n))}$. If, moreover, f is continuous, then $f(\mathrm{Sp}(N_n))$ is compact (3.17.9), and for each $\beta = f(\alpha)$, where $\alpha \in \mathrm{Sp}(N_n)$, every compact neighborhood of α has μ_n-measure > 0 (15.11.4), so that $f(\mathrm{Sp}(N_n)) = \mathrm{Sp}(f(N_n))$. To show that $\mathrm{Sp}(f(N)) \subset \overline{f(\mathrm{Sp}(N))}$ in general, and that $\mathrm{Sp}(f(N)) = f(\mathrm{Sp}(N))$ when f is continuous, we have only to use (15.11.5) and the fact that $f(\mathrm{Sp}(N))$ is compact if f is continuous. Finally, the assertions about eigenvalues are evident, again by reducing to the case of simple operators and using (15.11.6).

To prove (iii), note first that $g \circ f$ is a universally measurable mapping of $\mathrm{Sp}(N)$ into $\mathbf{C}$ (13.9.6). The relation

$$g(f(N)) = (g \circ f)(N)$$

follows from (15.11.1) in the case where $g(\zeta) = \zeta^p \bar{\zeta}^q$ (p, q being integers $\geqq 0$). Now, for each $\varepsilon > 0$, there exists a polynomial h in ζ and $\bar{\zeta}$ such that $|g(\zeta) - h(\zeta)| \leqq \varepsilon$ for all $\zeta \in \overline{f(\mathrm{Sp}(N))}$ (7.3.2). Since $|g(f(\zeta)) - h(f(\zeta))| \leqq \varepsilon$ for

$\zeta \in \mathrm{Sp}(N)$, it follows from (15.11.8.1) that $\|g(f(N)) - h(f(N))\| \leqq \varepsilon$ and that $\|(g \circ f)(N) - (h \circ f)(N)\| \leqq \varepsilon$. Since ε was arbitrary, this gives the required result.

Finally, to prove (iv), note that because the sequence of norms $\|f_k(N)\|$ is bounded, by virtue of (15.11.8.1), it is enough to prove the convergence of the sequence $(f_k(N) \cdot x)$ for x belonging to a total subset of E ((12.15.7.1) and (7.5.5)). With the notation of (15.11.3), we may therefore restrict ourselves to proving the assertion for each N_n. But since $f_k(N_n)$ may be identified with multiplication by the class of the restriction of f_k to $\mathrm{Sp}(N_n)$ in the space $\mathrm{L}^2_{\mathbf{C}}(\mathrm{Sp}(N_n), \mu_n)$, the result follows from (13.11.4(iii)).

(15.11.9) *For each universally measurable subset* M *of* $\mathrm{Sp}(N)$, *let* $\mathrm{E}(\mathrm{M})$ *be the closed subspace of* E, *stable with respect to* N *and* N^*, *which is the image of* E *under the orthogonal projector* $P_{\mathrm{E}(\mathrm{M})} = \varphi_{\mathrm{M}}(N)$ (15.10.6). *Then the spectrum of the restriction of* N *to* $\mathrm{E}(\mathrm{M})$ *is contained in* $\overline{\mathrm{M}}$ (*closure in* **C**).

For each n, let $\mathrm{E}_n(\mathrm{M})$ be the image of E_n under the orthogonal projector $\varphi_{\mathrm{M}}(N_n)$. It is immediately seen that $\mathrm{E}(\mathrm{M})$ is the Hilbert sum of the $\mathrm{E}_n(\mathrm{M})$, hence (15.11.5) it is enough to prove the proposition for each N_n. If $\alpha \notin \overline{\mathrm{M}}$, there exists a continuous function g on $\mathrm{Sp}(N_n)$ such that $g(\zeta)(\alpha - \zeta)\varphi_{\mathrm{M}}(\zeta) = \varphi_{\mathrm{M}}(\zeta)$ for all $\zeta \in \mathrm{Sp}(N_n)$ (4.5.1). It follows that, if N'_n is the restriction of N_n to $\mathrm{E}_n(\mathrm{M})$, then $\alpha 1_{\mathrm{E}_n(\mathrm{M})} - N'_n$ has an inverse equal to the restriction of $g(N_n)$ to $\mathrm{E}_n(\mathrm{M})$.

Remark

(15.11.10) Note that the argument which proves (ii) in (15.11.8) shows that, for each positive measure μ on **C** with compact support K, multiplication by the class of $1_{\mathbf{C}}$ in $\mathrm{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$ is a normal continuous operator N such that $\mathrm{Sp}(N) = \mathrm{K}$ (converse of (15.11.3)).

(15.11.11) *Let* f *be a homeomorphism of a closed subset* M *of* **C** *onto a closed subset* N *of* **C** *containing* $\mathrm{Sp}(N)$. *Then there exists a unique normal continuous operator* N' *on* E *whose spectrum is contained in* M *and which is such that* $f(N') = N$.

If h is the homeomorphism of N onto M which is the inverse of f, then by virtue of (15.11.8(iii)) we must have $N' = h(f(N')) = h(N)$; and since conversely the spectrum of $h(N)$ is $h(\mathrm{Sp}(N)) \subset \mathrm{M}$, it follows that $f(h(N)) = N$, and the result follows.

In particular:

(15.11.12) *If H is any positive, self-adjoint operator then there exists a unique-positive self-adjoint operator H' such that $H'^2 = H$.*

Apply (15.11.11) with $M = N = \mathbf{R}_+$ and $f(\zeta) = \zeta^2$.

The unique positive self-adjoint operator H' defined in (15.11.12) is denoted by $H^{1/2}$.

Example

(15.11.13) Let E be the Hilbert space $L^2_{\mathbf{C}}(\mathbf{R}; \lambda)$, where λ is Lebesgue measure. Since the function $f(t) = e^{-|t|}$ is λ-integrable, the convolution $g \mapsto f * g$ defines, on passing to the equivalence classes, a continuous operator H on E (14.10.6) with norm $N_1(f) = 2$. As in (11.6.1), it is immediately seen that H is self-adjoint. It can be shown directly (Problem 5) that the interval [0, 2] in $\mathbf{R}$ is equal to $\mathrm{Sp}(H)$; this also follows from the general theorems of harmonic analysis (Chapter XXII).

Note that, for each $\alpha \in \mathbf{R}$, the function $g_\alpha(t) = e^{i\alpha t}$ is such that the convolution $f * g_\alpha$ is defined and equal to $2g_\alpha/(1 + \alpha^2)$. However, it is not the case that the g_α are "eigenfunctions" of H, because they do not belong to $\mathscr{L}^2_{\mathbf{C}}(\mathbf{R}, \lambda)$. In Chapter XXIII we shall obtain a generalization and an interpretation of this phenomenon.

(15.11.14) *The case of normal operators whose spectrum contains no non-isolated point $\neq 0$.* In this case (which is that of *compact* normal operators (11.4.1)) let (λ_n) be the (finite or infinite) sequence of points of $\mathrm{Sp}(N)$, other than 0. These are the *eigenvalues* of N (15.11.6). The eigenspace $E(\lambda_n; N)$ corresponding to λ_n is just the space $E(\{\lambda_n\})$ defined in (15.11.9), and these closed subspaces are therefore pairwise orthogonal. Moreover, we have $E(\{0\}) = \mathrm{Ker}(N)$, the spectrum of the restriction of N to this subspace being reduced to 0 (15.11.9). Finally, E is the *Hilbert sum* of $E(\{0\})$ and the $E(\{\lambda_n\})$. For it is enough to apply (15.11.8(iv)) to the sequence of functions (f_n), where $f_n(\zeta) = 1$ for $\zeta = \lambda_k$ with $k \leqq n$, $f_n(\zeta) = 0$ for $\zeta = \lambda_k$ and $k > n$, and $f_n(0) = 1$; this shows in particular that every $x \in E$ is the limit of the sum of its projections on $E(\{0\})$ and the $E(\{\lambda_k\})$ with $k \leqq n$. Hence the result (6.4).

In particular, if E is *finite*-dimensional, a normal operator on E may be defined to be an operator whose matrix is *diagonal*, with respect to a suitably chosen orthonormal basis of E.

PROBLEMS

1. Show that a continuous operator N on a Hilbert space E is normal if and only if $\|N \cdot x\| = \|N^* \cdot x\|$ for all $x \in E$.

2. Let N be a continuous normal operator on a Hilbert space E.
(a) For each $x \in E$, consider the open sets $W \subset C$ such that there exists a continuous mapping f_W of W into E satisfying the relation
$$(\lambda \cdot 1 - N) \cdot f_W(\lambda) = x$$
for all $\lambda \in W$. Show that there exists a largest open set $\Omega(x)$ with this property, that all the functions f_W are restrictions of a unique mapping f of $\Omega(x)$ into E, and that f is analytic in $\Omega(x)$. (Use (15.5.6) to reduce to the case where the normal operator N is simple. In this case, E being identified with $L^2_{\mathbf{C}}(\mathrm{Sp}(U), \mu)$ and x with the class of a function g, the set $\Omega(x)$ is the interior of the set of all $\lambda \in \mathbf{C}$ such that the function $\zeta \mapsto (\lambda - \zeta)^{-1} g(\zeta)$ belongs to $\mathscr{L}^2_{\mathbf{C}}(\mathrm{Sp}(N), \mu)$.)
(b) Put $\Phi(x) = \mathbf{C} - \Omega(x)$. Show that, for each *closed* subset M of $\mathrm{Sp}(U)$, the space $E(M) = \varphi_M(U)(E)$ is the set of all $x \in E$ such that $\Phi(x) \subset M$. (Again reduce to the case of a simple normal operator.)
(c) Show that every continuous operator $V \in L(E)$ which commutes with N also commutes with all the operators $g(N)$, where $g \in \mathscr{U}_{\mathbf{C}}(\mathrm{Sp}(N))$, and in particular commutes with N^* (Fuglede's theorem). (First show that $\Omega(V \cdot x) \supset \Omega(x)$ for all $x \in E$, by considering the function f defined in (a) and the function $\lambda \mapsto V \cdot f(\lambda)$. Using (b), deduce that, for each closed subset M of E, the operator V commutes with the projector $P_{E(M)} = \varphi_M(N)$, and conclude that V commutes with $g(N)$ for all continuous functions g on $\mathrm{Sp}(N)$.)
(d) Deduce from (c) that, if N_1 and N_2 are *commuting* normal operators, then $N_1 N_2$ is normal.

3. Let μ be normalized Haar measure on the group $\mathbf{U} : |z| = 1$ (so that $d\mu(\theta) = (2\pi)^{-1}\, d\theta$) and let $E = L^2_{\mathbf{C}}(\mu)$. Then the operator $M_\mu(1_{\mathbf{C}}) = U$ is a *unitary* operator on E. Every closed subspace of E which is *stable* under N is either of the form $\varphi_M(N)(E)$, where $M \subset \mathbf{U}$ is of measure <1; or else is of the form $\tilde{q} \cdot H^2(\mu)$, where $|q| = 1$ (Section 15.3, Problem 15) (Beurling's theorem). Deduce that there exist closed subspaces F of E which are stable under U but *not* under $U^* = U^{-1}$, and are such that the orthogonal projector P_F does *not* commute with N (compare with (15.15.3)).

4. Without using Riesz theory (11.4), prove that the spectrum of a compact normal operator has only isolated points, except for the point 0 (reduce to the case of a simple normal operator).

5. Show that the spectrum of the self-adjoint operator H considered in (15.11.13) is the interval [0, 2] in **R**. (Observe that $\|H\| \leqq 2$; to prove that $H \geqq 0$, that is to say $(H \cdot u \mid u) \geqq 0$ for all $u \in E$, consider first the functions $u(t) = \int_a^b e^{itx}\, dx$ and their linear combinations. To show that every number of the form $2/(1 + \alpha^2)$ belongs to the spectrum of H, approximate g_α by functions $u_n g_\alpha$, where $u_n \in \mathscr{L}^2$ is $\geqq 0$ and the sequence (u_n) is increasing and tends to 1.)

6. Let E be a separable Hilbert space and T a continuous operator on E. Let R and L be the positive hermitian operators which are the square roots (15.11.12) of T^*T and TT^*, respectively. We write $R = \mathrm{abs}(T)$, and call it the "absolute value" of T. Then $L = \mathrm{abs}(T^*)$.

(a) Show that $\mathrm{Ker}(T) = \mathrm{Ker}(R)$ and that $\overline{L(\mathrm{E})} = \overline{T(\mathrm{E})}$. There exists a unique isometry V of $R(\mathrm{E})$ onto $T(\mathrm{E})$ such that $T = VR$. If we extend V by continuity to $\overline{R(\mathrm{E})}$, and then to an operator $U \in \mathscr{L}(\mathrm{E})$ by putting $U(x) = 0$ on the orthogonal supplement of $R(\mathrm{E})$, then we have also $T = UR$ (*polar decomposition* of T). Show that

$$R = U^*T = U^*UR = RU^*U, \qquad L = URU^*, \qquad T = LU^*.$$

(b) For T to be invertible it is necessary and sufficient that $R = \mathrm{abs}(T)$ and $L = \mathrm{abs}(T^*)$ are invertible. (To prove necessity, consider the spectra of R and L. To prove sufficiency, use the closed graph theorem.)

(c) N is normal if and only if $\mathrm{abs}(N) = \mathrm{abs}(N^*)$, and if this condition is satisfied there exists a unitary operator W such that $N = W \cdot \mathrm{abs}(N)$.

7. A compact operator T on a separable Hilbert space E is said to be *nuclear* if, denoting by (λ_n) the full sequence of eigenvalues of $\mathrm{abs}(T)$ (Section 11.5, Problem 8), we have $\sum_{n=1}^{\infty} \lambda_n < +\infty$.

(a) Use polar decomposition (Problem 6) to show that the product S_1S_2 of two Hilbert–Schmidt operators is nuclear. Conversely, if T is nuclear, then $\mathrm{abs}(T)^{1/2}$ is a self-adjoint Hilbert–Schmidt operator, and T is the product of two Hilbert–Schmidt operators. Consequently T^* is also nuclear. If A is any continuous operator on E, then AT and TA are nuclear.

(b) If A and B are two Hilbert–Schmidt operators and if (e_n) is a Hilbert basis of E, then the series $\sum_n (AB \cdot e_n \mid e_n)$ and $\sum_n (BA \cdot e_n \mid e_n)$ are absolutely convergent, and their sums are equal. (Write $B \cdot e_n = \sum_m (B \cdot e_n \mid e_m) e_m$.) Consequently, for every unitary operator U and every nuclear operator T, we have

$$\sum_n (U^{-1}TU \cdot e_n \mid e_n) = \sum_n (T \cdot e_n \mid e_n).$$

Deduce that, for a nuclear operator T, the sum $\sum_n (T \cdot e_n \mid e_n)$ is independent of the Hilbert basis (e_n) chosen. This sum is called the *trace* of T and is denoted by $\mathrm{Tr}(T)$. If A, B are two Hilbert–Schmidt operators, then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA) = (A \mid B^*)$.

(c) If T is nuclear, show that

$$\mathrm{Tr}(\mathrm{abs}(T)) = \sup\left(\sum_n |(T \cdot a_n \mid b_n)|\right)$$

where the supremum is taken over all pairs of Hilbert bases (a_n), (b_n) of E (use the polar decomposition of E). If we put $\|T\|_1 = \mathrm{Tr}(\mathrm{abs}(T))$, then the set $\mathscr{L}_1(\mathrm{E})$ of nuclear operators on E is a vector space on which $\|T\|_1$ is a norm, such that $\|T\|_2 \leqq \|T\|_1$.

(d) If (T_ν) is a sequence of nuclear (resp. Hilbert–Schmidt) operators on E which converges weakly (Section 12.15, Problem 9) to an operator T, and which is such that the sequence of norms $(\|T_\nu\|_1)$ (resp. $(\|T_\nu\|_2)$) is bounded, then T is a nuclear (resp. Hilbert–Schmidt) operator.

(e) Show that $\mathscr{L}_1(\mathrm{E})$ is a Banach space with respect to the norm $\|T\|_1$.

(f) Let (λ_n) be the sequence of eigenvalues of a nuclear operator T, each counted according to its algebraic multiplicity (11.4.1). Show that $\sum_n |\lambda_n| \leqq \|T\|_1$. (For each integer p, consider the sum V of the p spaces $N(\mu_k; T)$ $(1 \leqq k \leqq p)$, where $\mu_1, \ldots, \mu_p$ are the first p distinct eigenvalues in the sequence (λ_n). Take a Hilbert basis of V with respect to which the matrix of $T|V$ is triangular, and use (c).) Deduce that if T is a Hilbert–Schmidt operator and if (λ_n) is the sequence of its eigenvalues, each counted with its algebraic multiplicity, then $\sum_n |\lambda_n|^2 \leqq \|T\|_2^2$.

(g) If a continuous operator $T \in \mathscr{L}(E)$ is such that, for each pair of Hilbert bases (a_n), (b_n) of E, the series $\sum_n (T \cdot a_n \mid b_n)$ is convergent, then T is nuclear. (Write $T = LU^*$ (Problem 6(a)), and by choosing (a_n) and (b_n) suitably show that $L^{1/2}$ is a Hilbert–Schmidt operator.)

(h) For a continuous operator $T \in \mathscr{L}(E)$ to be nuclear, it is necessary and sufficient that, for at least one Hilbert basis (e_n) of E, the series $\sum_n \|T \cdot e_n\|$ should converge. (Write $T = UR$ (Problem 6(a)) and note that $(R \cdot e_n \mid e_n) \leqq \|T \cdot e_n\|$. This proves that the condition is sufficient. Conversely, take for (e_n) a basis consisting of eigenvectors of R.)

(i) In the space $E = l^2_{\mathbf{C}}$, let (e_n) be the canonical Hilbert basis, and let $a = \sum_n (1/n)e_n$. If F is the subspace $\mathbf{C} \cdot a$ of dimension 1, show that the projector P_F is nuclear, but that the series $\sum_n \|P_F \cdot e_n\|$ does not converge.

8. Show that, for every normal operator N on a separable Hilbert space E, there exists a normal operator N' such that $N'^2 = N$. Give examples where there exist infinitely many such operators.

9. Let T be a continuous operator on a separable Hilbert space E.

(a) For T to be a topological left zero-divisor (Section 15.2, Problem 3) in the algebra $\mathscr{L}(E)$, it is necessary and sufficient that there should exist a sequence (x_n) of vectors in E such that $\|x_n\| = 1$ for all n and such that $(T \cdot x_n)$ tends to 0. The complex numbers ζ such that $T - \zeta \cdot 1$ is a topological left zero-divisor in $\mathscr{L}(E)$ form what is called the *approximative point-spectrum* Spa(T). Thus $\zeta \notin \mathrm{Spa}(T)$ means that $T - \zeta \cdot 1$ is injective and a homeomorphism of E onto a closed subspace of E. Show that Spa(T) is closed in C, and contains the frontier of Sp(T). If P is any polynomial, show that

$$\mathrm{Spa}(P(T)) = P(\mathrm{Spa}(T)).$$

(b) Let $T \in \mathscr{L}(E)$. For each $\lambda \in \mathbf{C}$, let $m(T, \lambda)$ denote the dimension of $\mathrm{Ker}(T^* - \lambda \cdot 1)$ (so that $m(T, \lambda)$ is either a nonnegative integer or $+\infty$), equal also to the codimension of $\overline{(T - \lambda \cdot 1)(E)}$. Let $T_0 \in \mathscr{L}(E)$ and let λ_0 be a complex number not belonging to Spa(T_0). Show that there exists a number $\varepsilon > 0$ such that $m(T, \lambda) = m(T_0, \lambda_0)$ whenever $\|T - T_0\| \leqq \varepsilon$ and $|\lambda - \lambda_0| \leqq \varepsilon$.

(c) Deduce from (b) that if K is a compact subset of C which does not intersect Spa(T_0), then there exists $\varepsilon > 0$ such that $m(T, \lambda) = m(T_0, \lambda)$ for all $\lambda \in \mathbf{K}$ and all $T \in \mathscr{L}(E)$ such that $\|T - T_0\| \leqq \varepsilon$.

(d) Let $T \in \mathscr{L}(E)$ and let K be a compact subset of $\mathrm{Sp}(T) - \mathrm{Spa}(T)$ with the following properties: (1) $0 \notin \mathbf{K}$; (2) the inverse image K′ of K under the mapping $\zeta \mapsto \zeta^2$ is convex; (3) $m(T, \lambda) = 1$ for all $\lambda \in \mathbf{K}$. Then there exists *no* operator $T' \in \mathscr{L}(E)$ such that $T'^2 = T$. (Supposing the contrary, let $\mathbf{L} = \mathbf{K}' \cap \mathrm{Sp}(T')$; then we have $\mathbf{L} \subset \mathrm{Sp}(T') - \mathrm{Spa}(T')$, and $\mathbf{L} \cup (-\mathbf{L}) = \mathbf{K}'$. Show that $\lambda \in \mathbf{L}$ implies that $-\lambda \notin \mathbf{L}$

so that $L \cap (-L) = \varnothing$, which is a contradiction. Observe that if $\lambda \in L$, then $\bar{\lambda}^2$ is an eigenvalue of T^* and $m(T, \bar{\lambda}^2) = 1$.) Show also that, if T is invertible, there exists $\varepsilon > 0$ such that every operator $T_1 \in \mathscr{L}(E)$ satisfying $\|T_1 - T\| \leqq \varepsilon$ is invertible and such that there exists no operator $T' \in \mathscr{L}(E)$ for which $T'^2 = T_1$ (use (c).)

10. Let Ω be a bounded open subset of $\mathbf{C}$, and let H be the Hilbert space of analytic functions f on Ω such that

$$\iint_{\Omega} |f(x+iy)|^2 \, dx \, dy < +\infty$$

(cf. Problem in Section 9.13). Let T be the operator which maps each $f \in H$ to the function $\zeta \mapsto \zeta f(\zeta)$. Show that, for each $\lambda \in \Omega$ and each function $g \in H$ such that $g(\lambda) = 0$, there exists a unique $f \in H$ such that $(T - \lambda I) \cdot f = g$. Also, if the disk $|\zeta - \lambda| < \delta$ is contained in Ω, then $\|g\|^2 \geqq \frac{1}{2}\delta^2 \|f\|^2$. Deduce that $\mathrm{Sp}(T)$ is the closure $\bar{\Omega}$ of Ω in $\mathbf{C}$, and that Ω is contained in $\mathrm{Sp}(T) - \mathrm{Spa}(T)$.

Deduce from these results that if Ω is taken to be the open annulus $r_1 < |z| < r_2$ (where $r_1 > 0$), then the operator T is invertible and has no square root in $\mathscr{L}(H)$ (cf. Problem 9).

11. Let E be a separable Hilbert space, K a compact subset of $\mathbf{C}$, and $(x, y) \mapsto m_{x,y}$ a continuous sesquilinear mapping of $E \times E$ into the space $M_{\mathbf{C}}(K)$ of complex measures on K. Suppose that

$$(x \mid y) = \int dm_{x,y}(\zeta), \qquad \bar{m}_{x,y} = m_{y,x}$$

and that the measure $m_{x,x}$ is *positive* for all $x \in E$. Let T be the continuous operator on E such that

$$(T \cdot x \mid y) = \int \zeta \, dm_{x,y}(\zeta)$$

for all x, y in E. For each function $f \in \mathscr{U}_{\mathbf{C}}(K)$, let $f(T)$ denote the operator defined by

$$(f(T) \cdot x \mid y) = \int f(\zeta) \, dm_{x,y}(\zeta)$$

for all x, y in E (Section 15.10, Problem 1). The mapping $T \to f(T)$ of $\mathscr{U}_{\mathbf{C}}(K)$ into $\mathscr{L}(E)$ is linear and such that $T^* = c(T)$, where $c(\zeta) = \bar{\zeta}$, but this mapping is *not* in general an algebra homomorphism. Prove that there exists a separable Hilbert space H, the Hilbert sum of E and another Hilbert space F, and a *representation* $f \mapsto V(f)$ of $\mathscr{U}_{\mathbf{C}}(K)$ on $\mathscr{L}(H)$ such that, if P is the orthogonal projection of H on E, we have $f(T) = P\,V(f) \mid E$. (Apply Problem 6 of Section 15.9, by taking Γ to be the set of all finite products of characteristic functions $\varphi_{A_n} (n \in \mathbf{N})$ of universally measurable sets in K, chosen in such a way that these functions form a total set in *each* of the spaces $\mathscr{L}_{\mathbf{C}}(K, m_{x_n, x_n})$, where (x_n) is a dense sequence in E.) ("Neumark's theorem.")

12. Let E be a separable Hilbert space.

Let H be a self-adjoint operator on E such that $0 \leqq H \leqq 1_E$. Show that there exists a separable Hilbert space G which is the Hilbert sum of E and a Hilbert space F, and an orthogonal projector Q on G such that $H = PQ \mid E$, where P is the orthogonal projection of G on E. (For x, y in E, define $m_{x,y}$ to be the measure carried by the set of two points $\{0, 1\}$ such that $m_{x,y}(\{0\}) = ((1_E - H) \cdot x \mid y)$ and $m_{x,y}(\{1\}) = (H \cdot x \mid y)$, and apply Problem 11.)

13. Let E be a separable Hilbert space, and (H_n) a sequence of self-adjoint operators with the following property: there exists an interval $[-M, M]$ in $\mathbf{R}$ such that, if $P(X) = a_0 + a_1X + \cdots + a_nX^n$ is any polynomial with real coefficients and $P(\xi) \geqq 0$ for $-M \leqq \xi \leqq M$, then also $a_0I + a_1H_1 + \cdots + a_nH_n \geqq 0$ (which implies *inter alia* that $-M \cdot I \leqq H_n \leqq M \cdot I$ for all n). Show that there exists a separable Hilbert space G which is the Hilbert sum of E and a Hilbert space F, and a self-adjoint operator H on G such that $H_n = PH^n \mid E$ for all $n \geqq 1$, where P is the orthogonal projection of G on E. (Using Problem 5 of Section 13.20, prove that for each pair (x, y) of elements of E there exists a real measure $m_{x,y}$ on $[-M, M]$ such that

$$(H_n \cdot x \mid y) = \int \xi^n \, dm_{x,y}(\xi), \qquad (x \mid y) = \int dm_{x,y}(\xi)$$

for all $n \geqq 1$.) Deduce that

$$H_1^2 \leqq H_2, \qquad H_{2n+1}^2 \leqq \|H_{2n}\| H_{2n+2}.$$

14. Let E be a separable Hilbert space and T a continuous operator on E such that $\|T\| \leqq 1$. Then there exists a separable Hilbert space H, the Hilbert sum of E and a Hilbert space F, and a unitary operator U on H such that, if P is the orthogonal projection of H on E, then $T^n = PU^n \mid E$ for all $n \geqq 1$. (Apply Problem 6 of Section 15.9, taking $\Gamma = \mathbf{Z}$, the involution on Γ being $n \mapsto -n$, and the representation of Γ in H such that $U(n) = T^n$ for all $n \geqq 0$. Observe that, for all $y \in E$ and all $\zeta \in C$ such that $|\zeta| < 1$, we have

$$(*) \qquad \mathscr{R}((1 + \zeta T) \cdot y \mid (1 - \zeta T) \cdot y) \geqq 0,$$

and note that every $x \in E$ can be written in the form $(I - \zeta T) \cdot y$ for some $y \in E$. In particular, this is so for every linear combination $x = \sum_{-\infty}^{+\infty} x_n e^{-ni\varphi}$, where the $x_n \in E$ are zero except for finitely many indices n, and where $\zeta = re^{i\varphi}$ with $r > 0$. Then express

$$\sum_{m,n} (r^{|n-m|} U(n - m) \cdot x_n \mid x_m)$$

in terms of the left-hand side of (*), and let r tend to 1.)

Deduce that, if $\sum_{n=0}^{\infty} c_n z^n$ is a power series which converges absolutely on the circle $|z| = 1$, and if $u(T) = \sum_{n=0}^{\infty} c_n T^n$, then the relation $|u(z)| \leqq 1$ (resp. $\mathscr{R}u(z) \geqq 0$) for $|z| \leqq 1$ implies that $\|u(T)\| \leqq 1$ (resp. $u(T) + u(T^*) \geqq 0$). (Note that $u(T) = Pu(U) \mid E$.)

15. If N is a normal continuous operator on a separable Hilbert space E, and K a compact subset of C containing $\mathrm{Sp}(N)$, then we have a (nonfaithful) representation $f \mapsto f(N)$ of $\mathscr{C}_C(K)$ in E; the measures $m_{x,y}$ can be considered as measures on K satisfying (15.11.2) and (15.11.2.1). In particular, if U is a *unitary* operator on E, then $f \mapsto f(U)$ is a representation of $\mathscr{C}_C(U)$ in E. Let (e_n) be a Hilbert basis of E indexed by $\mathbf{Z}$, and let U be the unitary operator on E such that $U \cdot e_n = e_{n+1}$. Show that the representation $f \mapsto f(U)$ of $\mathscr{C}_C(U)$ on E is topologically cyclic and that e_0 is a totalizing vector; and that $\mu = m_{e_0, e_0}$ is the normalized Haar measure on U (cf. (7.4.2)). Deduce that $\mathrm{Sp}(U)$ is the whole of the circle U, and also give a direct proof of this fact.

Give examples of closed vector subspaces of E which are stable under U but not under $U^* = U^{-1}$.

16. Let X be a locally compact space, μ a bounded positive measure on X with total mass 1, and u a μ-measurable mapping of X into X such that μ is *invariant* with respect to u (Section 13.9, Problem 24). Let U be the unitary operator on $L^2_{\mathbf{C}}(X, \mu)$ such that $U \cdot \tilde{f} = (f \circ u)^{\sim}$ (Section 13.11, Problem 10).

(a) The mapping u is said to be *mixing* (resp. *weakly mixing*) with respect to μ if, for each pair of μ-measurable subsets A, B of X we have

$$\lim_{n \to \infty} (\mu(u^{-n}(A) \cap B)) = \mu(A)\mu(B)$$

(resp.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(u^{-k}(A) \cap B) - \mu(A)\mu(B)| = 0.)$$

Every mixing mapping is weakly mixing. Every weakly mixing mapping is ergodic (Section 13.9, Problem 13(d)).

(b) Show that u is mixing (resp. weakly mixing) if and only if, for each pair of functions f, g in $\mathscr{L}^2_{\mathbf{C}}(X, \mu)$, we have

$$\lim_{n \to \infty} (U^n \cdot \tilde{f} \mid \tilde{g}) = (\tilde{f} \mid \tilde{1})(\tilde{1} \mid \tilde{g})$$

(resp.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |(U^n \cdot \tilde{f} \mid \tilde{g}) - (\tilde{f} \mid \tilde{1})(\tilde{1} \mid \tilde{g})| = 0).$$

An equivalent condition is that, for each $f \in \mathscr{L}^2_{\mathbf{C}}(X, \mu)$ such that $(\tilde{f} \mid \tilde{1}) = 0$ (i.e., $\int f \, d\mu = 0$), we have

$$\lim_{n \to \infty} (U^n \cdot \tilde{f} \mid \tilde{f}) = 0$$

(resp.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |(U^k \cdot \tilde{f} \mid \tilde{f})|^2 = 0).$$

(Replace f by $f + g$, and remark that if a sequence (a_n) satisfies the condition $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |a_k|^2 = 0$, then also $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |a_k| = 0$, by the Cauchy–Schwarz inequality.)

(c) For u to be ergodic with respect to μ, it is necessary and sufficient that 1 should be an eigenvalue of U with multiplicity 1. If this is so, then all the eigenvalues of U have multiplicity 1 and form a subgroup of the group U of complex numbers of absolute value 1. For each eigenvector $\tilde{f} \in L^2_{\mathbf{C}}(X, \mu)$ of U, the function $|f|$ is constant almost everywhere. (Remark that if $U \cdot \tilde{f} = \lambda \tilde{f}$ and $U \cdot \tilde{g} = \lambda \tilde{g}$, then $U \cdot (g/f)^{\sim} = (g/f)^{\sim}$.)

(d) Show that the following properties are equivalent:

(α) u is weakly mixing with respect to μ.

(β) $u \times u$ is an ergodic mapping of $X \times X$ into $X \times X$ with respect to the measure $\mu \otimes \mu$.

(γ) The only eigenvalue of U is 1.

(To show that (α) implies (β), consider subsets $M \times N$ of $X \times X$, where M and N are μ-measurable subsets of X. To prove that (β) implies (γ), observe that if $\tilde{f}$ is an

eigenvector of U, then $(f \otimes \check{f})^{\sim}$ is an eigenvector of the unitary operator corresponding to $u \times u$, for the eigenvalue 1. To prove that (γ) implies (α), use the last criterion of (b); if $(\check{f} \mid \tilde{1}) = 0$, introduce the measure $\nu = m_{\check{f},\check{f}}$ (15.11.1), and observe that this measure on $\mathbf{U}$ is diffuse (15.11.6). Then we are reduced to proving that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \int_{\mathbf{U}} \zeta^k \, d\nu(\zeta) \right|^2 = 0.$$

Write this relation in the form

$$\lim_{n\to\infty} \iint \left(\frac{1}{n} \sum_{k=0}^{n-1} (\xi\bar{\eta})^k \right) d\nu(\xi)\, d\nu(\eta) = 0,$$

and remark that the diagonal of $\mathbf{U} \times \mathbf{U}$ is $(\nu \otimes \nu)$-negligible.)

(e) With the notation of Section 13.9, Problem 13(c), show that if θ is irrational, the mapping $z \mapsto e^{2\pi i\theta} z$ of $\mathbf{U}$ onto $\mathbf{U}$ is not weakly mixing. (Calculate the spectrum of the corresponding unitary operator U.)

(f) Suppose that the space $\mathrm{L}^2_{\mathbf{C}}(\mathrm{X}, \mu)$ is the Hilbert sum of the subspace $\mathbf{C} \cdot \tilde{1}$ (the classes of the constant functions) and an at most denumerable family $(\mathrm{H}_j)_{j \in \mathrm{J}}$ of Hilbert spaces, where each H_j has a Hilbert basis $(e_{nj})_{n \in \mathbf{Z}}$ such that $U \cdot e_{nj} = e_{n+1,j}$ for all $n \in \mathbf{Z}$ (cf. Problem 15). Then the mapping u is mixing. In particular, if. (X, μ, μ) is the Bernoulli scheme $\mathrm{B}(\frac{1}{2}, \frac{1}{2})$ (Section 13.21, Problem 18) then u is mixing. (If, for each $n \in \mathbf{Z}$, f_n is the function on $\mathrm{I}^{\mathbf{Z}}$ such that $f_n(x) = -1$ if $\mathrm{pr}_n x = 0$, and $f_n(x) = 1$ if $\mathrm{pr}_n x = 1$, then the classes of the finite products $f_{n_1} f_{n_2} \cdots f_{n_k}$, in which all the indices are distinct, form together with the class of 1 an orthonormal basis of $\mathrm{L}^2_{\mathbf{C}}(\mathrm{X}, \mu)$.) Likewise show that, if X is the torus $\mathbf{T}^2$, $\pi : \mathbf{R} \to \mathbf{T}$ the canonical homomorphism and μ the normalized Haar measure on X, then u defined by

$$u(\pi(x), \pi(y)) = (\pi(x + y), \pi(x + 2y))$$

is mixing.

(g) Suppose that u is ergodic with respect to μ, so that (by virtue of (c)) the eigenvalues of U form an at most denumerable subgroup G of $\mathbf{U}$, the eigenspace corresponding to an eigenvalue $\alpha \in \mathrm{G}$ being a line $\mathrm{D}(\alpha)$ in $\mathrm{L}^2_{\mathbf{C}}(\mathrm{X}, \mu)$. Show that there exists a family of eigenvectors $\check{f}_\alpha \in \mathrm{D}(\alpha)$ of U such that $|f_\alpha| = 1$ almost everywhere in X and such that $f_{\alpha\beta} = f_\alpha f_\beta$ almost everywhere, for all pairs (α, β) of points of G. (Let $\check{h}_\alpha \in \mathrm{D}(\alpha)$ be such that $|h_\alpha| = 1$ almost everywhere, for all $\alpha \in \mathrm{G}$; for each pair of points $\alpha, \beta \in \mathrm{G}$ we may write $h_{\alpha\beta} = r(\alpha, \beta) h_\alpha h_\beta$ almost everywhere, where $r(\alpha, \beta) \in \mathbf{U}$ is a constant. Denote by A the subgroup of $\mathbf{U}^{\mathrm{X}}$ generated by the h_α for $\alpha \in \mathrm{G}$ and the group of constant functions from X to $\mathbf{U}$ (which may be identified with $\mathbf{U}$). Show that there exists a homomorphism $\theta : \mathrm{A} \to \mathbf{U}$ such that $\theta(\zeta) = \zeta$ for all $\zeta \in \mathbf{U}$. For this purpose, arrange the h_α in a sequence (h_n) and proceed by induction: if θ is already defined on the subgroup A_n generated by $\mathbf{U}$ and the h_j such that $j < n$, distinguish two cases according as h_n^k is not constant almost everywhere for any nonzero $k \in \mathbf{Z}$, or on the contrary that there exists a smallest positive integer k such that h_n^k is constant almost everywhere; use the fact that for all $\zeta \in \mathbf{U}$ and all integers $k > 0$, there exists $\eta \in \mathbf{U}$ such that $\eta^k = \zeta$. Then take $f_\alpha = \overline{\theta(h_\alpha)} h_\alpha$.)

(h) Let $v : \mathrm{X} \to \mathrm{X}$ be another mapping which is ergodic with respect to μ, and let V be the corresponding unitary operator. Show that if u satisfies the hypotheses of (g) and if U, V have the *same eigenvalues*, then u and v are conjugate (Section 13.12, Problem 11).

17. Let E be an infinite-dimensional separable Hilbert space and N a normal continuous operator on E.

(a) Show that E is the Hilbert sum $E_1 + E_2$ of two infinite-dimensional subspaces, each of which is stable under N and N^*. (Reduce to the case of a simple normal operator $M_\mu(1_C)$; with the notation of (15.11.9), observe that if M is a closed subset of Sp(N), then E(M) can be finite-dimensional only if M is a finite set for which each point has μ-measure $\neq 0$. Then distinguish two cases, according as there exist infinitely many points of measure $\neq 0$ or not; in the second case, use Problem 3(b) of Section 13.18.)

(b) Deduce from (a) that there exists a decomposition of E as the Hilbert sum of an infinite sequence (E_n) of infinite-dimensional subspaces, each of which is stable under N and N^*.

18. (a) Let E be a separable Hilbert space which is the Hilbert sum of an infinite family $(E_n)_{n\in\mathbf{Z}}$ of infinite-dimensional subspaces. There exists a unitary operator S on E such that $S(E_n) = E_{n+1}$ for all $n \in \mathbf{Z}$. Show that if P (resp. Q) is the operator which is equal to S^{1-2n} (resp. S^{-2n}) on E_n for each n, then $P^2 = Q^2 = 1_E$, and $S = PQ$.

(b) Deduce from (a) and from Problem 17 that every unitary operator on an infinite-dimensional separable Hilbert space is the product of *four* involutory unitary operators.

(c) Let ω be a complex cube root of unity and let U be the homothety with ratio ω on E, which is a unitary operator. Show that U is not the product of three involutory unitary operators. (In general, in a group G, if t is in the center of G, and if there exist x, y, z in G such that $t = xyz$ and $x^2 = y^2 = z^2 = 1$, then also $t = yzx$, $t^2 = xyxy$ and $t^3 = xzy = t^{-1}$.)

19. (a) Let E be an infinite-dimensional separable Hilbert space and let $(e_n)_{n\geqslant 0}$ be a Hilbert basis of E. The continuous operator V such that $V \cdot e_n = e_{n+1}$ for all $n \geqq 0$ is called the *one-sided shift operator*; it is an isometry of E onto the hyperplane orthogonal to e_0; its spectrum is the disk $|\zeta| \leqq 1$ and contains no eigenvalue of V (Section 11.1, Problem 4); its approximative point-spectrum (Problem 9) is the circle U: $|\zeta| = 1$. The spectrum of the adjoint operator V^* is also the disk $|\zeta| \leqq 1$, and every ζ such that $|\zeta| < 1$ is an eigenvalue of V^*.

(b) Let T be a continuous operator on E which is an *isometry* of E onto a (necessarily closed) subspace T(E). Show that there exists a decomposition of E as the Hilbert sum of subspace L and an at most denumerable family $(F_i)_{i\in I}$ of subspaces, where L and each F_i is stable under T and are such that (1) $T \mid L$ is unitary and (2) each F_i is infinite-dimensional and $T \mid F_i$ is the one-sided shift operator, for a suitably chosen orthonormal basis. (Consider the orthogonal supplement N of T(E), and show that E is the Hilbert sum of the T^n(N) ($n \geqq 0$) and $L = \bigcap_n T^n(E)$.)

(c) Deduce from (a) and (b) that if T is any nonunitary isometry of E onto a subspace of E, then Sp(T) is the unit disk $|\zeta| \leqq 1$, and that $\|T - U\| = 2$ for all unitary operators U. (Observe that $\|T - U\| = \|U^*T - 1_E\|$ and that U^*T is not unitary, so that the point $\zeta = -1$ belongs to its spectrum.)

20. (a) Let E be a separable Hilbert space, T a continuous operator, and C a compact operator on E. Show that the points of Sp($T + C$) which do not belong to Sp(T) are eigenvalues of $T + C$. (Reduce to the case where $\zeta = 0$ is such a point and observe that, if T is bijective, we may write $T + C = T(1_E + T^{-1}C)$, where $T^{-1}C$ is compact; if $-1 \in$ Sp($T^{-1}C$), it follows that -1 is an eigenvalue of $T^{-1}C$.)

(b) With the notation of Problem 15, let C be the operator of rank 1 defined by $C \cdot x = -(x \mid e_{-1})e_0$. Show that $\mathrm{Sp}(U+C)$ is the disk $|\zeta| \leqq 1$ while $\mathrm{Sp}(U)$ is the circle $|\zeta| = 1$. (Consider separately the restrictions of $U+C$ to the subspace generated by the e_n with $n \geqq 0$ and to its orthogonal supplement.)

(c) Let N be a normal operator on E and C a compact operator on E. If $\mathrm{Sp}(N)$ is nondenumerable, show that the same is true of $\mathrm{Sp}((N+C)^*(N+C))$ (use (a) above and (15.11.8(i)). Deduce that the one-sided shift operator V (Problem 19(a)) cannot be of the form $N+C$ (observe that $V^*V = 1_E$).

21. Let E be an infinite-dimensional separable Hilbert space.

(a) Show that every nonzero two-sided ideal $\mathfrak{J}$ of the ring $\mathscr{L}(E)$ contains the ideal $\mathfrak{S}$ of operators of finite rank. (If $T \neq 0$ belongs to $\mathfrak{J}$, show that every operator of rank 1 can be written in the form BTC for suitably chosen operators B and C.)

(b) Show that the only closed two-sided ideal of the Banach algebra $\mathscr{L}(E)$, other than $\mathscr{L}(E)$ and $\{0\}$, is the ideal $\mathfrak{C}$ of compact operators. (First observe that $\mathfrak{C}$ is the closure of $\mathfrak{S}$, and then that if a two-sided ideal contains a noncompact operator, then it also contains a noncompact positive hermitian operator H (Problem 6). For such an operator, show that there exists an interval $M = [\alpha, +\infty[$ with $\alpha > 0$ such that the space E(M) (in the notation of (15.11.9)) is infinite-dimensional. If V is an isometry of E onto E(M), show that V^*HV is invertible.)

22. Let E be an infinite-dimensional separable Hilbert space. An *operator with index* on E is a continuous operator T such that (1) $T(E)$ is closed and of finite codimension; (2) $T^{-1}(0)$ is finite-dimensional.

(a) If T is an index operator, show that there exists a continuous operator A such that $1_E - AT$ and $1_E - TA$ are of finite rank. (Show that T is a homeomorphism of the orthogonal supplement F of $T^{-1}(0)$ onto $T(E)$, and take A to be the inverse homeomorphism on F and zero on the orthogonal supplement of F.)

(b) Conversely, suppose that T is a continuous operator on E, for which there exists a continuous operator A such that $1_E - AT$ and $1_E - TA$ are compact. Show that T is an index operator. (Using (11.3.2), show first that the kernels of T and T^* are finite-dimensional, and hence that $T(E)$ has finite codimension. Then use the fact that the restriction of AT to the orthogonal supplement F of the kernel of AT is a homeomorphism onto its image, and finally use (12.13.2(iii)).)

23. Let E be an infinite-dimensional separable Hilbert space.

(a) Let T be a continuous operator on E such that $T^{-1}(0)$ is of infinite dimension. Then E is the Hilbert sum of an infinite sequence $(E_n)_{n \geqslant 0}$ of infinite-dimensional subspaces, such that the E_n with $n \geqq 1$ are contained in $T^{-1}(0)$. For each $n \geqq 1$, let S_n denote an isometry of E_0 onto E_n. Let A be the continuous operator which on E_0 is equal to S_1, and on E_n is equal to $S_{n+1}S_n^{-1}$, for all $n \geqq 1$. Also let V be the operator which is zero on E_0, is equal to S_1^{-1} on E_1, and is equal to $S_{n-1}S_n^{-1}$ on E_n for all $n \geqq 2$. Let T_0 denote the restriction of $P_{E_0}T$ to E_0, and let B be the continuous operator which is equal to VT on E_0, to $-T_0S_1^{-1}$ on E_1, and to $-S_{n-1}T_0S_n^{-1}$ on E_n for all $n \geqq 2$. Prove that $T = AB - BA$.

(b) Deduce from (a) that for *any* continuous operator T on E there exist four operators A, B, C, D such that

$$T = (AB - BA) + (CD - DC).$$

(Write T as the sum of two continuous operators, each of which has an infinite-dimensional kernel.)

24. Let E be a separable Hilbert space and H a positive self-adjoint operator; then $I+\lambda H$ is invertible for all $\lambda > 0$. For $x \in E$ and $\lambda > 0$ let $F_\lambda(x) = (\lambda(I+\lambda H)^{-1} \cdot x \mid x)$. Show that $F_\lambda(x)$ is increasing as a function of λ, and that it is bounded if and only if $x \in H^{1/2}(E)$. (Reduce to the case where H is a simple operator (15.11.3).)

25. Let E_0 be a *real* Hilbert space and E the Hilbert space obtained by extending the field of scalars of E_0 to C, so that every element of E is uniquely of the form $x+iy$ with $x, y \in E_0$, and

$$(x'+iy' \mid x''+iy'') = (x' \mid x'') + (y' \mid y'') + i(y' \mid x'') - i(x' \mid y'').$$

Show that every self-adjoint operator H_0 on E_0 extends uniquely to a self-adjoint operator H on E, having the same spectrum.

26. With the notation of Problem 2 of Section 13.13, suppose that for each compact subset K of X there exists a constant $b_K \geqq 0$ such that

$$\text{(B)} \qquad \int_K |u|^2 \, d\mu \leqq b_K |u|^2$$

for all $u \in \mathscr{H}$. This condition implies condition (A) of Problem 2, Section 13.13, but is not equivalent to (A).

(a) In Problem 5(b) of Section 6.6, suppose that X is compact and that the functions f_n are real-valued, bounded and measurable with respect to a positive measure μ on X and that they satisfy the condition

$$\sum_n \|f_n\|^2 < +\infty,$$

where $\|f_n\| = \sup_{x \in X} |f_n(x)|$. Show that the space $\mathscr{H}$ of functions which are μ-equivalent to the functions belonging to the space denoted by E in the problem referred to, is such that $H = \mathscr{H}/\mathscr{N}$ is a Hilbert space isomorphic to E, and satisfying condition (B) above.

(b) For every function $f \in \mathscr{L}^2_{\mathbf{R}}(X, \mu)$ with *compact* support K, there exists a function $U^f \in \mathscr{H}$ such that $(U^f \mid u) = \int uf \, d\mu$ for all $u \in \mathscr{H}$. The class of U^f is uniquely determined by the class of f, and we have $|U^f| \leqq b_K^{1/2} N_2(f)$. Then the set $\mathscr{P}$ defined in Section 13.13, Problem 2(b) is also the closure of $\mathscr{H}$ in the set of the U^f for which f is $\geqq 0$, compactly supported and belongs to $\mathscr{L}^2_{\mathbf{R}}$. Generalize the result of part (e) of this problem to the case where $f \in \mathscr{L}^2_{\mathbf{R}}$ is compactly supported and $\geqq 0$ almost everywhere. Likewise, generalize part (f) of the same problem.

(c) Suppose that X is *compact*. Then U^f is defined for all functions $f \in \mathscr{L}^2_{\mathbf{R}}(X, \mu)$, and we have $N_2(U^f) \leqq b_X N_2(f)$. If $G \cdot \tilde{f}$ is the class of U^f, then G is a continuous *positive self-adjoint* operator on $L^2_{\mathbf{R}}(X, \mu)$. If F is the closure in $L^2_{\mathbf{R}}$ of $G^{1/2}(L^2_{\mathbf{R}})$ (which is the orthogonal supplement of $\mathrm{Ker}(G^{1/2}) = \mathrm{Ker}(G)$), then the restriction of $G^{1/2}$ to F is an *isometry* of the subspace F of $L^2_{\mathbf{R}}$ onto the Hilbert space H (equipped with the norm $|\tilde{u}|$). Hence $H = G^{1/2}(L^2_{\mathbf{R}})$.

(d) Suppose that X is compact and that the "domination principle" is satisfied, in the form of (b): that is to say, if $f \in \mathscr{L}^2_{\mathbf{R}}$ is $\geqq 0$ almost everywhere, and if $u \in \mathscr{P}$ is such that $U^f(x) \leqq u(x)$ almost everywhere *in the set of points x where* $f(x) > 0$, then $U^f(x) \leqq u(x)$ almost everywhere *in* X. For each $\lambda > 0$, put $R_\lambda = G(I+\lambda G)^{-1}$. If $f \in \mathscr{L}^2_{\mathbf{R}}$ is $\geqq 0$ almost everywhere, and if g is a function whose class is equal to

$R_\lambda \cdot \tilde{f}$, then $g(x) \geqq 0$ almost everywhere. (Observe that

$$\lambda U^{g^+}(x) \leqq U^f(x) + \lambda U^{g^-}(x)$$

almost everywhere in the set of x such that $g^+(x) > 0$.)

Deduce that, for all $u \in H$, we have $|u| \in H$ and $|(|u|)^\sim| \leqq |\tilde{u}|$. (If $F_\lambda(\tilde{v}) = (\lambda(I + \lambda G)^{-1} \cdot \tilde{v} \mid \tilde{v})$ for $v \in \mathscr{L}^2_{\mathbf{R}}$, show that $F_\lambda(|u|^\sim) \leqq |\tilde{u}|^2$, and use (c) above and Problem 24.)

(e) Generalize the results of (d) to the case where X is locally compact. (Let (K_n) be a sequence of compact subsets which cover X, and such that each is contained in the interior of the next. For each n, consider the space $\mathscr{H}_n$ of restrictions to K_n of functions of the form U^f, where $f \in \mathscr{L}^2_{\mathbf{R}}(X, \mu)$ and $\mathrm{Supp}(f) \subset K_n$; apply (d) to each of these spaces.)

12. UNBOUNDED NORMAL OPERATORS

(15.12.1) Let E be a separable Hilbert space, and let I denote the identity mapping of E. A (not necessarily continuous) linear mapping T of a subspace $\mathrm{dom}(T)$ of E (the "domain" of T, which need not be closed) into E will be called, by abuse of language, a *not necessarily bounded operator* on E, or simply an *unbounded operator* on E. The graph $\Gamma(T)$ (1.4) is a vector subspace of $E \times E$, and T is said to be a *closed* operator if $\Gamma(T)$ is closed in the product space $E \times E$. The *kernel* $\mathrm{Ker}(T)$ of a closed operator is *closed in* E, because it may be identified with the intersection of $\Gamma(T)$ and $E \times \{0\}$ in $E \times E$.

Throughout, it is to be understood that an equality $T_1 = T_2$ between two unbounded operators on E *implies the equality*

$$\mathrm{dom}(T_1) = \mathrm{dom}(T_2).$$

(15.12.2) *Let T be an unbounded operator on a Hilbert space* E. *Then, of the three properties*:

(i) $\mathrm{dom}(T)$ *is closed in* E; (ii) T *is closed*; (iii) T *is continuous*; *any two imply the third.*

If T is continuous, then $\Gamma(T)$ is closed in $\mathrm{dom}(T) \times E$ and therefore also in $E \times E$ if $\mathrm{dom}(T)$ is closed in E. Next, if T is continuous on $\mathrm{dom}(T)$, then it extends by continuity to a linear operator T' which is continuous on $\mathrm{dom}(T') = \overline{\mathrm{dom}(T)}$ (5.5.4), and $\Gamma(T')$ is the closure of $\Gamma(T)$ in $E \times E$. Hence if T is closed we have $\Gamma(T') = \Gamma(T)$, so that $\mathrm{dom}(T)$ is closed in E. Finally, if $\mathrm{dom}(T)$ is closed in E and if $\Gamma(T)$ is closed in $E \times E$, then T is continuous by the closed graph theorem (12.16.11).

(15.12.3) In what follows we shall be concerned with unbounded operators T such that $\mathrm{dom}(T)$ is *dense in* E. Let F be the set of all $y \in E$ such that the

linear form $x \mapsto (T \cdot x \mid y)$ is *continuous on* $\mathrm{dom}(T)$; when this is so, this linear form extends by continuity to the whole of E (5.5.4) and therefore can be written as $x \mapsto (x \mid T^* \cdot y)$ for a *uniquely* determined vector $T^* \cdot y$, because $\mathrm{dom}(T)$ is dense in E (6.3.2). This uniqueness shows that T^* is a linear mapping of F into E, and therefore an unbounded operator, called the *adjoint* of T. (When $\mathrm{dom}(T) = \mathrm{E}$ and T is continuous, this definition clearly agrees with that of (11.5).) Hence we have

$$(T \cdot x \mid y) = (x \mid T^* \cdot y) \tag{15.12.3.1}$$

for all $x \in \mathrm{dom}(T)$ and $y \in \mathrm{dom}(T^*)$.

If T_1 is an unbounded operator such that $\mathrm{dom}(T_1) \supset \mathrm{dom}(T)$, then $\mathrm{dom}(T_1^*) \subset \mathrm{dom}(T^*)$.

In what follows we shall endow $\mathrm{E} \times \mathrm{E}$ with the structure of a *Hilbert space* such that

$$((x_1, x_2) \mid (y_1, y_2)) = (x_1 \mid y_1) + (x_2 \mid y_2),$$

so that $\mathrm{E} \times \mathrm{E}$ is the Hilbert sum of its two subspaces $\mathrm{E} \times \{0\}$ and $\{0\} \times \mathrm{E}$, each of which is isomorphic to E. Also we denote by J the continuous operator $(x, y) \mapsto (y, -x)$; clearly J is a *unitary* operator on $\mathrm{E} \times \mathrm{E}$, and $J^2 = -I$.

(15.12.4) *Let T be an unbounded operator on* E, *such that* $\mathrm{dom}(T)$ *is dense in* E.

(i) *The adjoint operator T^* is closed, and its graph $\Gamma(T^*)$ is the orthogonal supplement of $J(\overline{\Gamma(T)})$ in the Hilbert space* $\mathrm{E} \times \mathrm{E}$.

(ii) *The following properties are equivalent*:

(a) *T can be extended to a closed operator.*

(b) $\mathrm{dom}(T^*)$ *is dense in* E.

If these conditions are satisfied, the graph of any closed operator extending T contains the graph of T^{**}, and $\Gamma(T^{**}) = \overline{\Gamma(T)}$. (Thus T^{**} is the *smallest* closed operator which extends T, and in particular, $T^{**} = T$ if T is closed.) *Moreover*, $(T^{**})^* = T^*$.

(i) If a sequence (y_n) of points in $\mathrm{dom}(T^*)$ converges to $y \in \mathrm{E}$ and is such that the sequence $(T^* \cdot y_n)$ converges to $z \in \mathrm{E}$, then the sequence of continuous linear forms $x \mapsto (x \mid T^* \cdot y_n)$ converges for all $x \in \mathrm{E}$ to the continuous linear form $x \mapsto (x \mid z)$. But if $x \in \mathrm{dom}(T)$, we have $(x \mid z) = \lim_{n \to \infty} (T \cdot x \mid y_n) = (T \cdot x \mid y)$, hence $y \in \mathrm{dom}(T^*)$ and $z = T^* \cdot y$ by definition, which shows that T^* is closed.

On the other hand, to say that $(y, z) \in \mathrm{E} \times \mathrm{E}$ is orthogonal to all the vectors $(T \cdot x, -x)$ with $x \in \mathrm{dom}(T)$ signifies that $(T \cdot x \mid y) = (x \mid z)$, i.e., that $x \mapsto (T \cdot x \mid y)$ is continuous, hence $y \in \mathrm{dom}(T^*)$ and $z = T^* \cdot y$.

(ii) A closed vector subspace G of $E \times E$ *is not* the graph of a closed operator if and only if, for some $x \in \mathrm{pr}_1(G)$, there exist at least two distinct points (x, y_1) and (x, y_2) belonging to G, or equivalently (since G is a vector subspace) that $(0, y_1 - y_2) \in G$. But to say that $\mathrm{dom}(T^*)$ *is not* dense in E means that there exists $z \neq 0$ in E orthogonal to $\overline{\mathrm{dom}(T^*)}$ (6.3.1), or equivalently that $(z, 0)$ is orthogonal to $\Gamma(T^*)$, or again that $(0, -z)$ belongs to $\overline{\Gamma(T)}$. It follows that $(0, -z)$ cannot be contained in the graph of any closed operator which extends T. Conversely, if $\mathrm{dom}(T^*)$ is dense in E, then T^{**} is defined, and $\Gamma(T^{**})$ is the orthogonal supplement of $J(\Gamma(T^*))$; but this orthogonal supplement is also equal to

$$J(J(\overline{\Gamma(T)})) = \overline{\Gamma(T)}$$

(6.3.1). We say then that T^{**} is the *closure* of T.

(15.12.5) Given two not necessarily bounded operators U, V on E, the vector $U \cdot x + V \cdot x$ is defined for all $x \in \mathrm{dom}(U) \cap \mathrm{dom}(V)$, and we denote by $U + V$ the linear mapping $x \mapsto U \cdot x + V \cdot x$ of $\mathrm{dom}(U) \cap \mathrm{dom}(V)$ into E. In particular, if V is everywhere defined, then $\mathrm{dom}(U + V) = \mathrm{dom}(U)$, and the graph $\Gamma(U + V)$ is the image of $\Gamma(U)$ under the linear mapping $(x, y) \mapsto (x, y + V \cdot x)$ of $E \times E$ into $E \times E$. If V is *continuous* and U is *closed*, it follows therefore that $U + V$ is *closed*, since the mapping

$$(x, y) \mapsto (x, y + V \cdot x)$$

and its inverse $(x, y) \mapsto (x, y - V \cdot x)$ are continuous.

Again, the vector $U \cdot (V \cdot x)$ is defined for the set of all $x \in E$ such that $x \in \mathrm{dom}(V)$ and $V \cdot x \in \mathrm{dom}(U)$. This set is a vector subspace which we denote by $\mathrm{dom}(UV)$, and UV denotes the linear mapping $x \mapsto U \cdot (V \cdot x)$ of $\mathrm{dom}(UV)$ into E. If T is a not necessarily bounded operator which is an *injective* mapping of $\mathrm{dom}(T)$ into E, we denote by T^{-1} the inverse mapping of $T(\mathrm{dom}(T)) = \mathrm{dom}(T^{-1})$ into E. The graph $\Gamma(T^{-1})$ is the image of $\Gamma(T)$ under the mapping $(x, y) \mapsto (y, x)$. Hence T^{-1} is *closed* if T is *closed* (and injective).

(15.12.6) (von Neumann) *Let T be a closed operator on* E *such that* $\mathrm{dom}(T)$ *is dense in* E. *Then* $\mathrm{dom}(T^*T)$ *is dense in* E; *the operator T^*T is closed; and the operator $I + T^*T$ (defined on* $\mathrm{dom}(T^*T)$*) is a bijection of* $\mathrm{dom}(T^*T)$ *onto* E. *The operator $B = (I + T^*T)^{-1}$ is defined on* E, *continuous, self-adjoint and injective, and its spectrum is contained in the interval* $[0, 1]$ *of* **R**. *Furthermore, the hermitian form $(x, y) \mapsto (B \cdot x \mid y)$ is positive and nondegenerate, and $C = TB$ is a continuous operator defined on* E, *such that* $C(E) \subset \mathrm{dom}(T^*)$. *Finally,* $(T^*T)^* = T^*T$.

We have seen in (15.12.4) that $\Gamma(T)$ and $J(\Gamma(T^*))$ are orthogonal supplements of each other in $E \times E$. Hence for each $x \in E$ there exists a unique $y \in \mathrm{dom}(T)$ and a unique $z \in \mathrm{dom}(T^*)$ such that

$$(x, 0) = (y, T \cdot y) + (T^* \cdot z, -z). \tag{15.12.6.1}$$

Put $y = B \cdot x$ and $z = C \cdot x$. Clearly B and C are linear operators defined on the whole of E, and we have $B(E) \subset \mathrm{dom}(T)$ and $C(E) \subset \mathrm{dom}(T^*)$. Also, by (15.12.6.1),

$$\|x\|^2 = \|y\|^2 + \|T \cdot y\|^2 + \|z\|^2 + \|T^* \cdot z\|^2,$$

so that $\|B \cdot x\| \leqq \|x\|$ and $\|C \cdot x\| \leqq \|x\|$. Hence B and C are continuous. The relation (15.12.6.1) is equivalent to

$$x = B \cdot x + T^*C \cdot x \qquad \text{and} \qquad 0 = -C \cdot x + TB \cdot x,$$

so that $C = TB$ and $T(B(E)) \subset \mathrm{dom}(T^*)$, hence $B(E) \subset \mathrm{dom}(T^*T)$. Consequently T^*TB is defined on all of E, and we have

$$I = B + T^*TB = (I + T^*T)B,$$

which shows that B is injective and $I + T^*T$ surjective. For each $w \in \mathrm{dom}(T^*T)$ we have

$$\begin{aligned}(w + T^*T \cdot w \mid w) &= \|w\|^2 + (T^*T \cdot w \mid w) \\ &= \|w\|^2 + \|T \cdot w\|^2\end{aligned} \tag{15.12.6.2}$$

because $T = T^{**}$; this shows that the relation $w + T^*T \cdot w = 0$ implies that $w = 0$, and hence that $I + T^*T$ is a *bijective* mapping of $\mathrm{dom}(T^*T)$ onto E. Also, since $\Gamma(B)$ is closed in $E \times E$ (15.12.2), the same is true of $\Gamma(I + T^*T)$ (15.12.5), and it follows immediately (15.12.5) that T^*T is closed. We next remark that, for all, u, v in E, we have

$$\begin{aligned}(B \cdot u \mid v) &= (B \cdot u \mid B \cdot v + T^*TB \cdot v) \\ &= (B \cdot u \mid B \cdot v) + (B \cdot u \mid T^*TB \cdot v) \\ &= (B \cdot u \mid B \cdot v) + (TB \cdot u \mid TB \cdot v) \\ &= (B \cdot u \mid B \cdot v) + (T^*TB \cdot u \mid B \cdot v) \\ &= ((I + T^*T)B \cdot u \mid B \cdot v) = (u \mid B \cdot v).\end{aligned}$$

Hence B is self-adjoint. Also, replacing w by $B \cdot x$ in (15.12.6.2) we obtain, for each $x \in E$,

$$(x \mid B \cdot x) = \|B \cdot x\|^2 + \|TB \cdot x\|^2 \geqq 0;$$

since $\|B \cdot x\| \leqq \|x\|$, it follows from the above and from (15.11.7) that $\mathrm{Sp}(B)$ is contained in $[0, 1]$. Moreover, the relation $(x \mid B \cdot x) = 0$ implies $B \cdot x = 0$, hence $x = 0$, and therefore the hermitian form $(x, y) \mapsto (B \cdot x \mid y)$ is non-degenerate.

We shall prove next that $\mathrm{dom}(T^*T)$ is dense in E. If T' is the restriction of T to $\mathrm{dom}(T^*T)$, it will be enough to show that $\Gamma(T')$ is dense in $\Gamma(T)$, since $\mathrm{dom}(T^*T)$ is the first projection of $\Gamma(T')$, and $\mathrm{dom}(T)$ is dense in E. To see that the subspace $\Gamma(T')$ of the Hilbert space $\Gamma(T)$ is dense in $\Gamma(T)$, it is enough to show that if a vector $(u, T \cdot u) \in \Gamma(T)$ is orthogonal to $\Gamma(T')$, then it is zero. Now this condition is

$$((u, T \cdot u) \mid (v, T \cdot v)) = 0$$

for all $v \in \mathrm{dom}(T^*T)$, or equivalently $(u \mid v) + (T \cdot u \mid T \cdot v) = 0$; or, since $T \cdot v \in \mathrm{dom}(T^*)$, $(u \mid v) + (u \mid T^*T \cdot v) = 0$, that is to say,

$$(u \mid (I + T^*T) \cdot v) = 0.$$

But $I + T^*T$ maps $\mathrm{dom}(T^*T)$ *onto* E. Hence $u = 0$, as required.

Finally, since B is self-adjoint, $\Gamma(B)$ is the orthogonal supplement of $J(\Gamma(B))$. Since $\Gamma(B)$ is the image of $\Gamma(I + T^*T)$ under the symmetry operator $S : (x, y) \mapsto (y, x)$, and since $JS = -SJ$, it follows that $\Gamma(I + T^*T)$ is the orthogonal supplement of $J(\Gamma(I + T^*T))$; in other words (15.12.4) $(I + T^*T)^* = I + T^*T$, or equivalently $(T^*T)^* = T^*T$.

(15.12.7) A not necessarily bounded operator T is said to be *normal* if it is closed, if $\mathrm{dom}(T)$ is dense in E and and if $T^*T = TT^*$ (which, we recall, implies by definition that $\mathrm{dom}(T^*T) = \mathrm{dom}(TT^*)$). We say that T is *self-adjoint* if $\mathrm{dom}(T)$ is dense in E and if $T^* = T$ (which implies that T is closed (15.12.4)). Clearly a self-adjoint operator is *normal*. It follows from (15.12.6) that, if T is any *closed* operator such that $\mathrm{dom}(T)$ is *dense* in E, then T^*T and TT^* are *self-adjoint*.

The following theorem reduces the problem of the structure of (unbounded) normal operators to that of *continuous* normal operators:

(15.12.8) *Let* E *be a separable Hilbert space.*

(i) *If N is a not necessarily bounded normal operator, then* $\mathrm{dom}(N) = \mathrm{dom}(N^*)$ *and* $\|N \cdot x\| = \|N^* \cdot x\|$ *for all* $x \in \mathrm{dom}(N)$. *The space* E *is the Hilbert sum of a family* (E_n) *of closed subspaces, such that* $\mathrm{E}_n \subset \mathrm{dom}(N)$ *and* E_n *is stable under N and N^* for all n, so that the restriction N_n of N to* E_n *is a continuous normal operator.*

(ii) *Conversely, let* E_n *be a sequence of closed subspaces of* E, *such that* E *is the Hilbert sum of the* E_n. *For each n, let N_n be a continuous normal operator on*

E_n. *Then there exists a unique normal operator* N *on* E *such that* $\mathrm{E}_n \subset \mathrm{dom}(N)$ *and* $N_n = N \mid \mathrm{E}_n$, *for all* n. *The set* $\mathrm{F} = \mathrm{dom}(N)$ *is the set of all* $x = \sum_n x_n$ (*where* $x_n \in \mathrm{E}_n$ *for all* n) *such that*

$$\sum_n \|N_n \cdot x_n\|^2 < +\infty,$$

and we have

$$N \cdot x = \sum_n N_n \cdot x_n \qquad \text{and} \qquad N^* \cdot x = \sum_n N_n^* \cdot x_n.$$

We shall begin by proving the first assertion of (i). We have seen in the course of the proof of (15.12.6) that the graph of the restriction of N to $\mathrm{dom}(N^*N)$ is dense in $\Gamma(N)$; hence, for each $x \in \mathrm{dom}(N)$, there exists a sequence (y_n) in $\mathrm{dom}(N^*N)$ such that $\lim_{n\to\infty} y_n = x$ and $\lim_{n\to\infty} N \cdot y_n = N \cdot x$. But for each $z \in \mathrm{dom}(N^*N)$ we have

$$\|N \cdot z\|^2 = (z \mid N^*N \cdot z) = (z \mid NN^* \cdot z) = \|N^* \cdot z\|^2,$$

because $N^*N = NN^*$. Applying this result with $z = y_n - y_m$, it follows that the sequence $(N^* \cdot y_n)$ is a Cauchy sequence and therefore converges in E. Since N^* is closed (15.12.4), we conclude that $x \in \mathrm{dom}(N^*)$ and $N^* \cdot x = \lim_{n\to\infty} N^* \cdot y_n$, whence $\|N \cdot x\| = \|N^* \cdot x\|$. We have therefore proved that $\mathrm{dom}(N) \subset \mathrm{dom}(N^*)$. Since $N^{**} = N$, the operator N^* is normal and $\mathrm{dom}(N^*) \subset \mathrm{dom}(N)$.

Now consider (ii). We shall first show that, if N satisfies the conditions of (ii), then $N^*(\mathrm{E}_n) \subset \mathrm{E}_n$. For all $m \neq n$, all $y_n \in \mathrm{E}_n$ and all $x_m \in \mathrm{E}_m$, we have $(x_m \mid N^* \cdot y_n) = (N \cdot x_m \mid y_n) = 0$, because $N(\mathrm{E}_m) \subset \mathrm{E}_m$. Hence $N^* \cdot y_n$ is orthogonal to all the E_m $(m \neq n)$, hence must lie in E_n. Next we shall show that, if $x \in \mathrm{dom}(N)$, then

$$P_{\mathrm{E}_n} N \cdot x = N P_{\mathrm{E}_n} \cdot x = N_n \cdot (P_{\mathrm{E}_n} \cdot x).$$

For all $y_n \in \mathrm{E}_n$, we have

$$(P_{\mathrm{E}_n} N \cdot x \mid y_n) = (N \cdot x \mid P_{\mathrm{E}_n} \cdot y_n) = (N \cdot x \mid y_n) = (x \mid N^* \cdot y_n),$$

and since $N^* \cdot y_n = P_{\mathrm{E}_n} N^* \cdot y_n$ from above,

$$(P_{\mathrm{E}_n} N \cdot x \mid y_n) = (P_{\mathrm{E}_n} \cdot x \mid N^* \cdot y_n) = (N P_{\mathrm{E}_n} \cdot x \mid y_n) = (N_n \cdot (P_{\mathrm{E}_n} \cdot x) \mid y_n);$$

since this holds for all $y_n \in \mathrm{E}_n$, it follows that the two elements $P_{\mathrm{E}_n} N \cdot x$ and $N_n \cdot (P_{\mathrm{E}_n} \cdot x)$ of E are equal.

We shall now prove the following lemma, which generalizes (15.10.8.1):

(15.12.8.1) *Let* (E_n) *be a sequence of closed subspaces of* E, *such that* E *is the Hilbert sum of the* E_n. *For each n, let* T_n *be a continuous operator on* E_n. *Then there exists a unique closed operator T on* E *such that, for all n,*

$$E_n \subset \mathrm{dom}(T), \qquad T \mid E_n = T_n, \qquad P_{E_n} T \cdot x = T P_{E_n} \cdot x \qquad (x \in \mathrm{dom}(T)).$$

Moreover, $\mathrm{dom}(T)$ *is the set* F *of all* $x = \sum_n x_n$ *(with* $x_n \in E_n$ *for all n) such that* $\sum_n \|T_n \cdot x_n\|^2 < +\infty$, *and we have* $T \cdot x = \sum_n T_n \cdot x_n$.

Since we must have $\sum_n \|P_{E_n} \cdot (T \cdot x)\|^2 = \|T \cdot x\|^2 < +\infty$, we see that the conditions stated imply that $\mathrm{dom}(T) \subset F$ and that $T \cdot x = \sum_n T_n \cdot x_n$ for all $x = \sum_n x_n \in \mathrm{dom}(T)$. Note that F is a vector subspace of E, by virtue of the inequality $\|z + z'\|^2 \leqq 2(\|z\|^2 + \|z'\|^2)$ in E. For each $x = \sum_n x_n \in F$, write $T' \cdot x = \sum_n T_n \cdot x_n$; we shall show that the operator $T' : F \to E$ so defined is *closed*. Since clearly $\Gamma(T)$ contains all finite sums $\sum_n (x_n, T_n \cdot x_n)$, it follows that $\Gamma(T)$ is dense in $\Gamma(T')$, and since T is a closed operator, this implies that $T = T'$. It is clear that T' is linear. Also, if a sequence (x_m) in F tends to $x \in E$ and if the sequence $(T' \cdot x_m)$ tends to $y \in E$, then $P_{E_n} T' \cdot x_m = T_n P_{E_n} \cdot x_m$ tends to $P_{E_n} \cdot y$ and also to $T_n P_{E_n} \cdot x$, by virtue of the continuity of T_n; this proves that $x \in F$ and that $y = T' \cdot x$, and completes the proof of the lemma.

This lemma, applied to the situation of (ii), proves the uniqueness of N, which therefore has to be the *closed* operator (15.12.7) described in the statement of (15.12.8(ii)). It remains to prove that N is *normal*. For each $x \in F$, we have $\|N_n^* P_{E_n} \cdot x\| = \|N_n P_{E_n} \cdot x\|$, and therefore the lemma (15.12.8.1), applied to the family of continuous operators (N_n^*), shows that there exists a unique closed operator N' such that, for all n, we have $E_n \subset \mathrm{dom}(N')$, $N' \mid E_n = N_n^*$, and $P_{E_n} N' \cdot x = N' P_{E_n} \cdot x$ for all $x \in \mathrm{dom}(N')$. Moreover, we have $\mathrm{dom}(N') = \mathrm{dom}(N) = F$, and if $x = \sum_n x_n \in F$, then $N' \cdot x = \sum_n N_n^* \cdot x_n$. From these formulas it follows immediately that $(N \cdot x \mid y) = (x \mid N' \cdot y)$ for all x, y in F: in other words (15.12.3), F is contained in $\mathrm{dom}(N^*)$, and $N' = N^* \mid F$. But also $P_{E_n} N^* \cdot y = N^* P_{E_n} \cdot y$ for all n, and all $y \in F$. For if $x_n \in E_n$, then $(x_n \mid P_{E_n} N^* \cdot y) = (x_n \mid N^* \cdot y)$ (15.5.4), and since $x_n \in \mathrm{dom}(N)$

$$(x_n \mid N^* \cdot y) = (N \cdot x_n \mid y) = (N \cdot x_n \mid P_{E_n} \cdot y)$$

because $N \cdot x_n = N_n \cdot x_n \in E_n$; hence finally, since $E_n \subset \mathrm{dom}(N^*)$, we have $(x_n \mid P_{E_n} N^* \cdot y) = (x_n \mid N^* P_{E_n} \cdot y)$, which proves our assertion. Since N^* is closed (15.12.4), it follows that $N' = N^*$ by virtue of (15.12.8.1). The preced-

ing results then imply that $(N \cdot x \mid N \cdot y) = (N^* \cdot x \mid N^* \cdot y)$ for all x, y in $\mathrm{dom}(N) = \mathrm{dom}(N^*) = \mathrm{F}$. This being so, if $z \in \mathrm{dom}(NN^*)$, then we have

$$(N \cdot x \mid N \cdot z) = (N^* \cdot x \mid N^* \cdot z) = (x \mid NN^* \cdot z)$$

for all $x \in \mathrm{dom}(N)$. This proves that $N \cdot z \in \mathrm{dom}(N^*)$, by the definition of the adjoint (15.12.3), and that $N^*N \cdot z = NN^* \cdot z$. Thus we have established that $\mathrm{dom}(NN^*) \subset \mathrm{dom}(N^*N)$ and that the operators NN^* and N^*N coincide on $\mathrm{dom}(NN^*)$. The proof of (ii) is now completed by interchanging the roles of N and N^* (since $N^{**} = N$).

Now return to the proof of (i). Consider the continuous hermitian operator $B = (I + N^*N)^{-1}$ (15.12.6), whose spectrum is contained in $\mathrm{I} = [0, 1]$. We shall first show that

$$f(B)(\mathrm{dom}(N)) \subset \mathrm{dom}(N) \qquad \text{and} \qquad f(B)N \cdot x = Nf(B) \cdot x$$

for all $f \in \mathscr{U}_{\mathbf{C}}(\mathrm{I})$ and all $x \in \mathrm{dom}(N)$. First of all take $f = 1_{\mathbf{C}}$; if $x \in \mathrm{dom}(N)$, we may write

$$BN \cdot x = BN(I + N^*N)B \cdot x = B(N + NN^*N)B \cdot x,$$

recalling that $NB(\mathrm{E}) \subset \mathrm{dom}(N^*)$ and $N^*NB \cdot x = x - B \cdot x \in \mathrm{dom}(N)$, because $B(\mathrm{E}) \subset \mathrm{dom}(N)$ (15.12.6). But $N^*N = NN^*$, hence

$$BN \cdot x = B(N + N^*NN)B \cdot x = B(I + N^*N)NB \cdot x = NB \cdot x;$$

that is, $BN \cdot x = NB \cdot x$ for all $x \in \mathrm{dom}(N)$. By induction on n, we obtain $B^n(\mathrm{dom}(N)) \subset \mathrm{dom}(N)$ and $B^nN \cdot x = NB^n \cdot x$ for all $x \in \mathrm{dom}(N)$ and all integers $n > 0$. This establishes our assertion when f is a *polynomial* with real coefficients. Hence, for such a polynomial f, we have

$$(f(B)N \cdot x \mid y) = (f(B) \cdot x \mid N^* \cdot y) \tag{15.12.8.2}$$

for all $x \in \mathrm{dom}(N)$ and $y \in \mathrm{dom}(N)$.

Having regard to (15.11.2) applied to B, we deduce that

$$\int f(\zeta)\, dm_{N \cdot x, y}(\zeta) = \int f(\zeta)\, dm_{x, N^* \cdot y}(\zeta),$$

and since the restrictions of polynomials with real coefficients to $\mathrm{Sp}(B)$ are dense in $\mathscr{C}_{\mathbf{R}}(\mathrm{Sp}(B))$ by the Weierstrass approximation theorem (7.4.1), the formula above shows that $m_{N \cdot x, y} = m_{x, N^* \cdot y}$ ((3.15.1) and (13.2)). Consequently, by applying (15.11.2) to B, the formula (15.112.8.) is valid for *all* functions $f \in \mathscr{U}_{\mathbf{C}}(\mathrm{I})$. But this formula shows that $f(B) \cdot x \in \mathrm{dom}(N^{**}) = \mathrm{dom}(N)$ and that $(f(B)N \cdot x \mid y) = (Nf(B) \cdot x \mid y)$ for all $y \in \mathrm{dom}(N)$. Since $\mathrm{dom}(N)$ is dense in E, it follows that $Nf(B) \cdot x = f(B)N \cdot x$ for all $x \in \mathrm{dom}(N)$.

Now let f_0 denote the characteristic function of the set $\{0\}$, and for $n \geqq 1$ let f_n denote the characteristic function of the interval $]1/(n+1), 1/n]$ in $\mathbf{R}$. For each $\zeta \in \mathbf{R}$, the series $f_0(\zeta) + f_1(\zeta) + \cdots + f_n(\zeta) + \cdots$ converges to the value at ζ of the characteristic function of the interval $[0, 1]$. For each $n \geqq 0$, let F_n be the image of E under the orthogonal projector $f_n(B) = P_{\mathrm{F}_n}$ (15.10.6). From the preceding discussion and from (15.11.8(iv)) it follows that E is the *Hilbert sum* of the F_n ($n \geqq 0$). Also, in fact, $\mathrm{F}_0 = \{0\}$, because F_0 is the *kernel* of the operator B (15.11.6), and by virtue of (15.12.6) the hermitian form $(x, y) \mapsto (B \cdot x \mid y)$ is nondegenerate, that is to say the relation $(B \cdot x \mid x) = 0$ implies $x = 0$, and so $\mathrm{F}_0 = \{0\}$.

It follows from above that $P_{\mathrm{F}_n}(\mathrm{dom}(N)) \subset \mathrm{dom}(N)$ and that $P_{\mathrm{F}_n} N \cdot x = NP_{\mathrm{F}_n} \cdot x$ for all $x \in \mathrm{dom}(N)$. On the other hand, the definition of the functions f_n for $n \geqq 1$ shows that the function $g_n(\zeta) = \zeta^{-1} f_n(\zeta)$ (with $g_n(0) = 0$) is a bounded regulated function on $\mathbf{R}$ and therefore universally measurable (7.6.1). Consequently we may write $P_{\mathrm{F}_n} = Bg_n(B)$, whence $NP_{\mathrm{F}_n} = NBg_n(B)$. But we have shown (15.12.6) that NB is a *continuous* operator defined on E, and hence the restriction N_n of N to $\mathrm{F}_n \cap \mathrm{dom}(N)$ is a *continuous* mapping of this subspace into F_n. Since $P_{\mathrm{F}_n}(\mathrm{dom}(N)) \subset \mathrm{F}_n \cap \mathrm{dom}(N)$, it follows that $\mathrm{F}_n \cap \mathrm{dom}(N)$ is dense in F_n; also N_n is clearly a closed operator on F_n, hence by (15.12.2) we conclude that $\mathrm{F}_n \subset \mathrm{dom}(N)$. Also, since $N^*N = NN^*$, we may interchange N and N^* in the above discussion, which does not change B; hence the restriction N'_n of N^* to F_n is also a continuous operator on this subspace, and is evidently equal to the adjoint N_n^* of N_n. It is now clear that N_n is normal, and the proof of (15.12.8) is complete.

(15.12.9) Let T be a *closed* (not necessarily bounded) operator on E. Generalizing (11.1), we say that $\zeta \in \mathbf{C}$ is a *regular value* of T if the operator $T - \zeta I$ is a *bijective* linear mapping of $\mathrm{dom}(T)$ onto E, and if the inverse linear mapping $R_T(\zeta) : \mathrm{E} \to \mathrm{dom}(T)$ is *continuous*.

(15.12.10) *For $\zeta \in \mathbf{C}$ to be a regular value of a closed operator T, it is sufficient that $T - \zeta I$ should be an injective linear mapping of* $\mathrm{dom}(T)$ *onto a dense subspace* L *of* E, *and that the inverse mapping* $(T - \zeta I)^{-1} : \mathrm{L} \to \mathrm{E}$ *should be continuous.*

For the graph of $T - \zeta I$ is closed, and so is the graph of $(T - \zeta I)^{-1}$ (15.12.5). Hence it follows from (15.12.2) that L is *closed* in E, and therefore $\mathrm{L} = \mathrm{E}$.

The complement in $\mathbf{C}$ of the set of regular values of T is again called the *spectrum* of T, and is denoted by $\mathrm{Sp}(T)$. If $\zeta \in \mathrm{Sp}(T)$, it follows from (15.12.10) that there are three possibilities:

(1) ζ is an eigenvalue of T (the set of these is called the *spectrum of* T);

(2) ζ is not an eigenvalue of T (which implies that $T - \zeta I$ is a mapping of $\mathrm{dom}(T)$ onto a subspace L of E), the subspace L is *dense* in $(T - \zeta I)^{-1}$ is not continuous on L (the set of these values is called the *tinuous spectrum* of T);

(3) ζ is not an eigenvalue of T and the subspace L is not dense in set of these values is called the *residual spectrum* of T).

For example, for the operator defined in (11.1.1), the point-spec empty, the continuous spectrum is the unit circle $\mathbf{U} : |\zeta| = 1$, and the spectrum is the open disk $|\zeta| < 1$ (Section 11.1, Problem 4).

(15.12.11) *The spectrum of a closed operator T on* E *is closed in* $\mathbf{C}$, *mapping* $\zeta \mapsto R_T(\zeta)$ *of* $\mathbf{C} - \mathrm{Sp}(T)$ *into* $\mathscr{L}(\mathrm{E})$ *is analytic.*

Let $\zeta_0 \in \mathbf{C} - \mathrm{Sp}(T)$, and put $a = \|R_T(\zeta_0)\|$. Then the continuous o $V(\zeta) = (I - (\zeta - \zeta_0)R_T(\zeta_0))^{-1}R_T(\zeta_0)$ is defined for all ζ satisfying $|\zeta - \zeta_0$ and is analytic in this disk (8.3.2.1). For each $x \in \mathrm{dom}(T)$, we have

$$R_T(\zeta_0)(T - \zeta I) \cdot x = x + (\zeta_0 - \zeta)R_T(\zeta_0) \cdot x$$

and therefore $V(\zeta)(T - \zeta I) \cdot x = x$. For each integer $n \geqq 0$, let

$$V_n(\zeta) = R_T(\zeta_0) + (\zeta - \zeta_0)R_T^2(\zeta_0) + \cdots + \zeta - (\zeta_0)^{n-1}R_T^n(\zeta_0),$$

so that $V(\zeta)$ is the limit of the sequence $(V_n(\zeta))$ in $\mathscr{L}(\mathrm{E})$. By definition, $V_n(\zeta) \cdot y \in \mathrm{dom}(T)$ for all $y \in \mathrm{E}$ and all $n > 0$, and

$$\begin{aligned}(T - \zeta I)V_n(\zeta) \cdot y &= (\zeta - \zeta_0)V_n(\zeta) \cdot y + (T - \zeta_0 I)V_n(\zeta) \cdot y \\ &= y - (\zeta - \zeta_0)^n R_T^n(\zeta_0) \cdot y,\end{aligned}$$

hence $(T - \zeta I)V_n(\zeta) \cdot y$ tends to y as $n \to +\infty$. Since by hypothesis T is this implies that $V(\zeta) \cdot y \in \mathrm{dom}(T)$ and $(T - \zeta I)V(\zeta) \cdot y = y$. This sho $T - \zeta I$ is a bijective linear mapping of $\mathrm{dom}(T)$ onto E and that the mapping is $V(\zeta)$. Hence $\zeta \notin \mathrm{Sp}(T)$ and $V(\zeta) = R_T(\zeta)$, which comple proof.

(15.12.12) *Let N be a (not necessarily bounded) normal operator on* H *the notation of* (15.12.8), *we have* $\mathrm{Sp}(N) = \overline{\bigcup_n \mathrm{Sp}(N_n)}$; *the point-spectru is the union of the point-spectra of the N_n, and the residual spectrum empty. For N to be self-adjoint, it is necessary and sufficient that its sp should be real.*

We have $\mathrm{Sp}(N_n) \subset \mathrm{Sp}(N)$ for all n, for if $N - \zeta I$ is a bijective mapping of $\mathrm{dom}(N)$ onto E, then each of the operators $N_n - \zeta I_{\mathrm{E}_n}$ is a bijective (and hence bicontinuous, by the closed graph theorem) mapping of E_n onto E_n. From this point onwards, the proof of the first assertion merely repeats that of (15.11.5), having regard to the facts that $\mathrm{Sp}(N)$ is closed (15.12.11) and that $(N_n - \alpha I_{\mathrm{E}_n})^{-1}$ is a continuous normal operator on E_n for all $\alpha \notin \mathrm{Sp}(N_n)$, and the proposition (15.12.10). The assertion about the point-spectra is immediate, and the eigenspaces of N are determined as in (15.11.6). To show that the residual spectrum of N is empty, it is enough to show that the same is true for each N_n, so that we reduce to the case where N is continuous; then, by using the decomposition of (15.11.3), we may assume that N is multiplication by the class of the function $1_{\mathbf{C}}$ in $\mathrm{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$, where K is a compact subset of **C**, and μ is a positive measure on K. We have then to show that, if the measure μ is not concentrated at the point 0, the set of functions $\zeta \mapsto \zeta g(\zeta)$, where $g \in \mathscr{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$, is *dense* in $\mathscr{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$. For this it is sufficient, by (6.3.1), to show that if a function $h \in \mathscr{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$ is such that $\int \zeta g(\zeta) h(\zeta)\, d\mu(\zeta) = 0$ for all $g \in \mathscr{L}^2_{\mathbf{C}}(\mathrm{K}, \mu)$, then h is μ-negligible. But this is clear, by taking $g(\zeta) = \overline{\zeta h(\zeta)}$, in view of the hypothesis on μ.

Finally, it is clear that if N is self-adjoint, then so is each N_n, hence $\mathrm{Sp}(N_n) \subset \mathbf{R}$ for all n, and therefore $\mathrm{Sp}(N) \subset \mathbf{R}$. Conversely, if $\mathrm{Sp}(N)$ is contained in **R**, then $\mathrm{Sp}(N_n) \subset \mathbf{R}$ for all n, hence N_n is self-adjoint for all n by (15.11.7), and therefore so is N by (15.12.8).

(15.12.13) Propositions (15.12.8) and (15.12.12) essentially serve to reduce problems relating to unbounded normal operators to the same problems for *bounded* (i.e., *continuous*) normal operators. In particular, let $f: \mathbf{C} \to \mathbf{C}$ be *universally measurable*. We shall show that for each unbounded normal operator N we can define an unbounded normal operator $f(N)$, having properties analogous to those of (15.11.1). Suppose first that the function f is *bounded*; then, with the notation of (15.12.8), $f(N_n)$ is a continuous normal operator on E_n (15.11.1) and moreover, $\|f(N_n)\| \leqq \sup_{\zeta \in \mathbf{C}} |f(\zeta)|$ for all n (15.11.8). The normal operator on E whose restriction to each E_n is $f(N_n)$ is therefore continuous (15.10.8.1), and it is this operator which is denoted by $f(N)$. It is immediately seen that if g is another function in $\mathscr{U}_{\mathbf{C}}(\mathbf{C})$, then $(f+g)(N) = f(N) + g(N)$, $(fg)(N) = f(N)g(N)$ and $\bar{f}(N) = (f(N))^*$, by virtue of (15.11.1.1). Now let $f: \mathbf{C} \to \mathbf{C}$ be any universally measurable function. For each $n \geqq 0$, let A_n be the universally measurable subset of **C** consisting of the complex numbers ζ such that $n \leqq |f(\zeta)| < n+1$, so that (A_n) is a partition of **C**. It follows from above that $P_n = \varphi_{\mathrm{A}_n}(N)$ is an orthogonal projection in E (15.5.3.1), and E is the Hilbert sum of the closed subspaces $\mathrm{H}_n = P_n(\mathrm{E})$; also,

putting $f_n = f\varphi_{A_n}$, the function f_n is universally measurable and bounded, and hence $f_n(N)$ is a continuous normal operator on E which leaves H_n stable and is zero on the orthogonal supplement of H_n. It follows now from (15.12.8) that there is a unique unbounded normal operator whose restriction to each H_n is $f_n(N)$, and it is this operator which is denoted by $f(N)$. It is clear that $\bar{f}(N) = (f(N))^*$; furthermore, if $g : \mathbf{C} \to \mathbf{C}$ is another universally measurable function, then again we have $(f+g)(N) = f(N) + g(N)$ and $(fg)(N) = f(N)g(N)$, these equalities being interpreted in the sense explained in (15.12.1) and (15.12.5). For the proof it is enough to consider, for $m \geqq 0$ and $n \geqq 0$, the sets $A_{m,n}$ of complex numbers ζ satisfying $m \leqq |f(\zeta)| < m+1$ and $n \leqq |g(\zeta)| < n+1$; if we put $P_{mn} = \varphi_{A_{m,n}}(N)$, $H_{mn} = P_{mn}(E)$, $f_{mn} = f\varphi_{A_{m,n}}$, $g_{mn} = g\varphi_{A_{m,n}}$, it is clear that E is the Hilbert sum of the H_{mn}, which are stable under $f_{mn}(N)$ and $g_{mn}(N)$, and that $f(N)$ (resp. $g(N)$) is the unique normal operator whose restriction to each H_{mn} is $f_{mn}(N)$ (resp. $g_{mn}(N)$). Our assertions therefore follow from the corresponding assertions for each pair (f_{mn}, g_{mn}).

PROBLEMS

1. Let E, F be two Hilbert spaces. A *not necessarily bounded operator from* E *to* F (or simply an *unbounded operator from* E *to* F) is a linear mapping T of a vector subspace $\mathrm{dom}(T)$ of E into F. We say that T is *closed* if its graph $\Gamma(T)$ is closed in $E \times F$.

(a) If T is closed, show that $\mathrm{Ker}(T)$ is closed in E. Generalize (15.12.2) to the present situation.

(b) If $\mathrm{dom}(T)$ is dense in E, define the *adjoint* T^* of T as in (15.12.3). It is an unbounded operator from F to E. Generalize (15.12.4).

(c) Suppose from now on that T is a *closed* operator from E to F, whose domain $\mathrm{dom}(T)$ is *dense* in E. Show that the closure in E of $\mathrm{Im}(T^*)$ is the orthogonal supplement of $\mathrm{Ker}(T)$.

(d) Suppose that $\mathrm{Ker}(T) = \{0\}$, in other words that T is injective. Then $G = \mathrm{Im}(T)$ is closed in F if and only if the inverse mapping $T^{-1} : G \to E$ is continuous (use (a)).

(e) Without any hypothesis on $\mathrm{Ker}(T)$, show that the following conditions are equivalent: (1) $\mathrm{Im}(T)$ is closed in F; (2) $\mathrm{Im}(T^*)$ is closed in E. (Let G be the closure of $\mathrm{Im}(T^*)$ in E. To show that (1) implies (2), consider the restriction T_1 of T to $G \cap \mathrm{dom}(T)$. Show that $\mathrm{Im}(T_1) = \mathrm{Im}(T)$ and use (d) to deduce that T_1^{-1} is continuous. Hence show that every $y \in G$ is of the form $T^* \cdot z$, by remarking that the mapping $x \mapsto (T_1^{-1} \cdot x \mid y)$ is a continuous linear form on the Hilbert space $\mathrm{Im}(T)$.)

(f) Put

$$\gamma(T) = \inf \frac{\|T \cdot x\|}{d(x, \mathrm{Ker}(T))},$$

the infimum being taken over the complement of $\mathrm{Ker}(T)$ in $\mathrm{dom}(T)$ (d denotes the distance in E). Show that $\mathrm{Im}(T)$ is closed in F if and only if $\gamma(T) > 0$. (Express $\gamma(T)$ in terms of T_1^{-1}, where T_1 is the operator considered in (e).) Show that $\gamma(T^*) \geqq \gamma(T)$.

(g) Suppose that $\mathrm{Im}(T)$ is closed in F. If $M \supset \mathrm{Ker}(T)$ is a closed subspace of E, show that $T(M \cap \mathrm{dom}(T))$ is closed in F. (Consider the restriction of T to $M \cap \mathrm{dom}(T)$ as an unbounded operator from M to F, and use (f).)
(h) Let N be a closed subspace of F such that $N \cap \mathrm{Im}(T) = \{0\}$. If $N + \mathrm{Im}(T)$ is closed in F, then $\mathrm{Im}(T)$ is closed in F. (Consider the operator T_2 from $E \times N$ to F defined on $\mathrm{dom}(T) \times N$ by $T_2 \cdot (x, y) = T \cdot x + y$, and remark that $\gamma(T) \geqq \gamma(T_2)$.) In particular, if $\mathrm{Im}(T)$ has finite codimension in F, then $\mathrm{Im}(T)$ is closed in F.

2. Let E, F, H be three Hilbert spaces, T an unbounded operator from E to F, and U an unbounded operator from H to E. Suppose that T is closed, $\mathrm{dom}(T)$ is dense in E, $\mathrm{Ker}(T)$ is finite-dimensional and $\mathrm{Im}(T)$ is closed in F.
(a) Show that if U is closed then so is TU (defined on $\mathrm{dom}(U) \cap U^{-1}(\mathrm{dom}(T))$.) (Suppose that $z_n \in H$ tends to z and that $TU \cdot z_n$ tends to $y \in F$. Using the relation $\gamma(T) > 0$ (Problem 1), show that there exists a sequence (x_n) in $\mathrm{Ker}(T)$ such that $U \cdot z_n + x_n$ has a limit x in E. Use the fact that $\mathrm{Ker}(T)$ is locally compact to show that the sequence (x_n) must be bounded (argue by contradiction) and then pick a convergent subsequence of (x_n).)

(b) If U is closed and $\mathrm{Im}(U)$ is closed in E, then $\mathrm{Im}(TU)$ is closed in F. (Use Problem 1(g) and (5.9.2).)

(c) Suppose that $\mathrm{Ker}(U)$ is finite-dimensional. Show that $\mathrm{Ker}(TU)$ is finite-dimensional and that

$$\dim(\mathrm{Ker}(TU)) = \dim(\mathrm{Ker}(U)) + \dim(\mathrm{Im}(U) \cap \mathrm{Ker}(T)).$$

(d) Suppose that U is closed, $\mathrm{dom}(U)$ dense in H and $\mathrm{Im}(U)$ of *finite* codimension in E (which implies, by virtue of Problem 1(h), that $\mathrm{Im}(U)$ is closed in E). Show that $\mathrm{dom}(TU)$ is dense in H. (Consider the orthogonal supplement H_1 of $\mathrm{Ker}(U)$ in H, and the restriction U_1 of U to the dense subspace $\mathrm{dom}(U) \cap H_1$ of H_1; observe that U^{-1} is continuous on $\mathrm{Im}(U_1) = \mathrm{Im}(U)$ and that $\mathrm{dom}(T) \cap \mathrm{Im}(U)$ is dense in $\mathrm{Im}(U)$.)

(e) Suppose that $\mathrm{Im}(T)$ and $\mathrm{Im}(U)$ have finite codimension in F and E, respectively, and let ν be the codimension of $\mathrm{Im}(U) \cap \mathrm{Ker}(T)$ in $\mathrm{Ker}(T)$. Show that $\mathrm{Im}(TU)$ has finite codimension in F and that

$$\mathrm{codim}(\mathrm{Im}(TU)) = \mathrm{codim}(\mathrm{Im}(T)) + \mathrm{codim}(\mathrm{Im}(U)) - \nu.$$

(Remark that E is the direct sum of $\mathrm{Im}(U)$, a supplement N_1 of $\mathrm{Im}(U) \cap \mathrm{Ker}(T)$ in $\mathrm{Ker}(T)$, and a subspace N_2 of finite dimension contained in $\mathrm{dom}(T)$, and that the restriction of T to N_2 is injective.)

3. Let E, F be two Hilbert spaces. An unbounded operator T from E to F is said to be an *operator with index* (Section **15.11**, Problem 22) if T is closed, $\mathrm{dom}(T)$ dense in E, $\mathrm{Ker}(T)$ finite-dimensional and $\mathrm{Im}(T)$ of finite codimension in F (in which case $T(E) = \mathrm{Im}(T)$ is *closed* in F by Problem 1(h)). The *index of* T is defined as the number

$$i(T) = \dim(\mathrm{Ker}(T)) - \mathrm{codim}(\mathrm{Im}(T)).$$

(a) If T is an index operator from E to F, then T^* is an index operator from F to E, and $i(T^*) = -i(T)$ (use Problem 1(e)).

(b) Deduce from Problem 2 that if $U : H \to E$ and $T : E \to F$ are index operators, then $TU : H \to F$ is an index operator and $i(TU) = i(T) + i(U)$.

(c) Let T_1 be an unbounded operator from E to F, which extends T and is such that $\mathrm{dom}(T_1) = \mathrm{dom}(T) \oplus M$, where M is a finite-dimensional subspace of E. If T is

closed, show that T_1 is closed; if $\mathrm{Im}(T)$ is closed in F, then so is $\mathrm{Im}(T_1)$; if T is an index operator, then so is T_1, and $i(T_1) = i(T) + \dim(\mathrm{M})$. (By induction on $\dim(\mathrm{M})$.)

4. Let E, F, be two Hilbert spaces and let T be an unbounded operator from E to F. Suppose that T is closed and $\mathrm{dom}(T)$ dense in E. Let B be a continuous operator from E to F.

(a) Show that, for each $x \in \mathrm{Ker}(T+B)$,

$$d(x, \mathrm{Ker}(T)) \leqq \frac{\|B\|}{\gamma(T)} \|x\|.$$

Deduce that, if $\mathrm{Ker}(T)$ has finite dimension and $\gamma(T) > 0$ and $\|B\| < \gamma(T)$, then $\dim(\mathrm{Ker}(T+B)) \leqq \dim(\mathrm{Ker}(T))$ (use Problem 9 of Section 6.3). Furthermore, $\mathrm{Im}(T+B)$ is closed in F (consider the restriction of $T+B$ to the orthogonal supplement of $\mathrm{Ker}(T) + \mathrm{Ker}(T+B)$ in E).

(b) Suppose that T is an index operator and that $\|B\| < \gamma(T)$. Show that $\mathrm{codim}(\mathrm{Im}(T+B)) \leqq \mathrm{codim}(\mathrm{Im}(T))$, and that $i(T+B) = i(T)$. (To prove the first inequality, consider $(T+B)^*$. For the equality of the indices, reduce to the case where T is injective, by considering the orthogonal supplement of $\mathrm{Ker}(T)$ and using Problem 3(c). Then observe that $\gamma(T+\lambda B)$ is a continuous function of λ for $0 \leqq \lambda \leqq 1$.)

5. With the same general hypotheses as in Problem 4, suppose that B is a *compact* operator from E to F.

(a) Show that, if $\mathrm{Ker}(T)$ is finite-dimensional and $\mathrm{Im}(T)$ is closed, then $\mathrm{Ker}(T+B)$ is finite-dimensional and $\mathrm{Im}(T+B)$ is closed. (Consider the orthogonal supplement M of $\mathrm{Ker}(T)$, and show first that $\mathrm{M} \cap \mathrm{Ker}(T+B) \subset \mathrm{M} \cap \mathrm{dom}(T)$ is finite-dimensional, by noting that there exists $m > 0$ such that $\|T \cdot x\| \geqq m\|x\|$ for all $x \in \mathrm{M} \cap \mathrm{dom}(T)$, and using (5.9.4). Then take the orthogonal supplement N of $\mathrm{M} \cap \mathrm{Ker}(T+B)$ in M, and prove that there exists no sequence (x_n) in $\mathrm{N} \cap \mathrm{dom}(T)$ such that $\|x_n\| = 1$ for all n and $T(x_n) + B(x_n)$ tends to 0.)

(b) Deduce from (a) that if T is an index operator then so is $T+B$, and that $i(T+B) = i(T)$. (To prove that $\mathrm{Im}(T+B)$ has finite codimension, consider $T^* + B^*$; then observe that $\lambda \mapsto i(T+\lambda B)$ is a finite continuous function of λ on $[0, 1]$ by virtue of Problem 4.)

(c) Suppose that $\mathrm{E} = \mathrm{F}$ and that there exists a regular value ζ_0 for T such that $(T - \zeta_0 I)^{-1}$ is a *compact* operator. Show that, for all $\zeta \in \mathbf{C}$, the operator $T - \zeta I$ has index 0; the spectrum of T is a denumerable discrete subset of $\mathbf{C}$, all of whose points are eigenvalues of T, and $(T-\zeta I)^{-1}$ is a compact operator for all regular values ζ of T. (Note that

$$\zeta I - T = (I + (\zeta - \zeta_0)(\zeta_0 I - T)^{-1})(\zeta_0 I - T)$$

and use the theory of Riesz (11.4.1).)

Hence give an example of an operator of index 0 whose spectrum is *empty* (use Problem 9 of Section 11.6).

6. With the notation of the proof of (15.12.8), prove that the spectrum of the operator N_n is contained in the annulus $\mathrm{S}_n : (n-1)^{1/2} \leqq |\zeta| \leqq n^{1/2}$. For each pair of points (x_n, y_n) in F_n, there corresponds to N_n a measure $m^{(n)}_{x_n, y_n}$ with support contained in S_n. If $x = \sum_n x_n$ and $y = \sum_n y_n$ are two points of E, with $x_n \in \mathrm{F}_n$ and $y_n \in \mathrm{F}_n$, then the

sum $m_{x,y} = \sum_n m^{(n)}_{x_n, y_n}$ is a bounded complex measure on $\mathbf{C}$, with norm $\leqq \|x\| \cdot \|y\|$, and $m_{x,x}$ is a bounded measure on $\mathbf{C}$ with mass $(x|x)$. The subspace $\mathrm{dom}(N)$ is the set of all $x \in \mathrm{E}$ such that the function $\zeta \mapsto |\zeta|^2$ is $m_{x,x}$-integrable. For each pair of points $x, y \in \mathrm{dom}(N)$, we have $m_{y,x} = \bar{m}_{x,y}$, $(x|y) = \int dm_{x,y}$ and $(N \cdot x|y) = \int f(\zeta)\, dm_{x,y}(\zeta)$.

7. With the notation of (15.12.13) and Problem 6, show that $\mathrm{dom}(f(N))$ is the set of all $x \in \mathrm{E}$ such that the function $\zeta \mapsto |f(\zeta)|^2$ is $m_{x,x}$-integrable, and that $(f(N) \cdot x|y) = \int f(\zeta)\, dm_{x,y}(\zeta)$ for all $x, y \in \mathrm{dom}(f(N))$. Show that $\mathrm{Sp}(f(N))$ is contained in the closure of $f(\mathrm{Sp}(N))$. If $g : \mathbf{C} \to \mathbf{C}$ is another universally measurable function, show that $(f \circ g)(N) = f(g(N))$ (cf. Section 13.9, Problem 21). When $N = M_\mu(1_\mathbf{C})$ is a simple continuous normal operator (15.11.3), $\mathrm{dom}(f(N))$ is the set of classes of functions $u \in \mathscr{L}^2_\mathbf{C}(\mu)$ such that $fu \in \mathscr{L}^2_\mathbf{C}(\mu)$, and $f(N)$ is multiplication by the class of f.

8. Let N be a normal operator and x a point of $\mathrm{dom}(N)$ such that $\|x\| = 1$. Put $\alpha_x = (N \cdot x|x)$ and $\beta_x = \|N \cdot x\|$. Show that, for each $\varepsilon > 0$, the open disk Δ_x with center α_x and radius $(\beta_x^2 - |\alpha_x|^2)^{1/2} + \varepsilon$ meets $\mathrm{Sp}(N)$ (Krylov–Weinstein theorem). (With the notation of Problem 6, show that $\beta_x^2 - |\alpha_x|^2 = \int |\zeta - \alpha_x|^2\, dm_{x,x}(\zeta)$, and hence obtain a contradiction of the hypothesis that Δ_x has measure zero with respect to $m_{x,x}$.)

9. Let H be a self-adjoint operator on E (bounded or not). For each (bounded or unbounded) open interval $\mathrm{J} \subset \mathbf{R}$, let P be the orthogonal projector $\varphi_\mathrm{J}(H)$. Show that

$$2\pi i P_\mathrm{J} \cdot x = \lim_{v \to 0,\ v > 0} \int_\mathrm{J} (((u - iv)I - H)^{-1} \cdot x - ((u + iv)I - H)^{-1} \cdot x)\, du$$

for all $x \in \mathrm{E}$ ("Titchmarsh–Kodaira formula"). (For each $y \in \mathrm{E}$ such that $\|y\| \leqq 1$, show that the scalar product of y and the difference between $2\pi i P_\mathrm{J} \cdot x$ and the integral on the right-hand side tends to 0 uniformly for $\|y\| \leqq 1$: express this difference in terms of the measure $m_{x,y}$ (Problem 6), and use the Lebesgue–Fubini theorem and the dominated convergence theorem.)

10. Show that, for each closed operator T on E such that $\mathrm{dom}(T)$ is dense in E, there exists a positive self-adjoint operator R such that $\mathrm{dom}(R) = \mathrm{dom}(T)$ and an isometry V of $\overline{R(\mathrm{dom}(R))}$ onto $\overline{T(\mathrm{dom}(T))}$, such that $T = VR$. (As in the proof of (15.12.8), define the spaces $\mathrm{F}_n \subset \mathrm{dom}(T^*T) \subset \mathrm{dom}(T)$ so that the restriction of T^*T to F_n is a continuous positive self-adjoint operator H_n, for which we may take the square root R_n. We have $\|T \cdot x_n\| = \|R_n \cdot x_n\|$ for all $x_n \in \mathrm{F}_n$, hence there exists an isometry V_n of $R_n(\mathrm{F}_n)$ onto $T(\mathrm{F}_n)$ such that $T|\mathrm{F}_n = V_n R_n$. Let R be the positive self-adjoint operator whose restriction to each F_n is R_n. Show that the V_n are the restrictions of an isometry V of $\overline{R(\mathrm{dom}(R))}$ onto $\overline{T(\mathrm{dom}(T))}$, by considering the finite sums of the F_n and using the facts that T is closed and $\mathrm{dom}(R)$ is dense in $\mathrm{dom}(T)$. Finally, show that $V^{-1}T$ is a closed operator which coincides with R_n on F_n for each n, and hence deduce that $\mathrm{dom}(R) = \mathrm{dom}(T)$ and $V^{-1}T = R$.)

11. Let T be a closed operator on E such that $\mathrm{dom}(T)$ is dense in E and such that $T(\mathrm{dom}(T)) \subset \mathrm{dom}(T^*)$. Show that $\mathrm{dom}(T) = \mathrm{E}$ and that E is continuous. (If

$B = (I + TT^*)^{-1}$ and if for each $y \in E$ we put $x = B \cdot y$, show that the hypotheses imply that $y \in \mathrm{dom}(T^*)$, by using the fact that $x \in \mathrm{dom}(T^*)$.)

12. Let N be a normal operator on E. If $\mathrm{dom}(N^2) = \mathrm{dom}(N)$, show that N is continuous. (If not, there would exist a closed subspace F of E, such that F is the Hilbert sum of an infinite sequence (F_n) of nonzero subspaces stable under N and N^*, such that the restriction N_n of N to F_n is a continuous operator and such that

$$a_n\|x_n\| \leqq \|N_n \cdot x_n\| \leqq a_{n+1}\|x_n\|$$

for $x_n \in F_n$, where the sequence (a_n) is increasing and tends to $+\infty$.)

13. EXTENSIONS OF HERMITIAN OPERATORS

(15.13.1) Let E be a separable Hilbert space. A not necessarily bounded operator H on E is said to be *hermitian* if: (1) $\mathrm{dom}(H)$ is *dense* in E; (2) $\mathrm{dom}(H) \subset \mathrm{dom}(H^*)$, and the restriction of H^* to $\mathrm{dom}(H)$ is equal to H; in other words, for all x, y in $\mathrm{dom}(H)$,

$$(H \cdot x \mid y) = (x \mid H \cdot y). \tag{15.13.1.1}$$

In particular, $(H \cdot x \mid x)$ is *real* for all $x \in \mathrm{dom}(H)$.

It should be noted that in general H^* *is not hermitian.* When H is *continuous*, this definition agrees with that of (11.5), and a continuous hermitian operator is the same thing as a continuous *self-adjoint* operator. An unbounded self-adjoint operator is always hermitian, but as we shall see below there exist *closed* hermitian operators *which are not self-adjoint.*

Example

(15.13.2) Let $\mathscr{D}$ be the vector space of indefinitely differentiable complex-valued functions on $\mathbf{R}$ with compact support. If β denotes Lebesgue measure on $\mathbf{R}$, then $\mathscr{D}$ may be identified with a subspace of $L^2_{\mathbf{C}}(\mathbf{R}, \beta)$ (13.19), which is *dense* in $L^2_{\mathbf{C}}(\mathbf{R}, \beta)$ (17.1.2). It follows immediately from the formula for integration by parts (8.7.5) and the fact that the functions belonging to $\mathscr{D}$ vanish outside a compact set, that for all x, y in $\mathscr{D}$

$$(Dx \mid y) = \int Dx(t)\overline{y(t)}\,dt = -\int x(t)\overline{Dy(t)}\,dt = -(x \mid Dy).$$

This may be expressed by saying that $iD = H$ is a *hermitian* operator. The conclusion is the same if we replace $L^2_{\mathbf{C}}(\mathbf{R}, \beta)$ by $L^2_{\mathbf{C}}(\mathrm{I}, \beta)$, where I is a (bounded

or unbounded) *open* interval in $\mathbf{R}$, and $\mathscr{D}$ by the subspace $\mathscr{D}(\mathrm{I})$ of indefinitely differentiable complex-valued functions on I with compact support. It is clear that the graph of D is not closed: for example, it contains the pairs $(x, \mathrm{D}x)$ where x is of class C^1 but not of class C^2, and $\mathrm{Supp}(x)$ is compact (17.1.2).

(15.13.3) Let H be a hermitian operator. Since $\mathrm{dom}(H^*)$ is dense in E, the closure H^{**} of H exists, and since its graph is the closure of the graph of H (15.12.4), it follows from (15.13.1.1) by continuity that H^{**} is *hermitian*. We may therefore restrict our considerations to *closed* hermitian operators.

(15.13.4) (i) *Let H be a closed hermitian operator on* E. *For each real number $\alpha \neq 0$, the closed operator $H + \alpha i I : x \mapsto H \cdot x + \alpha i x$ is an injective mapping of* $\mathrm{dom}(H)$ *into* E, *whose image F_α is a closed subspace of* E, *and the operator $(H + \alpha i I)^{-1}$* (15.12.5) *is continuous on F_α.*

(ii) *The linear mapping*

$$V : x \mapsto (H - iI)(H + iI)^{-1} \cdot x \tag{15.13.4.1}$$

of F_1 into E *is an isometry of F_1 onto the closed subspace $V(\mathrm{F}_1) = \mathrm{F}_{-1}$. The mapping $I - V : \mathrm{F}_1 = \mathrm{dom}(V) \to \mathrm{E}$ is a bijection of* $\mathrm{dom}(V)$ *onto* $\mathrm{dom}(H)$, *and we have*

$$H \cdot y = i(I + V)(I - V)^{-1} \cdot y \tag{15.13.4.2}$$

for all $y \in \mathrm{dom}(H)$.

(iii) *Conversely, let* F *be a closed subspace of* E, *and U an isometry of* F *onto a subspace of* E, *such that the image* G *of* $\mathrm{F} = \mathrm{dom}(U)$ *under $I - U$ is dense in* E. *Then the operator $I - U$ is a bijection of* F *onto* G; *if, for each $y \in \mathrm{G}$, we put*

$$H \cdot y = i(I + U)(I - U)^{-1} \cdot y, \tag{15.13.4.3}$$

then H is a closed hermitian operator, with $dom(H) = \mathrm{G}$; moreover, the operator V defined in (15.13.4.1) *is equal to U.*

By virtue of (15.13.1.1) we have

$$\begin{aligned} \|H \cdot y + \alpha i y\|^2 &= (H \cdot y \mid H \cdot y) + \alpha i (y \mid H \cdot y) \\ &\quad - \alpha i (H \cdot y \mid y) + \alpha^2 (y \mid y) \\ &= \|H \cdot y\|^2 + \alpha^2 \|y\|^2 \geqq \alpha^2 \|y\|^2 \end{aligned} \tag{15.13.4.4}$$

for all $y \in \mathrm{dom}(H)$, which shows that $H + \alpha iI$ is injective and that the inverse mapping of F_α onto $\mathrm{dom}(H)$ is continuous (5.5.1). Also, since the operator $(H + \alpha iI)^{-1}$ is continuous and its graph is closed (15.12.5), F_α is a *closed* subspace of E (15.12.2). If in (15.13.4.4) we put $\alpha = -1$ and $y = (H + iI)^{-1} \cdot x$, where $x \in \mathrm{F}_1$, then we get $\|(H - iI) \cdot y\|^2 = \|(H + iI) \cdot y\|^2$, hence $\|V \cdot x\| = \|x\|$ for all $x \in \mathrm{F}_1$. Since the subspace $V(\mathrm{F}_1)$ is therefore isometric to F_1, it is complete and therefore *closed* (3.14.4). On the other hand, the relations $H \cdot y + iy = x$ and $H \cdot y - iy = V \cdot x$ give

$$iy = \tfrac{1}{2}(x - V \cdot x), \qquad H \cdot y = \tfrac{1}{2}(x + V \cdot x)$$

for all $x \in \mathrm{F}_1$. This shows that, if $x - V \cdot x = 0$ for $x \in \mathrm{F}_1$, then we must have $x + V \cdot x = 0$, hence $x = 0$. Hence (i) and (ii) are proved.

We shall now prove (iii). The hypothesis that U is isometric implies that

$$(x \mid y - U \cdot y) + (x - U \cdot x \mid U \cdot y) = 0 \tag{15.13.4.5}$$

for all x, y in F; hence, if $x - U \cdot x = 0$, x is orthogonal to G, and since G is dense in E we have $x = 0$. Next, let us show that H defined by (15.13.4.3) is *hermitian*. If x, $y \in \mathrm{G}$, then we may write $x = u - U \cdot u$ and $y = v - U \cdot v$, with u, v in F; hence, as $(U \cdot u \mid U \cdot v) = (u \mid v)$, we obtain

$$\begin{aligned}(H \cdot x \mid y) = (i(u + U \cdot u) \mid v - U \cdot v) &= i((U \cdot u \mid v) - (u \mid U \cdot v)) \\ &= (u - U \cdot u \mid i(v + U \cdot v)) = (x \mid H \cdot y)\end{aligned}$$

which proves our assertion. Next we show that H is *closed*. If (y_n) is a sequence of points in G which tends to $y \in \mathrm{E}$ and is such that the sequence $(H \cdot y_n)$ converges to $z \in \mathrm{E}$, then the sequence of points

$$x_n = H \cdot y_n + iy_n = 2i(I - U)^{-1} \cdot y_n,$$

which belong to F, converges to $x = z + iy \in \mathrm{F}$; hence, as U is continuous on F, the sequence of points $(I - U) \cdot x_n = 2iy_n$ converges to a point in G. This shows that $y \in \mathrm{G}$ and that $(I - U) \cdot x = 2iy$. On the other hand, the sequence of points $(I + U) \cdot x_n = 2H \cdot y_n$ converges to $(I + U) \cdot x = 2z$ for the same reason, and therefore $H \cdot y = z$. Hence H is closed. Finally, if $x = u - U \cdot u$ with $u \in \mathrm{F}$, then we have $H \cdot x = i(u + U \cdot u)$, and it is immediately verified that $V \cdot u = U \cdot u$.

The isometric linear mapping V defined in (15.13.4.1) is called the *Cayley transform* of H. Proposition (15.13.4) reduces the study of hermitian operators to that of their Cayley transforms.

(15.13.5) With the notation of (15.13.4), let E_H^+ be the orthogonal supplement $(\mathrm{dom}(V))^\perp$ of the closed subspace $F_1 = \mathrm{dom}(V)$, and E_H^- the orthogonal supplement $(V(\mathrm{dom}(V)))^\perp$ of the closed subspace $F_{-1} = V(F_1)$.

(15.13.6) (*i*) *The subspace* E_H^+ (resp. E_H^-) *is the set of all* $x \in \mathrm{dom}(H^*)$ *such that* $H^* \cdot x = ix$ (resp. $H^* \cdot x = -ix$), *and* $\mathrm{dom}(H^*)$ *is the direct sum of* $\mathrm{dom}(H)$, E_H^+, *and* E_H^-.

(ii) *Let* G^+ (resp. G^-) *be the subspace of* $\Gamma(H^*)$ *whose first projection in* E *is* E_H^+ (resp. E_H^-). *Then* G^+ *and* G^- *are closed in* $E \times E$ *and* $\Gamma(H^*)$ *is the Hilbert sum of* $\Gamma(H)$, G^+, *and* G^-.

We have $x \in E_H^+$ if and only if

$$(H \cdot y + iy \mid x) = 2i((I - V)^{-1} \cdot y \mid x) = 0$$

for all $y \in \mathrm{dom}(H)$, or equivalently if and only if $(H \cdot y \mid x) = (y \mid ix)$; by definition (15.12.3), this signifies that $x \in \mathrm{dom}(H^*)$ and $H^* \cdot x = ix$. Similarly for $x \in E_H^-$, if we observe that $V(F_1)$ is the set of all $H \cdot y - iy$ with $y \in \mathrm{dom}(H)$.

Let x be an arbitrary point in $\mathrm{dom}(H^*)$; as E is the Hilbert sum of $\mathrm{dom}(V)$ and E_H^+, we may write $H^* \cdot x + ix = H \cdot y + iy + z$ with $y \in \mathrm{dom}(H)$ and $z \in E_H^+$. As we have $H \cdot y = H^* \cdot y$ and $H^* \cdot z = iz$, we may write $z = H^* \cdot z_1 + iz_1$, with $z_1 = z/2i$, hence $H^* \cdot x + ix = H^* \cdot y + iy + H^* \cdot z_1 + iz_1$; this shows that $z_2 = x - y - z_1$ satisfies $H^* \cdot z_2 = -iz_2$, in other words $z_2 \in E_H^-$, and therefore $\mathrm{dom}(H^*)$ is the *sum* of $\mathrm{dom}(H)$, E_H^+, and E_H^-. This implies that $\Gamma(H^*)$ is the *sum* of $\Gamma(H)$, G^+ and G^-. That G^+ (resp. G^-) is closed follows from the fact that this subspace is the intersection of the closed subspace $\Gamma(H^*)$ of $E \times E$ and of the subspace consisting of the points (x, y) such that $y = ix$ (resp. $y = -ix$), which is obviously closed. It remains to be shown that G^+, G^-, and $\Gamma(H)$ are *mutually orthogonal*. The fact that G^+ and G^- are orthogonal follows from the relation

$$((x_1, ix_1) \mid (x_2, -ix_2)) = (x_1 \mid x_2) + (ix_1 \mid -ix_2) = 0.$$

Furthermore, if $x \in \mathrm{dom}(H)$ and $y \in E_H^-$, we have

$$((x, H \cdot x) \mid (y, -iy)) = (x \mid y) + i(H \cdot x \mid y) = (x \mid y) + i(x \mid H^* \cdot y)$$

and as $H^* \cdot y = -iy$, we see that G^- is orthogonal to $\Gamma(H)$; the fact that G^+ is orthogonal to $\Gamma(H)$ is proved in the same way, and this ends the proof of (i) and (ii).

(15.13.7) The *defect* of a *closed* Hermitian operator H is defined to be the pair (m, n) where m (resp. n) is equal to the dimension of E_H^+ (resp. E_H^-) if this dimension is finite, and is $+\infty$ otherwise; m is called the *positive defect* and n the *negative defect* of H.

(15.13.8) *For a closed hermitian operator H to be self-adjoint, it is necessary and sufficient that its defect should be* $(0, 0)$, *or in other words that its Cayley transform V should be unitary. For H to be capable of extension to a self-adjoint operator, it is necessary and sufficient that its defect should be of the form* (m, m).

The first assertion follows trivially from (15.13.6). Next, if H is the restriction of a self-adjoint operator A, then the Cayley transform V of H is the restriction of the Cayley transform U of A, which is unitary. Hence by definition we have $E_H^- = U(E_H^+)$ because U is an automorphism of the Hilbert space E, and the defect of H is therefore of the form (m, m). Conversely, if this is so, then there exists an automorphism U of the Hilbert space E which coincides with V on $\mathrm{dom}(V)$ and is such that $U(E_H^+) = E_H^-$ (because two Hilbert spaces of the same finite dimension, or two separable infinite-dimensional Hilbert spaces, are isomorphic (6.6.2)). Since U is a unitary operator on E, and the image of E under $I - U$ (which contains $\mathrm{dom}(H)$) is dense in E, it follows that U is the Cayley transform of a self-adjoint operator which extends H.

A hermitian operator H whose closure H^{**} is self-adjoint is said to be *essentially self-adjoint*.

Examples

(15.13.9) Let $(e_n)_{n\geqq 0}$ be a Hilbert basis of an infinite-dimensional separable Hilbert space E. Let F be the closed hyperplane in E generated by the e_n with $n \geqq 1$ (the orthogonal supplement of the line $\mathbf{C}e_0$) and let U' be the isometry of F onto E such that $U' \cdot e_n = e_{n-1}$ for all $n \geqq 1$. Then the image G' of $I - U'$ is *dense* in E. For G' contains the vectors $e_n - e_{n+1}$ for all $n \geqq 0$, and if a vector $x = \sum_{n=0}^{\infty} \xi_n e_n$ in E is orthogonal to all the vectors $e_m - e_{m+1}$ $(m \geqq 0)$, then we have $\xi_m = \xi_{m+1}$ for all $m \geqq 0$, so that all the ξ_n $(n \geqq 0)$ are equal; and this is not compatible with the convergence of the series $\sum_{n=0}^{\infty} |\xi_n|^2$ unless

$\xi_n = 0$ for all n. It follows that U' is the Cayley transform of a closed Hermitian operator H' of defect (1, 0), and consequently $H'' = -H'$ is a closed Hermitian operator of defect (0, 1) (15.13.6). It can be shown that there exist closed Hermitian operators of defect (m, n) for all choices of m and n in the set $\mathbf{N} \cup \{+\infty\}$ (Problem 7).

In Chapter XXIII we shall see, by means of the theory of distributions, how to describe the closure of the Hermitian operator $H = i\mathrm{D}$ defined in (15.13.2). We shall show that the defect of H^{**} is (1, 1) if the interval I is bounded, (1, 0) if I is bounded above but not below, (0, 1) if I is bounded below but not above, and finally (0, 0) if $I = \mathbf{R}$.

Theorem (15.13.6) generalizes as follows.

(15.13.10) *For every complex number λ such that $\mathscr{I}(\lambda) > 0$, the subspace* $\mathrm{E}^+(\lambda)$ (resp. $\mathrm{E}^-(\lambda)$) *of* $\mathrm{dom}(H^*)$ *consisting of the solutions of* $H^* \cdot x = \lambda x$ (resp. *of* $H^* \cdot x = -\lambda x$) *is isomorphic to* E_H^+ (resp. E_H^-).

Since the subspace $\mathrm{G}^- + \Gamma(H)$ of $\Gamma(H^*)$ is closed in $\mathrm{E} \times \mathrm{E}$ (15.13.6), the restriction H_1 of H^* to $\mathrm{dom}(H_1) = \mathrm{E}_H^- + \mathrm{dom}(H)$ is a *closed* operator. Put $\lambda = \mu + i\nu$, where μ and ν are real and $\nu > 0$, and calculate $\|(H_1 - \lambda I) \cdot x\|^2$ for some $x = y + z \in \mathrm{dom}(H_1)$, where $y \in \mathrm{dom}(H)$ and $z \in \mathrm{E}_H^-$. We have

$$(H_1 \cdot x \,|\, x) = (H \cdot y - iz \,|\, y + z) = (H \cdot y \,|\, y) + (H \cdot y \,|\, z) - i(z \,|\, y) - i\|z\|^2,$$

and since $H^* \cdot z = -iz$, we have $(H \cdot y \,|\, z) = (y \,|\, H^* \cdot z) = i(y \,|\, z)$. Now,

$$\begin{aligned} \|(H_1 - \lambda I) \cdot x\|^2 &= \|(H_1 - \mu I) \cdot x\|^2 + \nu^2 \|x\|^2 \\ &\quad + i\nu((H_1 \cdot x \,|\, x) - (x \,|\, H_1 \cdot x)) \\ &= \|(H_1 - \mu I) \cdot x\|^2 + \nu^2 \|x\|^2 + 2\nu \|z\|^2 \\ &\geqq \nu^2 \|x\|^2 \end{aligned} \tag{15.13.10.1}$$

by the preceding calculation and the hypothesis $\nu > 0$. If we put $\mathrm{F} = \mathrm{Im}(H_1 - \lambda I)$, it follows that $H_1 - \lambda I$ is a bijection of $\mathrm{dom}(H_1)$ onto F, and the inverse mapping $(H_1 - \lambda I)^{-1}$ is a *continuous* linear mapping of the normed space F into E; since it is also a *closed* operator (because H_1 is closed), we conclude (15.12.2) that the subspace F of E is *closed* in E. We shall show that in fact $\mathrm{F} = \mathrm{E}$. We assert first that this is the case when $\lambda = i$: for this we have to prove that if $((H_1 - iI) \cdot x \,|\, z) = 0$ for all $x \in \mathrm{dom}(H_1)$, then z must be

0. Now, by the definition of the adjoint (15.12.3), we see first that $z \in \mathrm{dom}(H_1^*)$ and next that $(x \mid H_1^* \cdot z + iz) = 0$ for all $x \in \mathrm{dom}(H_1)$, and since $\mathrm{dom}(H_1)$ is dense in E, we have $H_1^* \cdot z + iz = 0$. Since $\mathrm{dom}(H) \subset \mathrm{dom}(H_1)$, we have $\mathrm{dom}(H_1^*) \subset \mathrm{dom}(H^*)$, and H_1^* is the restriction of H^*. Hence $H^* \cdot z + iz = 0$, in other words $z \in \mathrm{E}_H^-$. But $\mathrm{E}_H^- \subset \mathrm{dom}(H_1)$ by definition, so that

$$0 = (z \mid (H_1^* + iI) \cdot z) = ((H_1 - iI) \cdot z \mid z) = ((H^* - iI) \cdot z \mid z) = -2i\|z\|^2,$$

and therefore finally $z = 0$.

To prove that $\mathrm{F} = \mathrm{E}$ for all λ such that $\mathscr{I}\lambda > 0$, we observe that if this is true for $\lambda = \lambda_0$, then the operator $(H_1 - \lambda_0 I)^{-1}$ is everywhere defined and continuous, and satisfies

$$\|(H_1 - \lambda_0 I)^{-1}\| \leqq |\mathscr{I}\lambda_0|^{-1}$$

by virtue of (15.13.10.1). It follows (15.12.11) that $(H_1 - \lambda I)^{-1}$ is everywhere defined and continuous for $|\lambda - \lambda_0| < |\mathscr{I}\lambda_0|$. By induction, $(H_1 - \lambda I)^{-1}$ is everywhere defined and continuous for $|\lambda - (\frac{3}{2})^n i| < (\frac{3}{2})^n$, and since the integer n is arbitrary, we have shown that $(H_1 - \lambda I)^{-1}$ is everywhere defined for all λ in the half-plane $\mathscr{I}\lambda > 0$.

We have thus defined, for all $\lambda \in \mathbf{C}$ such that $\mathscr{I}\lambda > 0$, a *continuous operator* $A(\lambda) = (H_1 - \lambda I)^{-1}$ on E, with image equal to $\mathrm{dom}(H_1) \subset \mathrm{dom}(H^*)$. Since $(H_1 - \lambda I)A(\lambda) = I$, it follows that, for any two complex numbers α, β such that $\mathscr{I}\alpha > 0$ and $\mathscr{I}\beta > 0$, the operator

$$K(\alpha, \beta) = (H_1 - \alpha I)A(\beta) = I + (\beta - \alpha)A(\beta)$$

is everywhere defined and *continuous*; also the operator $A(\beta)(H_1 - \beta I)$ is the identity on $\mathrm{dom}(H_1)$, and therefore

(15.13.10.2) $$K(\alpha, \beta)K(\beta, \gamma) = K(\alpha, \gamma), \qquad K(\alpha, \alpha) = I.$$

Hence the operators $K(\alpha, \beta)$ are invertible and bicontinuous.

This being so, for every $x \in \mathrm{dom}(H^*)$ we have $K(i, \lambda) \cdot x \in \mathrm{dom}(H^*)$ and

$$(H^* - \lambda I) \cdot (K(i, \lambda) \cdot x) = (H^* - \lambda I) \cdot x + (\lambda - i)(H^* - \lambda I)A(\lambda) \cdot x$$

$$= (H^* - iI) \cdot x$$

and likewise $(H^* - iI) \cdot (K(\lambda, i) \cdot x) = (H^* - \lambda I) \cdot x$. This shows that $K(i, \lambda)$ is a *bicontinuous bijection* of E_H^+ onto $\mathrm{E}^+(\lambda)$. Similarly for $\mathrm{E}^-(\lambda)$.

PROBLEMS

1. An unbounded self-adjoint operator H on a separable Hilbert space E is said to be *simple* if there exists $x \in \mathrm{dom}(H)$ such that the vectors $H^n \cdot x$ for $n \geqq 1$ belong to $\mathrm{dom}(H)$ and form a total set in $\mathrm{dom}(H)$.

(a) If H is simple, show that there exists a bounded positive measure μ on $\mathbf{R}$ and an isomorphism T of $L^2_{\mathbf{C}}(\mu)$ onto E such that $H = TM_\mu T^{-1}$, where M_μ is multiplication by the class of the function $1_{\mathbf{R}}$ in $L^2_{\mathbf{C}}(\mu)$: thus the subspace $\mathrm{dom}(M_\mu)$ consists of the classes of functions $f \in \mathscr{L}^2_{\mathbf{C}}(\mu)$ such that $\xi \mapsto \xi f(\xi)$ is square μ-integrable. (Use Problem 7 of Section 15.12, by considering the bounded operators $\varphi_{\mathrm{I}_n}(H)$, where $\mathrm{I}_n = [n, n+1[$ and $n \in \mathbf{Z}$.)

(b) Let μ be a bounded positive measure on $\mathbf{R}$. Show that the class $\tilde{g}$ of the function $\xi \mapsto \exp(-\xi^2)$ is such that the classes $\tilde{g}_n = M_\mu^n \cdot \tilde{g}$ $(n \geqq 0)$ belong to $\mathrm{dom}(M_\mu)$ and form a total set in $L^2_{\mathbf{C}}(\mu)$. (Let f be a function belonging to $\mathscr{L}^2_{\mathbf{C}}(\mu)$ such that

$$\int t^n \exp(-t^2) f(t)\, d\mu(t) = 0$$

for all $n \geqq 0$. Every continuous regularization $\mathrm{F} = \rho * (f \cdot \mu)$ of the measure $f \cdot \mu$ (14.11) is then orthogonal to the functions $\xi \mapsto \xi^n \exp(-\xi^2)$ for all $n \geqq 0$, with respect to Lebesgue measure on $\mathbf{R}$, and therefore F is identically zero.* Deduce that $f * \mu = 0$ and hence that f is μ-negligible.)

(c) If we apply the orthonormalization process (6.6) to the vectors $\tilde{g}_n$, we obtain a Hilbert basis $(e_n)_{n \geqq 0}$ of $L^2_{\mathbf{C}}(\mu)$, where e_n is the class of a function of the form $\mathrm{P}_n(\xi)\exp(-\xi^2)$, with P_n a uniquely determined *polynomial of degree n*, with real coefficients and strictly positive leading coefficient. Deduce that, if we put $a_{mn} = (M_\mu \cdot e_n \mid e_m)$, we have $a_{mn} = 0$ if $|m-n| > 1$, and $a_{n,n+1} > 0$. To simplify the notation, put $a_n = a_{nn}$, $b_n = a_{n,n+1} = a_{n+1,n}$; then the operator M_μ has as its infinite matrix with respect to the orthonormal basis (e_n) the *Jacobi matrix*

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & \cdots \\ 0 & 0 & b_2 & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{1}$$

(d) Let ν be the measure with density $\exp(-t^2)$ with respect to μ, and put

$$c_n = \int t^n\, d\nu(t)$$

for $n \geqq 0$ (the "moments" of ν, cf. Section 13.20, Problem 5). By multiplying ν by a suitable constant, we may assume that $c_0 = 1$. If we form the Gram determinants (Section 6.6, Problem 3)

$$\mathrm{D}_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} \tag{2}$$

* See for example my "Calcul infinitésimal," Chapter V, Exercise 12. Hermann, Paris, 1968.

for $n \geqq 0$, which are all strictly positive, show that $P_0(\xi) = 1$ and that

$$(3) \qquad P_n(\xi) = \frac{1}{(D_{n-1}D_n)^{1/2}} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & \xi & \cdots & \xi^n \end{vmatrix}$$

for all $n \geqq 1$. Also we have

$$(4) \qquad b_n = \frac{(D_{n-1}D_{n+1})^{1/2}}{D_n}.$$

(e) Conversely, for every (bounded or unbounded) positive measure μ on $\mathbf{R}$, show that the operator M_μ (which is in general unbounded) is self-adjoint and simple.

2. Conversely, let J be a Jacobi matrix (formula (1) of Problem 1) in which the a_n are real and the b_n are real and >0, but are not restricted in any other way. Let $(e_n)_{n\geqq 0}$ be the canonical Hilbert basis of the Hilbert space $l^2_{\mathbf{C}}$ (6.5). Define a hermitian operator H, with domain the subspace G of $l^2_{\mathbf{C}}$ generated by the (finite) linear combinations of the e_n, by the conditions

$$H \cdot e_n = b_{n-1}e_{n-1} + a_n e_n + b_n e_{n+1} \qquad (n \geqq 0)$$

(with $e_{-1} = 0$, $b_{-1} = 0$). By abuse of notation, let H also denote the closure of the hermitian operator H.

(a) Let ζ be a complex number. If $y = \sum_{n=0}^{\infty} y_n e_n$ is an eigenvector of H^* corresponding to the eigenvalue ζ, show that $y_n = P_n(\zeta)y_0$, where P_n is a polynomial of degree n, determined recursively by the formulas $P_0(\zeta) = 1$ and

$$(1) \qquad b_n P_{n+1}(\zeta) = (\zeta - a_n)P_n(\zeta) - b_{n-1}P_{n-1}(\zeta)$$

for all $n \geqq 0$. Deduce that the defect of H is (1, 1) if, for some $\zeta \in \mathbf{C}$ such that $\mathscr{I}\zeta \neq 0$, we have

$$\sum_{n=0}^{\infty} |P_n(\zeta)|^2 < +\infty:$$

in which case this inequality is true for *all* $\zeta \in \mathbf{C}$ such that $\mathscr{I}\zeta \neq 0$. In the contrary case, the defect of H is (0, 0), in other words H is self-adjoint (having regard to the abuse of language mentioned above). In every case there exist self-adjoint extensions of H.

(b) Let F be the subspace of $\mathscr{C}_{\mathbf{C}}(\mathbf{R})$ consisting of polynomials with complex coefficients. Show that there exists a unique linear form α_F on F such that

$$\alpha_F(P_n P_m) = \delta_{mn} \qquad \text{(Kronecker index)}.$$

(If a polynomial is of the form RS, where $R = \sum_i u_i P_i$ and $S = \sum_i v_i P_i$, then we must have $\alpha_F(RS) = \sum_i u_i v_i$, which shows that α_F is unique. For the existence, it is enough to show that, if we write the product $U(x) = (x - z_0)R(x)S(x)$ in the two forms $((x - z_0)R(x))S(x)$ and $R(x)((x - z_0)S(x))$, where z_0 is any complex number, then the calculation of $\alpha_F(U)$ from each of these two products must give the same value. Reduce to the case where each of R, S is one of the polynomials P_n, and use the relation (1).)

(c) Deduce from (b) that there exists at least one bounded positive measure ν on $\mathbf{R}$ with respect to which the powers t^n ($n \geqq 0$) are integrable, and which extends the linear form α_F, so that the P_n form a (not necessarily total) *orthonormal system* in $\mathscr{L}^2_{\mathbf{C}}(\nu)$. (Use Problem 5 of Section 13.20.) All such measures ν have the same "moments"

$$c_n = \int t^n \, d\nu(t) \qquad (n \geqq 0);$$

each c_n is a rational fraction in the a_j and b_j ($1 \leqq j \leqq n-1$) with integer coefficients, and $P_n(\xi)$ is given as a function of the c_n by the formulas (3) of Problem 1.

(d) For each of the measures ν defined in (c), if $\mathscr{I}\zeta \neq 0$ we may write

$$\int \frac{P_n(t)}{t-\zeta} \, d\nu(t) = Q_n(\zeta) + w_\nu(\zeta) P_n(\zeta),$$

where

$$Q_n(\zeta) = \int \frac{P_n(t) - P_n(\zeta)}{t-\zeta} \, d\nu(t) \tag{2}$$

is a polynomial of degree $\leqq n-1$, and

$$w_\nu(\zeta) = \int \frac{d\nu(t)}{t-\zeta}. \tag{3}$$

Deduce that, if $\mathscr{I}\zeta \neq 0$,

$$\sum_{n=0}^{\infty} |Q_n(\zeta) + w_\nu(\zeta) P_n(\zeta)|^2 \leqq \int \frac{d\nu(t)}{|t-\zeta|^2} = \frac{w_\nu(\zeta) - \overline{w_\nu(\zeta)}}{\zeta - \bar{\zeta}}. \tag{4}$$

In order that there should be equality in (4) for one value of ζ such that $\mathscr{I}\zeta \neq 0$ (in which case there is equality for all such ζ), it is necessary and sufficient that polynomials should be *dense* in $\mathscr{L}^2_{\mathbf{C}}(\nu)$. (To show that this condition is sufficient, show that it implies that if a function $g \in \mathscr{L}^2_{\mathbf{C}}(\nu)$ is orthogonal to all polynomials, then $\int (t-\zeta)^{-1} g(t) \, d\nu(t) = 0$ for $\mathscr{I}\zeta > 0$, and use Problem 16 of Section 14.11.)

3. (a) With the hypotheses and notation of Problem 2, consider two sequences (y_n), (z_n) of complex numbers satisfying the recurrence relations

$$\lambda y_n = b_{n-1} y_{n-1} + a_n y_n + b_n y_{n+1}, \tag{1}$$

$$\mu z_n = b_{n-1} z_{n-1} + a_n z_n + b_n z_{n+1}, \tag{2}$$

where λ, μ are any complex numbers. Show that

$$(\mu - \lambda) \sum_{k=m}^{n-1} y_k z_k = b_{n-1}(y_{n-1} z_n - y_n z_{n-1}) - b_{m-1}(y_{m-1} z_m - y_m z_{m-1}). \tag{3}$$

(b) Deduce the following formulas from (a):

$$P_{n-1}(\lambda) Q_n(\lambda) - P_n(\lambda) Q_{n-1}(\lambda) = \frac{1}{b_{n-1}} \tag{4}$$

(show that the $Q_n(\lambda)$ satisfy (1));

(5) $$(\mu - \lambda)\sum_{k=0}^{n-1} P_k(\lambda)P_k(\mu) = b_{n-1}(P_{n-1}(\lambda)P_n(\mu) - P_n(\lambda)P_{n-1}(\mu))$$

("Christoffel–Darboux formula");

(6) $$\sum_{k=0}^{n-1} (P_k(\lambda))^2 = b_{n-1}(P_{n-1}(\lambda)P'_n(\lambda) - P_n(\lambda)P'_{n-1}(\lambda)).$$

(c) Denote by $K_n(\lambda)$, where $\lambda \in \mathbf{C}$ and $\mathscr{I}\lambda > 0$, the set of complex numbers w such that

$$\sum_{k=0}^{n-1} |wP_k(\lambda) + Q_k(\lambda)|^2 \leqq \frac{w - \overline{w}}{\lambda - \overline{\lambda}}.$$

Show that $K_n(\lambda)$ is a closed disk contained in the half-plane $\mathscr{I}w > 0$, with center at the point

$$\frac{Q_{n-1}(\lambda)\overline{P_n(\lambda)} - Q_n(\lambda)\overline{P_{n-1}(\lambda)}}{P_n(\lambda)\overline{P_{n-1}(\lambda)} - P_{n-1}(\lambda)\overline{P_n(\lambda)}}$$

and radius

$$\frac{1}{(\lambda - \overline{\lambda})\sum_{k=0}^{n-1} |P_k(\lambda)|^2},$$

the frontier of $K_n(\lambda)$ being the closure of the set of points

$$w_n(\lambda, t) = -\frac{Q_n(\lambda) - tQ_{n-1}(\lambda)}{P_n(\lambda) - tP_{n-1}(\lambda)}.$$

where $t \in \mathbf{R}$. We have $K_{n+1}(\lambda) \subset K_n(\lambda)$, and the frontiers of these two disks have a point in common. The intersection $K_\infty(\lambda)$ of these disks consists of a single point if

$$\sum_{n=0}^{\infty} |P_n(\lambda)|^2 = +\infty,$$

and otherwise is a disk of radius

$$\frac{1}{(\lambda - \overline{\lambda})\sum_{n=0}^{\infty} |P_n(\lambda)|^2}.$$

(d) Show that the following three properties are equivalent:

(α) The operator H deduced from the Jacobi matrix J is self-adjoint.

(β) There exists only one positive measure ν which extends the linear form α_F (in other words, the "moment problem" for the sequence (c_n) has only one solution).

(γ) The set $K_\infty(\lambda)$ consists of a single point, for some $\lambda \in \mathbf{C}$ with $\mathscr{I}\lambda > 0$ (and consequently for all $\lambda \in \mathbf{C}$ with $\mathscr{I}\lambda > 0$).

(To show that (β) and (γ) are equivalent, consider the matrices J_n with n rows and n columns obtained by deleting the rows and columns with indices $> n$ in J; consider the corresponding problem in $\mathbf{C}^n$, and then pass to the limit.)

(e) If the equivalent conditions in (d) are satisfied, then the P_n form a total system in $L^2_{\mathbf{C}}(\nu)$.

4. With the notation of Problems 2 and 3, show that if the moment problem for the sequence (c_n) has two distinct solutions, then both the series $\sum_n |P_n(\lambda)|^2$ and $\sum_n |Q_n(\lambda)|^2$ converge for $\mathcal{I}\lambda \neq 0$ (use formula (4) of Problem 2).

Deduce that the moment problem has only one solution in each of the following cases:

(a) $\sum_n \frac{1}{b_n} = +\infty$ (Use formula (4) of Problem 3);

(b) $\sum_n \frac{|a_{n+1}|}{b_n b_{n+1}} = +\infty$

(use the relation

$$P_n(\lambda)Q_{n+2}(\lambda) - P_{n+2}(\lambda)Q_n(\lambda) = \frac{\lambda - a_{n+1}}{b_n b_{n+1}});$$

(c) there exists a finite number r such that

$$b_{n-1} + a_n + b_n < r$$

for all n. (Note that equation (1) of Problem 3, with $\lambda = r$, takes the form

$$b_n(y_{n+1} - y_n) - b_{n-1}(y_n - y_{n-1}) = (r - b_{n-1} - a_n - b_n)y_n,$$

and deduce that the sequence $(Q_n(r))$ is an increasing sequence of numbers >0.)

5. (a) If (u_n) is a convergent series of real numbers >0, prove *Carleman's inequality*

$$\sum_{n=1}^{\infty} (u_1 u_2 \cdots u_n)^{1/n} < e \sum_{n=1}^{\infty} u_n,$$

by writing $u_1 u_2 \cdots u_n = (u_1 a_1 u_2 a_2 \cdots u_n a_n)/(n+1)^n$ for suitably chosen a_n, to be determined, and using the inequality of the means (Section 13.8, Problem 14).

(b) With the notation of Problem 2, show that if

$$\sum_{n=1}^{\infty} c_{2n}^{-1/2n} = +\infty,$$

then the corresponding moment problem has only one solution ("Carleman's criterion"). (Observe that

$$b_0 b_1 \cdots b_{n-1} \int (P_n(t))^2 \, d\nu(t) = \int t^n P_n(t) \, d\nu(t),$$

and deduce that $b_0 b_1 \cdots b_{n-1} \leqq (c_{2n})^{1/2}$; then use (a).)

6. Let H be a *simple* unbounded self-adjoint operator (Problem 1). Show that every closed subspace of $\mathrm{dom}(H)$ which is stable under H is of the form $\mathrm{dom}(H) \cap \mathrm{E}(\mathrm{A})$, where $\mathrm{E}(\mathrm{A})$ is the closed subspace of E which is the image of E under the orthogonal projector $\varphi_A(H)$, and A is a universally measurable subset of E. (By using 3(d) and 3(e), remark that we may assume that H is of the form M_v, where the bounded measure ν on $\mathbf{R}$ is such that polynomials are dense in $\mathscr{L}^2_{\mathbf{C}}(\nu)$.)

7. By using the existence of closed hermitian operators with defects $(1, 0)$ or $(0, 1)$ (15.13.9), give examples of closed hermitian operators with defect (m, n) where m, n are arbitrary integers $\geqq 0$, or $+\infty$.

8. A *conjugation* in a complex Hilbert space E is a semilinear bijection C of E onto E (i.e., such that $C \cdot (\alpha x + \beta y) = \bar{\alpha} C \cdot x + \bar{\beta} C \cdot y$) satisfying $(C \cdot x \mid C \cdot y) = (y \mid x)$ and $C^{-1} = C$. Show that if a closed operator T on E commutes with C (which implies that $C(\mathrm{dom}(T)) \subset \mathrm{dom}(T)$), and if $\mathrm{dom}(T)$ is dense in E, then C also commutes with T^*. Deduce that, if a hermitian operator H commutes with C, then $C(\mathrm{E}_H^+) = \mathrm{E}_H^-$, and consequently the defects of H are *equal*. Apply this result in particular to the case where E is obtained by extension of scalars from a real Hilbert space E_0, and H is the extension to E of an unbounded operator H_0 defined on a dense subspace $\mathrm{dom}(H_0)$ of E_0, such that $(H_0 \cdot x \mid y) = (x \mid H_0 \cdot y)$ for all x, y in $\mathrm{dom}(H_0)$.

9. Show that a *surjective* hermitian operator H is self-adjoint. (For each $y \in \mathrm{dom}(H^*)$, remark that there exists $z \in \mathrm{dom}(H)$ such that $H^* \cdot y = H \cdot z$, and show that z must equal y.)

10. Let E be an infinite-dimensional separable Hilbert space.

(a) If (a_n) is any infinite sequence in E, then the vector subspace generated by the a_n (i.e. the set of all (finite) linear combinations of the a_n) is not equal to E (cf. (12.16.1)).

(b) Show that there exist two Hilbert bases (a_n), (b_n) of E such that, if F and G are the vector subspaces of E generated by the a_n and the b_n respectively, we have $\mathrm{F} \cap \mathrm{G} = \{0\}$. (Starting with an arbitrary Hilbert basis (a_n), construct inductively a total sequence (c_n) such that the subspace G it generates is such that $\mathrm{F} \cap \mathrm{G} = \{0\}$, by using (a). Then orthonormalize the sequence (c_n).)

(c) Suppose that the Hilbert bases (a_n), (b_n) have the property stated in (b). Consider two compact self-adjoint operators A, B on E, defined by $A \cdot a_n = \lambda_n a_n$ and $B \cdot b_n = \lambda_n b_n$, where (λ_n) is a sequence of real numbers >0 and tending to 0. Show that the sequence (λ_n) can be chosen so that $A(\mathrm{E}) \cap B(\mathrm{E}) = \{0\}$. (Proceed by induction: let S_n be the set of points of F of the form $\sum_{k=1}^{n} \xi_k \lambda_k a_k$ with $\sum_{k=1}^{n} |\xi_k| = 1$, and let d_n be the distance (strictly positive) of S_n from the subspace G_n generated by $b_1, \ldots, b_n$; choose the λ_k with $k > n$ such that $\sum_{k>n} \lambda_k < d_n/n$. Then prove that, if U is the closed ball with center 0 and radius 1 in E, we have $A(\mathrm{U}) \cap B(\mathrm{U}) = \{0\}$.)

(d) The operators A and B are injective; A^{-1} and B^{-1} are therefore self-adjoint closed operators (Problem 8) such that $\mathrm{dom}(A^{-1}) \cap \mathrm{dom}(B^{-1}) = \{0\}$.

(e) Deduce from (d) an example of a self-adjoint operator H and a unitary operator U on E such that $U^2 = 1_{\mathrm{E}}$ and $\mathrm{dom}(H) \cap \mathrm{dom}(U^{-1}HU) = \{0\}$. (Take $\mathrm{E} = \mathrm{F} \oplus \mathrm{F}$ and $U \cdot (x, y) = (y, x)$, and take H to be equal to A^{-1} on one of the summands F, and to B^{-1} on the other, where A and B are defined as above.) Hence give an example of a closed operator T such that $\mathrm{dom}(T)$ is dense in E, but $\mathrm{dom}(T^2) = \{0\}$.

11. Let T be a closed operator on a separable Hilbert space E. The *essential spectrum* of T is the set of $\xi \in \mathbf{C}$ such that $\mathrm{Im}(T - \xi I)$ is not closed; it is a subset of the spectrum of T.

(a) Show that if N is an unbounded normal operator on E, then the isolated points of $\mathrm{Sp}(N)$ do not belong to the essential spectrum of N. (If λ is isolated in $\mathrm{Sp}(N)$ and if $\mathrm{M} = \mathrm{Sp}(N) - \{\lambda\}$, show that $\mathrm{Im}(N - \lambda I)$ is the image of E under the projector $P = \varphi_{\mathrm{M}}(N)$ (Section 15.12, Problem 7), by observing that there exists a continuous bounded function f on $\mathbf{C}$ such that $f(N) \cdot y \in \mathrm{dom}(N)$ and $(N - \lambda I) \cdot (f(N) \cdot y) = y$ for all $y \in P(\mathrm{E})$.

(b) Conversely, show that for each unbounded self-adjoint operator A on E, a point $\lambda \in \mathrm{Sp}(A)$ which does not belong to the essential spectrum of A is isolated in $\mathrm{Sp}(A)$

and is an eigenvalue of A. The essential spectrum of A is therefore the set of *non-isolated* points of $\mathrm{Sp}(A)$. (Reduce to the case $\lambda = 0$, and by using Problem 1 of Section 15.12 reduce further to the case where $\mathrm{Ker}(A) = \{0\}$; then show that 0 is a regular value for A, and use (15.12.11).)

(c) If H is a closed hermitian operator of defect (m, n), then $\mathrm{Sp}(H)$ contains the half-plane $\mathscr{I}z \geqq 0$ (resp. $\mathscr{I}z \leqq 0$) if $m > 0$ (resp. $n > 0$). The essential spectrum of H is contained in $\mathbf{R}$. If m and n are finite, and if H_1 is a closed hermitian operator which extends H, then the essential spectra of H and H_1 are the same (Section 15.12, Problem 1(h)).

12. Let H be an unbounded hermitian operator on a separable Hilbert space E.

(a) If H_1 is the restriction of H^* to the subspace $\mathrm{dom}(H_1) = \mathrm{dom}(H) + \mathrm{Ker}(H^*)$, show that H_1 is hermitian.

(b) Show that if $\mathrm{Im}(H)$ is closed in E, then H_1 is self-adjoint. (If $x \in \mathrm{dom}(H_1^*)$, show that $H_1^* \cdot x$ is orthogonal to $\mathrm{Ker}(H^*)$, and therefore $H_1^* \cdot x \in \mathrm{Im}(H)$ (Section 15.12, Problem 1(c)); deduce that $x \in \mathrm{dom}(H_1)$.) If H is closed and if there exists a real number which does not belong to the essential spectrum of H, deduce that the defects of H are *equal*.

(c) Show that $E_{H_1}^+$ (resp. $E_{H_1}^-$) (notation of (15.13.6) is the intersection of E_H^+ (resp. E_H^-) with the orthogonal supplement of $\mathrm{Ker}(H^*)$. Deduce that if there exists $\lambda \in \mathbf{R}$ not belonging to the essential spectrum of H, and if the defects of H (which are necessarily equal) have a finite value m, then $\dim(\mathrm{Ker}(H^* - \lambda I)) \geqq m$.

(d) Suppose that H is closed, and let x be an eigenvector of H^* corresponding to a *real* eigenvalue λ. Put $x = x_0 + y + z$, where $x_0 \in \mathrm{dom}(H)$, $y \in E_H^+$ and $z \in E_H^-$. Show that $\|y\| = \|z\|$ (reduce to the case $\lambda = 0$).

(e) Suppose that H is closed, and let λ be a real number which does not belong to the essential spectrum of H; suppose that the defects of H are equal and finite, say m, and that $\dim(\mathrm{Ker}(H - \lambda I))$ is finite, say k. Then $\dim(\mathrm{Ker}(H^* - \lambda I)) = m + k$. (We may assume that $\lambda = 0$. By considering the restriction of H to the orthogonal supplement of $\mathrm{Ker}(H)$, reduce to the case where $k = 0$. Then deduce from (d) that $\mathrm{Ker}(H^*)$ cannot have dimension $> m$, by using the hypothesis $\mathrm{Ker}(H^*) \cap \mathrm{dom}(H) = \{0\}$, and complete the proof by using (c).)

13. Let H be a closed hermitian operator whose defects are equal and finite, say m.

(a) If V is defined as in (15.13.4), then the self-adjoint extensions A of H are of the form $A = i(I + U)(I - U)^{-1}$, where U is a unitary operator extending V, such that $U(E_H^+) = E_H^-$; $\mathrm{dom}(A)$ is therefore the direct sum of $\mathrm{dom}(H)$ and the subspace $(I - U)(E_H^+)$ of dimension m, contained in $\mathrm{dom}(H^*)$.

(b) For a real number λ to be an eigenvalue of a self-adjoint extension A of H, it is necessary and sufficient that λ should be an eigenvalue of H^* (use Problem 12(d)). Show that if $\lambda \in \mathbf{R}$ is not an eigenvalue of H, then there exists a self-adjoint extension A of H for which λ is not an eigenvalue. (Use Problem 12(d) and (e), and choose suitably the unitary operator U of (a).)

(c) Suppose that $m > 0$. For each $\lambda \in \mathbf{R}$, show that there exists a self-adjoint extension A of H such that $\lambda \in \mathrm{Sp}(A)$. (Remark that, if A_0 is a self-adjoint extension of H and $\lambda \notin \mathrm{Sp}(A_0)$, then it follows from Problems 12(e) and 11(c) that λ is an eigenvalue of H^*.)

(d) Let A_1, A_2 be two self-adjoint extensions of H. If P^+ (resp. P^-) is the orthogonal projection onto E_H^+ (resp. E_H^-), show that the continuous operator

$$D = (A_2 + iI)^{-1} - (A_1 + iI)^{-1}$$

is such that $D = P^- D = DP^+$, and is therefore of rank m (note that, if $y \in \mathrm{E}$ and $x_1 = (A_1 + iI)^{-1} \cdot y$, then $x_1 \in \mathrm{dom}(H^*)$ and $y = (H^* + iI) \cdot x_1$.)

14. A closed unbounded hermitian operator H on E is said to be *majorized* (resp. *minorized*) if there exists a real number c such that $(H \cdot x \mid x) \leqq c(x \mid x)$ (resp. $(H \cdot x \mid x) \geqq c(x \mid x)$) for all $x \in \mathrm{dom}(H)$. The operator H is said to be *positive* if $(H \cdot x \mid x) \geqq 0$ for all $x \in \mathrm{dom}(H)$. Thus H is majorized (resp. minorized) if and only if there exists $c \in \mathbf{R}$ such that $cI - H$ (resp. $H - cI$) is positive.
(a) Let H be a positive closed hermitian operator on E. The hermitian form $f(x, y) = (x + H \cdot x \mid y)$ on $\mathrm{dom}(H)$ makes it a prehilbert space, which may be considered as a dense subspace of a Hilbert space G. Show that the canonical injection j of $\mathrm{dom}(H)$ into E extends by continuity to an injection of G into E, so that G may be identified with a subspace of E. We denote the scalar product on G again by $f(x, y)$. Show that there exists a continuous linear mapping B of E into G (where G is considered as a Hilbert space) such that $(x \mid y) = f(B \cdot x, y)$ for all $x \in \mathrm{E}$ and $y \in \mathrm{G}$. Considered as a mapping of E into E, the operator B is positive, continuous, self-adjoint and of norm $\leqq 1$; we have $(B \cdot x \mid x) \geqq \|B \cdot x\|^2$, and $B(I + H) \cdot y = y$ for all $y \in \mathrm{dom}(H)$. Deduce that $A = B^{-1} - I$ is an unbounded positive self-adjoint operator, such that $\mathrm{dom}(A) = B(\mathrm{E})$, which *extends* H. Hence the defects of H are equal.
(b) Suppose that the defects of H are equal and finite, say m. If A_1 is any self-adjoint extension of H, show that the intersection of $\mathrm{Sp}(A_1)$ and the interval $]-\infty, 0[$ consists of eigenvalues of A_1, and that the total number of such eigenvalues, each counted according to its multiplicity, is $\leqq m$. (For each compact interval $\mathrm{J} \subset]-\infty, 0[$, show that the image of E under the projector $\varphi_{\mathrm{J}}(A_1)$ (Section 15.12, Problem 7), which is contained in $\mathrm{dom}(A_1)$, cannot have dimension $> m$, otherwise it would contain a nonzero vector $x \in \mathrm{dom}(H)$ (Problem 13(a)); show that this conclusion would contradict the positivity of H.)

15. Let H be a closed hermitian operator on E, with defect (m, n), so that $-H$ has defect (n, m). Show that H is the restriction to E of an unbounded self-adjoint operator A on the Hilbert sum $\mathrm{E} \oplus \mathrm{E}$.

16. (a) Let G be an analytic function on the disk $\mathrm{B} : |z| < 1$, such that $\mathscr{R}\mathrm{G}(z) \geqq 0$ in this disk. Show that the function

$$\mathrm{K}(u, v) = (\mathrm{G}(u) + \overline{\mathrm{G}(v)})/(1 - u\bar{v})$$

on $\mathrm{B} \times \mathrm{B}$ is of positive type (Section 6.3, Problem 4). (Use Section 14.11, Problem 18.)
(b) Let F be an analytic function on the half-plane $\mathrm{D} : \mathscr{I} z > 0$, such that $\mathscr{I}\mathrm{F}(z) \geqq 0$ for all $z \in \mathrm{D}$. Show that the function

$$\mathrm{K}(u, v) = (\mathrm{F}(u) - \overline{\mathrm{F}(v)})/(u - \bar{v})$$

is of positive type on $\mathrm{D} \times \mathrm{D}$ (map D conformally on the disk B, and use (a)).
(c) Let S be a denumerable infinite subset of the half-plane $\mathscr{I} z > 0$. Let $\mathrm{C}(\varphi)$ denote the angular sector in this half-plane, defined by $|\mathscr{R} z| \leqq (\mathscr{I} z) \cos \varphi$, for $0 < \varphi < \frac{1}{2}\pi$. Suppose that there exists a sequence (σ_n) of points of S contained in a sector $\mathrm{C}(\varphi_0)$ and tending to 0. Let f be a mapping of S into the half-plane $\mathscr{I} z > 0$. In order that there should exist an analytic function F, defined on the half-plane $\mathscr{I} z > 0$, such that $\mathscr{I}\mathrm{F}(z) > 0$ on this half-plane and such that $|\mathrm{F}(z)/z|$ is bounded in every sector $\mathrm{C}(\varphi)$,

it is necessary and sufficient that f satisfy the following two conditions: (1) the sequence $(f(\sigma_n)/\sigma_n)$ is bounded; (2) the function

$$k(s, t) = (f(s) - \overline{f(t)})/(s - \bar{t})$$

on $S \times S$ is of positive type. (To prove sufficiency, note that there exists a separable Hilbert space E and a mapping $s \mapsto u_s$ of S into E such that $k(s, t) = (u_s \mid u_t)$ (Section 6.3, Problem 8(c)), and we may even assume that the set of points u_s is total in E. Show then that the sequence (u_{σ_n}) in E is bounded and converges weakly to a point u_0 (use (12.15.7.1) and (7.5.5)); we have $(u_s \mid u_0) = f(s)/s$ for all $s \in S$. Show that there exists an unbounded hermitian operator H on E such that $H \cdot u_s = (u_s - u_0)/s$ for all $s \in S$; using Problem 14, show that there exists a Hilbert space F containing E, and an unbounded self-adjoint operator A on F such that, for $\mathscr{I}z > 0$, the function $F(z) = z((I - zA)^{-1} \cdot u_0 \mid u_0)$ satisfies the required conditions. Putting

$$v_z = (I - zA)^{-1} \cdot u_0,$$

show that $\mathscr{I}F(z) = \mathscr{I}(z\|v_z\|^2)$ and that $\|v_z\| \sin \varphi \leqq \|u_0\|$ for $z \in C(\varphi)$.)

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